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# ON 3-DIMENSIONAL CONTACT SLANT SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Recently, B.-Y. Chen obtained an inequality for slant surfaces in complex space forms. Further, B.-Y. Chen and one of the present authors proved the non-minimality of proper slant surfaces in non-flat complex space forms. In the present paper, we investigate 3-dimensional proper contact slant submanifolds in Sasakian space forms. A sharp inequality is obtained between the scalar curvature (intrinsic invariant) and the main extrinsic invariant, namely the squared mean curvature.

It is also shown that a 3-dimensional contact slant submanifold $M$ of a Sasakian space form $\widetilde{M}(c)$, with $c \neq 1$, cannot be minimal.

## 1. Introduction.

In [**3**], Chen proved that the squared mean curvature $\|H\|^2$ and the Gauss curvature $K$ of a proper slant surface $M$ in a complex space form $\widetilde{M}(c)$ satisfy the following basic inequality:

$$\|H(p)\|^2 \geqslant 2K(p) - 2(1 + 3\cos^2\theta)c, \tag{1.1}$$

at each point $p \in M$.

The equality sign of (1.1) holds at a point $p \in M$ if and only if with respect to some suitable orthonormal basis $\{e_1, e_2, e_3, e_4\}$ at $p$, the shape operators at $p$ take the following forms:

$$A_{e_3} = \begin{pmatrix} 3\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}. \tag{1.2}$$

The purpose of the present paper is to establish a sharp inequality for 3-dimensional proper contact slant submanifolds in Sasakian space forms, involving the scalar curvature $\tau$ and the squared mean curvature $\|H\|^2$.

More precisely, we prove that the following estimate holds.

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**Theorem 1.** *Let $M$ be a 3-dimensional proper contact slant submanifold of a 5-dimensional Sasakian space form $\widetilde{M}(c)$. Then, we have*

$$\|H\|^2 \geqslant \frac{8}{9}\tau - \frac{2}{9}\left[c+3+(3c+5)\cos^2\theta\right].$$

The case in which equality holds is investigated.

In [4], B.-Y. Chen and one of the present authors proved that there do not exist minimal proper slant surfaces in a non-flat complex space form. We show that there do not exist 3-dimensional minimal proper contact slant submanifolds in a 5-dimensional Sasakian space form $\widetilde{M}(c)$, with $c \neq 1$.

Finally, we obtain another inequality between an intrinsic invariant (scalar curvature) and extrinsic invariants (scalar normal curvature and squared mean curvature) of a 3-dimensional proper contact slant submanifold in a 5-dimensional Sasakian space form, and investigate the case in which equality holds.

## 2. Submanifolds of a Sasakian space form.

Let $(\widetilde{M}, g)$ be a $(2m+1)$-dimensional Riemannian manifold endowed with an endomorphism $\phi$ of its tangent bundle $T\widetilde{M}$, a vector field $\xi$ and a 1-form $\eta$ such that

$$\begin{cases} \phi^2 X = -X + \eta(X)\xi\,,\ \phi\xi = 0\,,\ \eta\circ\phi = 0\,,\ \eta(\xi) = 1\,, \\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)\,,\ \eta(X) = g(X,\xi)\,, \end{cases} \tag{2.1}$$

for all vector fields $X, Y \in \Gamma(T\widetilde{M})$.

If, in addition, $d\eta(X,Y) = g(\phi X, Y)$, then $\widetilde{M}$ is said to have a contact Riemannian structure $(\phi, \xi, \eta, g)$. If, moreover, the structure is normal, that is, if

$$[\phi X, \phi Y] + \phi^2[X,Y] - \phi[X,\phi Y] - \phi[\phi X, Y] = -2d\eta(X,Y)\xi\,,$$

then the contact Riemannian structure is called a *Sasakian* structure and $\widetilde{M}$ is called a Sasakian manifold. On a Sasakian manifold one has

$$(\widetilde{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.2}$$

where $\widetilde{\nabla}$ is the Riemannian connection with respect to $g$. For more details and background, we refer to the standard references [**1, 8**].

A plane section $\sigma$ in $T_p\widetilde{M}$ of a Sasakian manifold $\widetilde{M}$ is called a $\phi$-section if it is spanned by $X$ and $\phi X$, where $X$ is a unit tangent vector orthogonal to $\xi$. The sectional curvature $\overline{K}(\sigma)$ with respect to a $\phi$-section $\sigma$ is called a $\phi$-sectional curvature. If a Sasakian manifold $\widetilde{M}$ has constant $\phi$-sectional curvature $c$, then it is called a *Sasakian space form* and is denoted by $\widetilde{M}(c)$.

