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VARIETIES OF STEINER LOOPS AND STEINER QUASIGROUPS

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1. Introduction. A *Steiner Triple System* (*STS*) is a pair $\langle P, B\rangle$ where P is a set of points and B is a set of 3-element subsets of P called *blocks* (or triples) such that for distinct $p, q \in P$ there is a unique block $b \in B$ with $\{p, q\} \subset b$. There are two well-known methods for turning Steiner Triple Systems into algebras; both methods are due to R. H. Bruck [**1**]. Each method gives rise to a variety of algebras; in this paper we will study these varieties.

The first method turns the *STS* $\langle P, B\rangle$ into a *Steiner quasigroup* (squag) based on P: define $x^2 = x$ and otherwise $xy = z$ where $\{x, y, z\} \in B$. Note that $\langle P; \cdot\rangle$ satisfies: i) $x^2 = x$, ii) $xy = yx$, iii) $x(xy) = y$. Conversely, any algebra $\langle P; \cdot\rangle$ satifying i) – iii) generates a *STS* $\langle P, B\rangle$ where B is the set of 3-element subalgebras of $\langle P; \cdot\rangle$. Thus the class of all squags forms a variety which is denoted by Sq.

The second method turns the *STS* $\langle P, B\rangle$ into a *Steiner loop* (sloop) based on $P \cup \{1\}$ (where $1 \notin P$): define $1x = x1 = x$, $x^2 = 1$ and otherwise $xy = z$ where $\{x, y, z\} \in B$. Note that $\langle P \cup \{1\}; \cdot\rangle$ satisfies a) $1x = x$, b) $x^2 = 1$, c) $xy = yx$, d) $x(xy) = y$. Conversely, any algebra $\langle P'; \cdot, 1\rangle$ satisfying a) – d) generates a *STS* $\langle P' - \{1\}, B\rangle$ where B is the set of all 3-element subsets of $P' - \{1\}$ which together with 1 form a subalgebra. Thus the class of all sloops forms a variety which is denoted by Sl.

We assume that the reader is familiar with the basic concepts of universal algebra; for details and notational conventions, see [**3**].

2. Two-variable identities. Sloops are closely related to boolean groups (i.e., groups satisfying $x^2 = 1$). In fact, all that we are missing is the associative law; in its place we have the special case of the associative law in two variables: $x(xy) = (x^2)y$. Let $\mathfrak{A} \in$ Sl. Then $\mathfrak{A}$ has the property that every 2-generated subsloop is a boolean group. Conversely, let $\mathfrak{B}$ be an algebra of type $\langle 2, 0\rangle$ with the property that every 2-generated subalgebra is a boolean group. Then $\mathfrak{B}$ necessarily satisfies a) – d) and so $\mathfrak{B} \in$ Sl. Thus Sl $= \mathscr{B}_{(2)} =$ the class of all algebras of type $\langle 2, 0\rangle$ such that every 2-generated subalgebra is in $\mathscr{B}$ (= boolean groups). Equivalently, Sl $= \mathscr{B}_{(2)}$ is the variety of all algebras of type $\langle 2, 0\rangle$ satisfying all 2-variable identities true in $\mathscr{B}$.

Thus we see that $\mathscr{B}$ is a subvariety of Sl. In fact, since every non-trivial sloop

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contains a non-trivial boolean group and $\mathscr{B}$ is generated by each of its non-trivial members, we see that $\mathscr{B}$ is the unique atom in the lattice of subvarieties of Sl.

Let us now turn our attention to squags. Any two distinct elements of a squag generate a 3-element subsquag and all 3-element squags are isomorphic. As a canonical example, consider the set $\{0, 1, 2\}$ and the operation $x \oplus y = 2x + 2y \pmod 3$; thus let $I_3 = \langle\{0, 1, 2\}; \oplus\rangle$. In a similar manner, the variety generated by I_3, $\mathscr{I}$, is the unique atom in the lattice of subvarieties of Sq. Moreover, Sq $= \mathscr{I}_{(2)} =$ the variety of all algebras of type $\langle 2\rangle$ which satisfy all 2-variable identities true in $\mathscr{I}$.

