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COMMUTATIVITY CONDITIONS ON RINGS

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We prove the following result: let R be an arbitrary ring with centre Z such that for every $x, y \in R$, there exists a positive integer $n = n(x, y) \geqslant 1$ such that $(xy)^n - y^n x^n \in Z$ and $(yx)^n - x^n y^n \in Z$; then, if R has no non-zero nil ideals, R is commutative. We also prove a result on commutativity of general rings: if R is $r!$-torsion free and for all $x, y \in R$, $[x^r, y^s] = 0$ for fixed integers $r \geqslant s \geqslant 1$, then R is commutative. As a corollary we obtain that if R is $(n+1)!$-torsion free and there exists a fixed $n \geqslant 1$ such that $(xy)^n - y^n x^n = (yx)^n - x^n y^n \in Z$ for all $x, y \in R$, then R is commutative.

In this note we study certain relations among the elements of a ring R forcing the commutativity of the ring under suitable conditions. In [3] Herstein proved that if R is a ring without nil ideals and there exists an integer $n \geqslant 1$ such that for all $x, y \in R$, $(xy)^n = x^n y^n$ then R must be commutative. In this direction, in [1] it was proved that if R has no nonzero nil ideals and if for each finite subset F of R there exists an integer $n = n(F) \geqslant 1$ such that $(xy)^n - y^n x^n \in Z$, where Z is the centre of R, for all $x, y \in F$ then R is commutative. Here we shall improve this result by proving the following more natural theorem: let R be an arbitrary ring with centre Z such that for every $x, y \in R$, there exists a positive integer $n = n(x, y) \geqslant 1$ such that

$$(xy)^n - y^n x^n \in Z \text{ and } (yx)^n - x^n y^n \in Z;$$

then, if R has no nonzero nil ideals, R is commutative. We also prove a theorem concerning the commutativity of general rings; more precisely, if R is $r!$-torsion free and, for all $x, y \in R$,

$$[x^r, y^s] = 0$$

for fixed integers $r \geqslant s \geqslant 1$, then R is commutative.

As a corollary we obtain that if R is $(n+1)!$-torsion free and there exists a fixed $n \geqslant 1$ such that $(xy)^n - y^n x^n = (yx)^n - x^n y^n \in \mathbb{Z}$, for all $x, y \in R$, then R is commutative.

We start with:

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THEOREM 1. *Let R be a ring with no nonzero nil ideals such that for all $x, y \in R$, there exists a positive integer $n = n(x, y) \geqslant 1$ such that*

$$(xy)^n - y^n x^n \in Z \text{ and } (yx)^n - x^n y^n \in Z.$$

Then R is commutative.

PROOF: Let $x, y \in R$ and let $n \geqslant 1$ be such that $(xy)^n - y^n x^n \in Z$ and $(yx)^n - x^n y^n \in Z$. Hence, for suitable $z_1, z_2 \in Z$ we have

$$(xy)^n x - y^n x^n x = z_1 x$$

and

$$x(yx)^n - x x^n y^n = z_2 x.$$

Subtracting the first equality from the second it follows that

$$y^n x^{n+1} - x^{n+1} y^n = (z_2 - z_1)x;$$

hence, by commuting with x, we get

$$[y^n, x^{n+1}, x] = 0.$$

But then, by a result of Bell-Klein-Nade [2], the ring R is commutative. ∎

We now prove a result on commutators:

THEOREM 2. *Let R be a ring with identity 1, such that, for every $x, y \in R$*

$$[x^r, y^s] = 0,$$

for fixed integers $r \geqslant s \geqslant 1$. Then, if R is $r!$-torsion free, R is commutative.

PROOF: We will make use of a Vandermonde determinant argument. Since R has identity 1, the elements $i + x$ for $1 \leqslant i \leqslant r$ are defined and we have

$$(i+x)^r y^s - y^s (i+x)^r = 0.$$

By expanding out the sums, we obtain

$$\binom{r}{1} i^{r-1}(xy^s - y^s x) + \ldots + \binom{r}{r-1} i (x^{r-1} y^s - y^s x^{r-1}) = 0$$

for all $1 \leqslant i \leqslant r$, where the $\binom{r}{j}$'s are the usual binomial coefficients.

It follows that

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 2^{r-1} \\ . & . & . & . \\ 1 & r & \ldots & r^{r-1} \end{pmatrix} \begin{pmatrix} x^r y^s - y^s x^r \\ \binom{r}{r-1}(x^{r-1} y^s - y^s x^{r-1}) \\ \ldots \\ \binom{r}{1}(xy^s - y^s x) \end{pmatrix} = 0.$$

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Since R is $r!$-torsion free the Vandermonde matrix has non-zero determinant; hence the column vector on the right must be zero; in particular

$$\binom{r}{1}(xy^s - y^s x) = 0$$

and, since R is $r!$-torsion free, we get

$$xy^s = y^s x.$$

Now, by using the same argument applied to y, remembering that R is $s!$-torsion free, we get the desired result, that is

$$xy = yx, \text{ for all } x, y \in R.$$

□

From Theorem 2, it follows:

Corollary. *Let R be a ring, n a fixed positive integer such that for all $x, y \in R$*

$$(xy)^n - y^n x^n = (yx)^n - x^n y^n \in Z.$$

If R is $(n+1)!$-torsion free then R is commutative.

Proof: Let $x, y \in R$; then

$$(xy)^n - y^n x^n = z = (yx)^n - x^n y^n$$

for a suitable $z \in Z$. As in the proof of Theorem 1 this easily leads to

$$y^n x^{n+1} - x^{n+1} y^n = 0,$$

that is

$$[y^n, x^{n+1}] = 0.$$

As this point, to reach the conclusion it is enough to make use of the previous theorem. □

References

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