The curvature tensor $\widetilde{R}$ of a Sasakian space form $\widetilde{M}(c)$ is given by ([1]):

$$
\begin{aligned}
(2.3)\quad \widetilde{R}(X,Y)Z = {} & \frac{c+3}{4}\big(g(Y,Z)X - g(X,Z)Y\big) \\
& + \frac{c-1}{4}\big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\
& - g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\big),
\end{aligned}
$$

for any tangent vector fields $X, Y, Z$ to $\widetilde{M}(c)$.

An $n$-dimensional submanifold $M$ of a Sasakian space form $\widetilde{M}(c)$ is called a *contact $\theta$-slant* submanifold if the structure vector field $\xi$ is tangent to $M$ and for each non-zero vector $X$ tangent to $M$ at $p \in M$ and orthogonal to $\xi$, the angle $\theta(X)$ between $\phi X$ and $T_pM$ is independent of the choice of $X$ and $p$ (see, for instance, [3] and [2]). Moreover, $M$ is a *proper* contact slant submanifold if $0 < \theta < \pi/2$, that is, $M$ is neither invariant nor anti-invariant submanifold.

It is easily seen that the minimum codimension of an $n$-dimensional proper contact slant submanifold is $n-1$. The anti-invariant submanifolds have the same property (see [7]).

## 3. Main results.

Let $M$ be an $n$-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of the plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space $T_pM$, the scalar curvature $\tau$ at $p$ is defined by

$$\tau(p) = \sum_{1 \leqslant i < j \leqslant n} K(e_i \wedge e_j)\,.$$

We consider a 3-dimensional proper contact $\theta$-slant submanifold $M$ in a 5-dimensional Sasakian space form $\widetilde{M}(c)$. For any vector $X$ tangent to $M$, we put

$$\phi X = PX + FX,$$

where $PX$ and $FX$ denote the tangential and normal components of $\phi X$, respectively.

Let $e_1$ be a unit vector tangent to $M$ and orthogonal to $\xi$. We construct a canonical orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ defined by

$$
\begin{aligned}
e_2 &= \frac{1}{\cos\theta}Pe_1, & e_3 &= \xi, \\
e_4 &= \frac{1}{\sin\theta}Fe_1, & e_5 &= \frac{1}{\sin\theta}Fe_2.
\end{aligned}
$$

We call such a basis an adapted slant orthonormal basis.

**Theorem 1.** *Let $M$ be a 3-dimensional proper contact slant submanifold of a 5-dimensional Sasakian space form $\widetilde{M}(c)$. Then, we have*

$$\|H\|^2 \geqslant \frac{8}{9}\tau - \frac{2}{9}\left[c+3+(3c+5)\cos^2\theta\right]. \tag{3.1}$$

*Moreover, the equality sign of (3.1) holds at a point $p \in M$ if and only if with respect to some suitable adapted slant orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ at $p$, the shape operators at $p$ take the following forms:*

$$A_{e_4} = \begin{pmatrix} 3\lambda & 0 & \sin\theta \\ 0 & \lambda & 0 \\ \sin\theta & 0 & 0 \end{pmatrix}, \qquad A_{e_5} = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}. \tag{3.2}$$

Proof: Let $p \in M$ and $\{e_1, e_2, e_3, e_4, e_5\}$ an adapted slant orthonormal basis. We have

$$\tau(p) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + K(e_2 \wedge e_3).$$

We recall the Gauss equation for the submanifold $M$ in the Sasakian space form $\widetilde{M}(c)$:

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g\big(h(X,W), h(Y,Z)\big) - g\big(h(X,Z), h(Y,W)\big),$$

for all vector fields $X, Y, Z, W$ tangent to $M$, where $h$ denotes the second fundamental form and $R$ the curvature tensor of $M$. Then, by using (2.3) and Gauss equation, it follows that

$$\begin{aligned} K(e_1 \wedge e_2) = R(e_1, e_2, e_1, e_2) = \frac{c+3}{4} &+ \frac{3}{4}(c-1)\cos^2\theta \\ &+ g\big(h(e_1,e_1), h(e_2,e_2)\big) - g\big(h(e_1,e_2), h(e_1,e_2)\big), \end{aligned}$$

or equivalently,

$$K(e_1 \wedge e_2) = \frac{c+3}{4} + \frac{3}{4}(c-1)\cos^2\theta + h^4_{11}h^4_{22} + h^5_{11}h^5_{22} - (h^4_{12})^2 - (h^5_{12})^2, \tag{3.3}$$

where $h^r_{ij} = g(h(e_i, e_j), e_r), i, j \in \{1, 2, 3\}, r \in \{4, 5\}$.