But what is $\mathscr{I}$? Let $C_3 = \langle\{0, 1, 2\}; \oplus, 0\rangle$; Then C_3 is equivalent to the 3-element group since $x + y \pmod 3 = 4x + 4y \pmod 3 = 2(2x + 0) + 2(2y + 0) \pmod 3 = (x \oplus 0) \oplus (y \oplus 0)$. Now if $\mathscr{C}_3$ is the variety generated by the 3-element group and $\langle A; +\rangle \in \mathscr{C}_3$ then $\langle A; \oplus\rangle \in \mathscr{I}$ where $x \oplus y = 2x + 2y$. In fact, it is straightforward to show that $\mathscr{I}$ is the class of all algebras $\langle A; \oplus\rangle$ such that $\langle A; +\rangle \in \mathscr{C}_3$.

Finally, note that i) – iii) and iv) $x(yz) = (xy)z$ is an equational base for $\mathscr{B}$. Less obvious, but still easy to prove, is the fact that a) – d) and e) $(xy)(uv) = (xu)(yv)$ is an equational base for $\mathscr{I}$.

3. Algebraic properties of Sl **and** Sq**.** Since Sl $= \mathscr{B}_{(2)}$ and Sq $= \mathscr{I}_{(2)}$ we know that Sl and Sq inherit those properties of $\mathscr{B}$ and $\mathscr{I}$ which depend on only two variables.

THEOREM 3.1. (a) Sl *and* Sq *each have permutable congruences.*

(b) Sl *and* Sq *each have uniform congruences* (*i.e. if* $\mathfrak{A} \in$ Sl *or* Sq *and* $\theta \in \mathscr{C}(\mathfrak{A})$ *then all* θ*-classes have the same cardinality*).

(c) Sl *and* Sq *each have regular congruences* (*i.e. if* $\mathfrak{A} \in$ Sl *or* Sq *and* $\theta \in \mathscr{C}(\mathfrak{A})$ *then any* θ*-class uniquely determines* θ).

Proof. It is well known that an equational class has permutable congruences if and only if it has a Mal'cev polynomial (i.e. a ternary polynomial $p(x, y, z)$ such that $p(x, y, y) = p(y, y, x) = x$; see [**3**]). In both Sl and Sq consider the polynomial $p(x, y, z) = y(xz)$. Then $p(x, y, z)$ is a Mal'cev polynomial in both Sl and Sq so a) is proved. Let $\mathfrak{A} \in$ Sl or Sq and $\theta \in \mathscr{C}(\mathfrak{A})$. Pick $a, b \in A$ and define $\tau_{a,b} : [a]\theta \longrightarrow [b]\theta$ by $\tau_{a,b}(x) = x(ab)$. Clearly $x \equiv a \bmod (\theta)$ implies $\tau_{a,b}(x) \equiv b \bmod (\theta)$. But $\tau_{b,a}(\tau_{a,b}(x)) = (x(ab))(ba) = (x(ab))(ab) = x$ so by symmetry $\tau_{a,b}$ is a bijection from $[a]\theta$ onto $[b]\theta$. Thus Sl and Sq each have uniform congruences. Moreover, for any $x, y \in A$ we see that $x \equiv y \bmod (\theta)$ if and only if $ax \equiv ay \bmod (\theta)$ if and only if $a = x(ax) \equiv x(ay) \bmod (\theta)$. Thus $[a]\theta$ uniquely determines θ so Sl and Sq have regular congruences.

Definition. Let $\mathfrak{A} \in$ Sl or Sq and let $\mathfrak{B} \subseteq \mathfrak{A}$. We say $\mathfrak{B}$ is *normal* in $\mathfrak{A}$ if $\mathfrak{B}$ is a congruence class of a congruence of $\mathfrak{A}$ and in this case we write $\mathfrak{B} \ll \mathfrak{A}$.

PROPOSITION 3.2. a) *If* $\mathfrak{A} \in \mathrm{Sl}$ *then* $\mathscr{C}(\mathfrak{A})$ *is isomorphic to the lattice of normal subloops of* $\mathfrak{A}$.

b) *If* $\mathfrak{A} \in \mathrm{Sq}$ *and* $a \in A$ *then* $\mathscr{C}(\mathfrak{A})$ *is isomorphic to the lattice of normal subsquags of* $\mathfrak{A}$ *containing* a.