It is easily seen that

$$A_{Fe_1}e_2 = A_{Fe_2}e_1,$$

which implies $h^5_{12} = h^4_{22}$.

We choose the unit normal vector $e_4 \in T^\perp_p M$ parallel to the mean curvature vector $H(p)$ of $M$ in $p$. Then one has $H(p) = \|H(p)\| e_4$, which leads to

$$\|H(p)\|^2 = \frac{1}{9}(h^4_{11} + h^4_{22})^2, \quad h^5_{11} + h^5_{22} = 0.$$

The relation (3.3) becomes

$$K(e_1 \wedge e_2) = \frac{c+3}{4} + \frac{3}{4}(c-1)\cos^2\theta + h^4_{11}h^4_{22} - (h^5_{11})^2 - (h^4_{12})^2 - (h^4_{22})^2. \tag{3.4}$$

The trivial inequality $(\mu - 3\lambda)^2 \geqslant 0$ is equivalent to $(\mu+\lambda)^2 \geqslant 8(\lambda\mu - \lambda^2)$. If we put

$$\mu = h^4_{11}, \quad \lambda = h^4_{22},$$

the above inequality and the equation (3.4) imply

$$\|H(p)\|^2 \geqslant \frac{8}{9}\Big[K(e_1 \wedge e_2) - \frac{c+3}{4} - \frac{3}{4}(c-1)\cos^2\theta\Big]. \tag{3.5}$$

On the other hand, using Gauss equation we find

$$K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = 1 - \sin^2\theta = \cos^2\theta. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain the inequality (3.1) to prove.

Moreover, equality holds in (3.1) at a point $p \in M$ if and only if

$$h^4_{11} = 3h^4_{22}, \quad h^4_{12} = 0, \quad h^5_{11} = 0.$$

Then the shape operators take the desired forms. □

Next, we shall prove the non-minimality of 3-dimensional proper contact slant submanifolds in 5-dimensional Sasakian space forms $\widetilde{M}(c)$, with $c \neq 1$.

**THEOREM 2.** *Let $M$ be a 3-dimensional proper contact slant subnmanifold in a 5-dimensional Sasakian space form $\widetilde{M}(c)$, with $c \neq 1$. Then $M$ is not minimal.*

PROOF: We assume that $M$ is a 3-dimensional minimal proper contact slant submanifold in a 5-dimensional Sasakian space form $\widetilde{M}(c)$, with $c \neq 1$. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be an adapted slant orthonormal local frame.

For any normal vector $U$, we put $\phi U = tU + fU$, where $tU$ and $fU$ denote the tangential and normal components of $\phi U$, respectively. Clearly one has

$$te_4 = -(\sin\theta)e_1, \quad te_5 = -(\sin\theta)e_2,$$
$$fe_4 = -(\cos\theta)e_5, \quad fe_5 = (\cos\theta)e_4.$$

Taking the normal part of the relation (2.2), we get

$$\nabla^\perp_X FY - F\nabla_X Y = fh(X,Y) - h(X,PY),$$

where $\nabla^\perp$ is the normal connection of $M$.

In particular, one has

$$\nabla^\perp_{e_1} e_4 = \frac{1}{\sin\theta}\big\{\omega^2_1(e_1)Fe_2 + h^4_{11}fe_4 + h^5_{11}fe_5 - \cos\theta(h^4_{12}e_4 + h^5_{12}e_5)\big\},$$

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where $\{\omega_A^B\}$ denote the connection 1-forms on $\widetilde{M}(c)$.

The last equation implies

$$\omega_4^5(e_1) = \omega_1^2(e_1) - (\cot\theta)(h_{11}^4 + h_{22}^4).$$

Since $M$ is minimal, it follows that $\omega_4^5(e_1) = \omega_1^2(e_1)$.