THEOREM 3.3. *Let* $\mathfrak{B} \subseteq \mathfrak{A} \in \mathrm{Sl}$. *Then* $\mathfrak{B} \ll \mathfrak{A}$ *if and only if one of the following equivalent conditions holds*:

i) *For all* $x, y \in A$, $x(yB) = (xy)B$.

ii) *For all* $x, y, z \in A$, $xy \in B$ *if and only if* $(xz)(yz) \in B$.

Proof. From the general theory of loops (see [**1**]) it is known that $\mathfrak{B}$ is normal if and only if a) $xB = Bx$, b) $x(By) = (xB)y$, c) $(Bx)y = B(xy)$, d) $(xy)B = x(yB)$. Since our loops are commutative, a) is trivial and c) and d) are equivalent. Moreover, if d) holds then $x(By) = x(yB) = (xy)B = (yx)B = y(xB) = (xB)y$. Hence i) is equivalent to $\mathfrak{B} \ll \mathfrak{A}$.

i) $\Rightarrow$ ii). Using the fact that $u(vB) = (uv)B$ for all $u, v \in A$ we see that $xy \in B \Leftrightarrow (xy)B = B \Leftrightarrow x(yB) = B \Leftrightarrow xB = yB \Leftrightarrow z(xB) = z(yB) \Leftrightarrow (zx)B = (zy)B \Leftrightarrow (xz)B = (yz)B \Leftrightarrow B = (xz)((yz)B) \Leftrightarrow B = ((xz)(yz))B \Leftrightarrow (xz)(yz) \in B$.

ii) $\Rightarrow$ i). Using the fact that $uv \in B \Leftrightarrow (uw)(vw) \in B$ for all $u, v, w \in A$ we see that for $x, y, b \in A$, $x(yb) \in x(yB) \Leftrightarrow b = y(yb) \in B \Leftrightarrow (yx)((yb)x) = (xy)(x(yb)) \in B \Leftrightarrow x(yb) = (xy)((xy)(x(yb))) \in (xy)B$.

THEOREM 3.4. *Let* $\mathfrak{B} \subseteq \mathfrak{A} \in \mathrm{Sq}$. *Then* $\mathfrak{B} \ll \mathfrak{A}$ *if and only if for all* $x, y, z \in A$ *and for all all* $b \in B$, $(xb)y \in B$ *implies* $((xz)b)(yz) \in B$.

Proof. Let $\mathfrak{B} \ll \mathfrak{A}$ and let θ be the congruence induced by $\mathfrak{B}$. Thus let $(xb)y \in B$; $(xb)y \equiv b \bmod (\theta)$ so $xb \equiv yb \bmod (\theta)$ so $x \equiv y \bmod (\theta)$ so $xz \equiv yz \bmod (\theta)$ so $(xz)b \equiv (yz)b \bmod (\theta)$ so $((xz)b)(yz) \equiv b \bmod (\theta)$ so $((xz)b)(yz) \in B$. Conversely let $\mathfrak{B}$ satisfy the condition and define $x \equiv y \bmod (\theta)$ if $(xb)y \in B$ for all $b \in B$. Since $(xb)x = b$, θ is reflexive. Let $x \equiv y \bmod (\theta)$ so $(xb)y \in B$. Hence B contains $((x(yb))b)(y(yb)) = ((x(yb))b)b = x(yb) = (yb)x$ so $y \equiv x \bmod (\theta)$ and θ is symmetric. Now let $x \equiv y \bmod (\theta)$ and $y \equiv z \bmod (\theta)$. Thus $(xb)y = b_1 \in B$ and $(yb_1)z = b_2 \in B$. But then $xb = yb_1 = zb_2$ and so $b_2 = (xb)z \in B$. Hence $x \equiv z \bmod (\theta)$ and θ is transitive. To prove that θ is a congruence we need only show that $x \equiv y \bmod (\theta)$ implies that $xz \equiv yz \bmod (\theta)$. But $x \equiv y \bmod (\theta)$ means $(xb)y \in B$ so $((xz)b)(yz) \in B$ so $xz \equiv yz \bmod (\theta)$. Finally, let $b \in B$. Then for all $b', b'' \in B$, $(b'b'')b \in B$ so $b' \equiv b \bmod (\theta)$. Conversely if $a \equiv b \bmod (\theta)$ then $(ab)b = a \in B$. Thus $[b]\theta = B$ and the theorem is proved.