Similarly $\omega_4^5(e_2) = \omega_1^2(e_2)$. Then, one finds

$$\omega_4^5 = \omega_1^2.$$

Let $p \in M$ be a non-totally geodesic point. Consider the function

$$\gamma_p : T_p^1 M \to \mathbf{R}, \quad \gamma_p(v) = g\big(h(v,v), Fv\big),$$

where $T_p^1 M = \{v \in T_pM \mid g(v,v) = 1\}$. Since $T_p^1 M$ is a compact set, there exists a vector $v \in T_p^1 M$ such that $\gamma_p(v) = \inf \gamma_p(T_p^1 M) = -\mu < 0, \mu \in \mathbf{R}$. It is easily seen that $v$ is an eigenvector of the shape operator $A_{Fv}$. Then we can choose an orthonormal basis $\{e_1, e_2, e_3\}$ of $T_pM$, with $e_1 = v$ and $e_3 = \xi$, such that

$$h(e_1,e_1) = -\mu e_4, \quad h(e_1,e_2) = \mu e_5, \quad h(e_2,e_2) = \mu e_4.$$

Consequently, there exists a local adapted slant orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ such that the second fundamental form $h$ satisfies

$$h(e_1,e_1) = -\lambda e_4, \quad h(e_1,e_2) = \lambda e_5, \quad h(e_2,e_2) = \lambda e_4,$$

for a certain smooth function $\lambda$ on $M$.

Using (2.3), a straightforward calculation leads to

$$(\widetilde{R}(e_2,e_1)e_1)^\perp = \frac{3}{4}(c-1)(\sin\theta\cos\theta)e_4,$$
$$(\widetilde{R}(e_1,e_2)e_2)^\perp = -\frac{3}{4}(c-1)(\sin\theta\cos\theta)e_5.$$

Therefore the Codazzi equation gives

$$e_2\lambda = 3\lambda\omega_1^2(e_1) - \frac{3}{4}(c-1)\sin\theta\cos\theta,$$
$$e_2\lambda = 3\lambda\omega_1^2(e_1) + \frac{3}{4}(c-1)\sin\theta\cos\theta.$$

Thus, we obtain $(c-1)\sin\theta\cos\theta = 0$, which is a contradiction. ∎

It is known that any invariant submanifold of a Sasakian manifold is minimal. Combining this result with Theorem 2, we find the following.

**Corollary 3.** *Let $M$ be a 3-dimensional minimal contact slant submanifold of a 5-dimensional Sasakian space form $\widetilde{M}(c)$. Then either $c = 1$, or $M$ is invariant, or $M$ is anti-invariant.*

A Sasakian space form $\widetilde{M}(1)$ is locally isometric to a sphere.

We characterise the 3-dimensional minimal proper contact slant submanifold in $S^5$.

**Proposition 4.** *A 3-dimensional proper contact slant submanifold in the 5-dimensional sphere $S^5$ is minimal if and only if with respect to some suitable local adapted slant orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$, the shape operators take the following forms:*

$$A_{e_4} = \begin{pmatrix} -\lambda & 0 & \sin\theta \\ 0 & \lambda & 0 \\ \sin\theta & 0 & 0 \end{pmatrix}, \qquad A_{e_5} = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}.$$

Proof: Let $M$ be a 3-dimensional minimal proper contact slant submanifold in $S^5$. Then, as in the proof of Theorem 2, we can construct a local adapted slant orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ such that the second fundamental form $h$ satisfies

$$h(e_1, e_1) = -\lambda e_4, \quad h(e_1, e_2) = \lambda e_5, \quad h(e_2, e_2) = \lambda e_4,$$

for a certain smooth function $\lambda$ on $M$. Then the shape operators take the desired forms.

The converse statement is obvious. ∎

## 4. Another inequality.

In this section, we prove another inequality between an intrinsic invariant, namely the scalar curvature $\tau$, and extrinsic invariants, namely the scalar normal curvature $\tau^\perp$ and squared mean curvature $\|H\|^2$, for a 3-dimensional proper contact slant submanifold $M$ in a 5-dimensional Sasakian space form $\widetilde{M}(c)$.

Let $p \in M$ and $\{e_1, e_2, e_3, e_4, e_5\}$ an adapted slant orthonormal basis of $T_pM$. We define the *scalar normal curvature* $\tau^\perp$ at $p$ by

$$\tau^\perp(p) = g\big(R^\perp(e_1, e_2)e_4, e_5\big),$$

where $R^\perp$ denotes the curvature tensor of $\nabla^\perp$.

This definition is formally similar to the definition of the normal curvature of a surface in a 4-dimensional space form (see [**6**]). Also, since, in the case under consideration,

$$g\big(R^\perp(e_1, \xi)e_4, e_5\big) = g\big(R^\perp(e_2, \xi)e_4, e_5\big) = 0,$$

it follows that the above definition agrees, up to a constant factor, to the definition introduced in [**5**].