The next result shows that, as with groups, large subalgebras are normal. Note that a slight reformulation extends these results to infinite algebras. Part ii) is due to B. Ganter.

THEOREM 3.5. i) *Let* $\mathfrak{A}$ *be a finite sloop with* $\mathfrak{B} \subseteq \mathfrak{A}$ *and* $|B| = \frac{1}{2}|A|$; *then* $\mathfrak{B} \ll \mathfrak{A}$.

ii) *Let* $\mathfrak{A}$ *be a finite squag with* $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathfrak{A}$, $B_1 \cap B_2 = \emptyset$, $|B_1| = |B_2| = |A|/3$. *Then* $\mathfrak{B}_i \ll \mathfrak{A}$.

Proof. i) is well known for loops. For ii) let $B_3 = A - B_1 - B_2$. Let $b_i \in B_i$; then $b_1 b_2 \in B_3$. Since A is finite, this means that $b_2 B_1 = b_1 B_2 = B_3$ so $(B_3)^2 = B_3$; i.e. B_3 is also a subalgebra. But now it is easy to see that for $\{i, j, k\} = \{1, 2, 3\}$, $B_i B_j = B_k$; i.e. $\mathfrak{B}_i \ll \mathfrak{A}$ for $i = 1, 2, 3$.

4. Simplicity and functional completeness. It is well known that if $\langle P, B\rangle$ is a finite STS then $|P| \equiv 1$ or $3 \bmod (6)$. Thus if $\mathfrak{A} \in \mathrm{Sl}$ is finite then $|A| \equiv 2$ or $4 \bmod (6)$ while if $\mathfrak{A} \in \mathrm{Sq}$ is finite then $|A| \equiv 1$ or $3 \bmod (6)$. What does this say about possible congruences on $\mathfrak{A}$? Because of the uniformity of congruences and the fact that congruences correspond to normal subalgebras, $\mathfrak{A}$ can have a proper non-trivial congruence only if $|A|$ factors appropriately. Thus if $\mathfrak{A} \in \mathrm{Sl}$ is finite and not simple then $|A| = (6m + i)(6n + j)$ for some m, n and some $i, j \in \{2, 4\}$; in particular if $|A|$ is not divisible by 4 then $\mathfrak{A}$ is simple. If $\mathfrak{A} \in \mathrm{Sq}$ is finite but not simple then we must have $|A| = (6m + i)(6n + j)$ for some m, n and some $i, j \in \{1, 3\}$; in particular if $|A|$ is prime then $\mathfrak{A}$ is simple.

Thus there are very many simple sloops and squags. In fact, the next theorem indicates that most finite squags and sloops are simple. For it we need a lemma due to R. Metz and H. Werner [**4**].

LEMMA 4.1. *Let* $\langle P, B\rangle$ *be a finite STS with* $\mathfrak{A}$ *the corresponding sloop and* $\mathfrak{B}$ *the corresponding squag. If* $\mathfrak{A}$ *has a homomorphism onto* C_2 *then* $\mathfrak{B}$ *is simple.*

Proof. Let $\mathfrak{A}'$ be the kernel of a homomorphism of $\mathfrak{A}$ onto C_2. Thus for all a, $a' \in A - A'$, $aa' \in A'$. Let θ be a non-trivial congruence on $\mathfrak{B}$. For $a \in A - A'$ look at $[a]\theta$; by assumption it is non-trivial and hence has at least 3 elements. Thus $\sigma_a(x) = ax$ is a bijection between $[a]\theta - (\{a\} \cup A')$ and $[a]\theta \cap A'$ (note that as $\mathfrak{A}'$ is a subsloop of $\mathfrak{A}$, $A' - \{1\}$ is a subsquag of $\mathfrak{B}$). This is true for all θ-classes intersecting $A - A'$; but because of the uniformity property, no θ-class can be contained entirely in $A' - \{1\}$. Since $|A'| = \frac{1}{2}|A|$ there must be only one θ-class; i.e. $\mathfrak{B}$ is simple.

THEOREM 4.2 *Let* $\langle P, B\rangle$ *be a finite STS with* $\mathfrak{A}$ *the corresponding sloop and* $\mathfrak{B}$ *the corresponding squag. Then either* $\mathfrak{A}$ *is simple or* $\mathfrak{B}$ *is simple.*

Proof. Suppose $\mathfrak{A}$ is not simple; let $\mathfrak{A}'$ be a proper non-trivial normal subsloop of $\mathfrak{A}$ and θ the corresponding congruence. Let $a \in A - A'$; thus $[a]\theta \cup \mathfrak{A}' = \mathfrak{A}''$ is a subalgebra of $\mathfrak{A}$ and has a homomorphism onto C_2. Hence $\mathfrak{B}_a$, the corresponding squag is simple. Also note that $\mathfrak{B} = \bigcup \{\mathfrak{B}_a | a \in A - A'\}$. Now we have two cases: i) $|A'| = 2$, and ii) $|A'| > 2$. In the first case let $A' = \{1, b\}$. Pick $a, a' \in A - A'$ with $a \not\equiv a' \bmod (\theta)$; then $[a]\theta \cup [a']\theta \cup [aa']\theta \cup A'$ is an 8-element subsloop of $\mathfrak{A}$ which has a homomorphism onto C_2. But the corresponding 7-element squag is simple. Thus we see in $\mathfrak{B}$ that for any two elements a, a' with $aa' \neq b$, $\{a, a', b\}$ generates a 7-element simple squag. Now let λ be a non-trivial congruence on $\mathfrak{B}$. Thus we have $a \equiv b \bmod (\lambda)$ for some $a \in A - \{1\}$. Let $a' \in A - \{1, a\}$; if $a' = ab$ then $a' \equiv b \bmod (\lambda)$. Otherwise $\{a, a', b\}$ generates a 7-element simple squag and the restriction of λ to it is a non-trivial congruence. Hence $a' \equiv b \bmod (\lambda)$; since this is now true for all a', $\mathfrak{B}$ is simple. In the second case we again let $b \in A' - \{1\}$ and let λ be a non-trivial congruence on $\mathfrak{B}$. Thus $a \equiv b \bmod (\lambda)$ for some $a \in A - \{1, b\}$. But then λ is a non-trivial congruence on the simple squag $\mathfrak{B}_a$ so all of $\mathfrak{B}_a$ is collapsed. In particular, all of $A' - \{1\}$ is collapsed (and $|A' - \{1\}| > 1$). Hence λ collapses all of $\mathfrak{B}_{a'}$ for every a' so that $\mathfrak{B}$ is simple.

Example 4.3. A STS $\langle P, B\rangle$ is *planar* if it is generated by every triangle and contains a triangle (a triangle is a set of 3 elements not in a block). In [**2**] J. Doyen constructed a planar STS of cardinality n for each $n \geqq 7$ with $n \equiv 1$ or $3 \bmod (6)$.

THEOREM 4.4. *Let $\langle P, B\rangle$ be a finite planar STS; let $\mathfrak{A}$ be the corresponding sloop and $\mathfrak{B}$ the corresponding squag. Then:*

a) *Either $\mathfrak{A}$ is simple or $|A| = 8$.*

b) *Either $\mathfrak{B}$ is simple or $|B| = 9$.*

Proof. Suppose $\mathfrak{A}$ ($\mathfrak{B}$) has a proper non-trivial congruence θ. Then for every $a \in A(B)$ we have $|[a]\theta| > 1$ and there are at least 2 θ-classes. Hence we may choose distinct $a, b, c \in A(B)$ such that $a \equiv b \bmod (\theta)$ but $b \not\equiv c \bmod (\theta)$. In particular, $\{a, b, c\}$ is a triangle and so generates $\mathfrak{A}$ ($\mathfrak{B}$). Hence $\mathfrak{A}/\theta$ ($\mathfrak{B}/\theta$) is generated by $[a]\theta$ and $[b]\theta$ so $|\mathfrak{A}/\theta| = 2$ or 4 ($|\mathfrak{B}/\theta| = 3$); i.e. θ has either 2 or 4 congruence classes (θ has 3 congruence classes). But in $\mathfrak{B}$ each θ-class contains 3 elements since otherwise a θ-class would contain a triangle whereas each θ-class is a subalgebra. Therefore $|B| = 9$. In $\mathfrak{A}$ similar reasoning tells us that each θ-class contains at most 4 elements. Thus $|A| \leqq 16$. But if $|A| = 16$ then $\mathfrak{A}$ would have an 8-element subsloop and so would have a non-generating triangle. One the other hand if $|A| = 4$ then $\mathfrak{A}$ has no triangle. Thus $|A| = 8$.