We observe that the normal connection of $M$ is flat if and only if $\tau^\perp = 0$, which is equivalent to the simultaneous diagonalisability of all shape operators (see, for instance, [**5**]).

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**THEOREM 5.** *Let $M$ be a 3-dimensional proper contact slant submanifold of a 5-dimensional Sasakian space form $\widetilde{M}(c)$. Then, we have*

$$\|H\|^2 \geqslant \frac{4}{9}(\tau + \tau^{\perp}) - \frac{2}{9}(c+1) - \frac{8}{9}\cos^2\theta. \tag{4.1}$$

*Moreover, the equality sign of (4.1) holds at a point $p \in M$ if and only if with respect to some suitable adapted slant orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ at $p$, the shape operators at $p$ take the following forms:*

$$A_{e_4} = \begin{pmatrix} -\lambda & \mu & \sin\theta \\ \mu & \lambda & 0 \\ \sin\theta & 0 & 0 \end{pmatrix} \qquad A_{e_5} = \begin{pmatrix} \mu & \lambda & 0 \\ \lambda & -\mu & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}. \tag{4.2}$$

PROOF: Let $p \in M$ and $\{e_1, e_2, e_3, e_4, e_5\}$ an adapted slant orthonormal basis. By the definition of the mean curvature vector, one has

$$\begin{aligned} 9\|H\|^2 &= (h^4_{11} + h^4_{22})^2 + (h^5_{11} + h^5_{22})^2 \\ &= (h^4_{11} - h^4_{22})^2 + (h^5_{11} - h^5_{22})^2 + 4(h^4_{11}h^4_{22} + h^5_{11}h^5_{22}). \end{aligned} \tag{4.3}$$

By using equation (3.3), (4.3) becomes

$$\begin{aligned} 9\|H\|^2 &= (h^4_{11} - h^4_{22})^2 + (h^5_{11} - h^5_{22})^2 + 4(\tau - 2\cos^2\theta) \\ &\quad - (c+3) - 3(c-1)\cos^2\theta + 4(h^4_{12})^2 + 4(h^5_{12})^2. \end{aligned} \tag{4.4}$$

We choose $e_4$ in the direction of the mean curvature vector. Then tr $A_{e_5} = 0$, and thus the shape operators have the following forms:

$$A_{e_4} = \begin{pmatrix} \alpha & \mu & \sin\theta \\ \mu & \lambda & 0 \\ \sin\theta & 0 & 0 \end{pmatrix}, \qquad A_{e_5} = \begin{pmatrix} \mu & \lambda & 0 \\ \lambda & -\mu & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}.$$

It follows that (4.4) is equivalent to

$$9\|H\|^2 - 4\tau + (c+3) + (3c+5)\cos^2\theta = 8\mu^2 + (\alpha - \lambda)^2 + 4\lambda^2. \tag{4.5}$$

On the other hand, by the definition of the scalar normal curvature and Ricci equation, we get

$$\begin{aligned} \tau^{\perp} &= g(R^{\perp}(e_1, e_2)e_4, e_5) = g\big(\widetilde{R}(e_1, e_2)e_4, e_5\big) + g\big([A_{e_4}, A_{e_5}]e_1, e_2\big) \\ &= \frac{c-1}{4}(1 - 3\cos^2\theta) + h^5_{11}h^4_{12} + h^5_{12}h^4_{22} - h^4_{11}h^5_{12} - h^4_{12}h^5_{22} \\ &= \frac{c-1}{4}(1 - 3\cos^2\theta) + 2\mu^2 + \lambda(\lambda - \alpha). \end{aligned}$$

Using (4.5) and the trivial inequality $4\lambda(\lambda-\alpha) \leqslant 4\lambda^2 + (\lambda-\alpha)^2$, the above equation implies

$$4\tau^{\perp} \leqslant 9\|H\|^2 - 4\tau + 2(c+1) + 8\cos^2\theta,$$

which is equivalent to (4.1).

Equality holds in (4.1) at a point $p \in M$ if and only if $\alpha = -\lambda$, that is, the shape operators take the forms (4.2). ∎

**Corollary 6.** *Each 3-dimensional proper contact slant submanifold $M$ of a 5-dimensional Sasakian space form $\widetilde{M}(c)$ which satisfies the equality case of (4.1) at every point $p \in M$ is a minimal submanifold.*

The proof follows from (4.2).

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