Definition. A finite non-trivial algebra $\mathfrak{B}$ is *functionally complete* if every function $f : B^n \to B$ is an algebraic function of $\mathfrak{B}$ (i.e. a polynomial with perhaps some variables replaced by members of B).

A functionally complete algebra is necessarily simple; in fact if $\mathfrak{A}$ is functionally complete then $\mathscr{C}(\mathfrak{A}^n) \cong 2^n$, the boolean algebra with exactly n atoms. The next theorem shows that for squags and sloops, this property is sufficient. The proof can be found in [**8**].

THEOREM 4.5. *Let $\mathfrak{B}$ be a finite non-trivial algebra in a variety with permutable congruences. If $\mathscr{C}(\mathfrak{B}^n) \cong 2^n$ for all positive integers n then $\mathfrak{B}$ is functionally complete.*

Thus in order to determine which sloops and squags are functionally complete we must examine the congruences of direct powers of simple squags and

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sloops. For this we need to make use of some recent results of H. Werner [**8**]. Notice that neither C_2 nor I_3 are functionally complete.

THEOREM 4.6 [**8**]. *Let $\mathscr{K}$ be a variety with permutable congruences and let $\mathscr{K}$ either be idempotent or have* 0*-regular congruences with $\mathfrak{F}_{\mathscr{K}}(\emptyset) = \{0\}$* (i.e. 0 *is a nullary operation; it is a subalgebra of every algebra of $\mathscr{K}$, and every congruence is determined by the class containing* 0). *Then for a simple algebra $\mathfrak{A} \in \mathscr{K}$ the following are equivalent*:

a) $\mathscr{C}(\mathfrak{A}) \not\cong 2^n$ *for some n.*

b) *For any polynomial $p(x_1, \ldots, x_n)$ of $\mathfrak{A}$ and $a_1, a_2, \ldots, a_n, b_2, \ldots, b_n \in A$ if $p(a_1, a_2, \ldots, a_n) = p(a_1, b_2, \ldots, b_n)$ then for all $x \in A$, $p(x, a_2, \ldots, a_n) = p(x, b_2, \ldots, b_n)$.*

Now we are ready to characterize functionally complete squags and sloops. Part of the next theorem is mentioned in [**8**].

THEOREM 4.7. i) *If $\mathfrak{A}$ is a finite simple sloop with $|A| > 2$ then $\mathfrak{A}$ is functionally complete.*

ii) *If $\mathfrak{B}$ is a finite simple squag with $|B| > 3$ then $\mathfrak{B}$ is functionally complete.*

Proof. i) Let $a \in A$ with $a \neq 1$. Thus $\{a, 1\}$ is a proper subsloop of $\mathfrak{A}$ and so is not normal. Hence by 3.3 there are $x, y, z \in A$ with $xy \in \{a, 1\}$ if and only if $(xz)(yz) \notin \{a, 1\}$; in particular, $xy \neq (xz)(yz)$. Let $u = xz$ so $x = uz$. Thus $(uz)y \neq u(yz)$. Let $p(x_1, x_2, x_3) = x_2(x_1x_3)$. Then $p(y, y, u) = y(yu) = u = u(yy) = p(y, u, y)$ while $p(z, y, u) = y(zu) = (uz)y \neq u(yz) = u(zy) = p(z, u, y)$. Hence by 4.5 and 4.6, $\mathfrak{A}$ is functionally complete.

ii) Since $|B| > 3$ and $\mathfrak{B}$ is simple, $\mathfrak{B} \notin \mathscr{I}$. Hence $\mathfrak{B}$ does not satisfy $(xy)(uv) = (xu)(yv)$. Thus there are $a, b, c, d \in B$ with $(ab)(cd) \neq (ac)(bd)$. Let $p(x_1, x_2, x_3, x_4) = (x_1x_2)(x_3x_4)$. Then $p(a, b, c, d) \neq p(a, c, b, d)$ while $p(bc, b, c, d) = ((bc)b)(cd) = c(cd) = d = b(bd) = ((bc)c)(bd) = p(bc, c, b, d)$. Hence by 4.5 and 4.6, $\mathfrak{B}$ is functionally complete.

In light of theorems 4.2 and 4.7 we see that if $\langle P, B\rangle$ is a finite STS with $|P| \geqq 7$ then any function $f : P^n \to P$ can be expressed as an algebraic function (where, of course, we must indicate how to evaluate x^2).

5. Varieties of squags and sloops. Throughout this section $\mathfrak{A}$ will denote a finite simple planar squag or sloop; we wish to determine the variety generated by $\mathfrak{A}$. Let $\mathfrak{A}_0$ denote C_2 if $\mathfrak{A}$ is a sloop and I_3 if $\mathfrak{A}$ is a squag. Let $\mathscr{K}$ be a class of algebras and denote by $P_f(\mathscr{K})$ the class of all algebras isomorphic to a direct product of finitely many copies of members of $\mathscr{K}$.

LEMMA 5.1. *A subdirect product of finitely many simple algebras in a permutable variety is isomorphic to a direct product of some of these simple algebras.*

COROLLARY 5.2. $SP_f(\mathfrak{A}) = P_f(\{\mathfrak{A}, \mathfrak{A}_0\})$.

Proof. Since a subalgebra of a direct product of algebras from $\mathscr{K}$ is a sub-

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direct product of algebras from $S(\mathscr{K}) = \{\mathfrak{A}, \mathfrak{A}_0, 0\}$ (where 0 is the trivial 1-element algebra), the result follows from 5.1.

Definition. Let $\mathfrak{B} = \Pi_{i \in I}\mathfrak{B}_i$. A congruence θ on $\mathfrak{B}$ is *skew* if it is not of the form $(\theta_i)_{i \in I}$ where $\theta_i \in \mathscr{C}(\mathfrak{B}_i)$.

LEMMA 5.3 [**8**]. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be members of a permutable variety. Then $\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n$ has a skew congruence if and only if for some $i \neq j$, $\mathfrak{A}_i \times \mathfrak{A}_j$ has a skew congruence.*

LEMMA 5.4 [**8**]. *Let $\mathscr{K}$ be a permutable variety with $\mathfrak{B}, \mathfrak{D} \in \mathscr{K}$. Then $\mathfrak{B} \times \mathfrak{D}$ has a skew congruence if and only if there are homomorphic images $\mathfrak{B}'$ of $\mathfrak{B}$ and $\mathfrak{D}'$ of $\mathfrak{D}$ and a* 1-1 *map μ with* dom $(\mu) \subseteq B'$, $|\text{dom } (\mu)| > 1$, range $(\mu) \subseteq D'$ *such that $\{(b, \mu(b)) | \, b \in$ dom $(\mu)\}$ is a congruence class on $\mathfrak{B}' \times \mathfrak{D}'$.*

LEMMA 5.5 [**9**]. *Let $\mathscr{K}$ be a permutable variety and $\mathfrak{B}, \mathfrak{D} \in \mathscr{K}$. If $\{(b_i, c_i) | i \in I\}$ is a congruence class of $\mathfrak{B} \times \mathfrak{D}$ then $\{b_i | i \in I\}$ is a congruence class of $\mathfrak{B}$ (i.e. the projection of a congruence class is a congruence class).*

THEOREM 5.6. $HSP_f(\mathfrak{A}) = P_f(\mathfrak{A}, \mathfrak{A}_0)$.

Proof. Every member of $SP_f(\mathfrak{A})$ is of the form $\mathfrak{A}^m \times \mathfrak{A}_0{}^n$; we want to show that every homomorphic image is of the form $\mathfrak{A}^p \times \mathfrak{A}_0{}^q$ for some $p \leqq m$, $q \leqq n$. We know that a homomorphic image of $\mathfrak{A}_0{}^n$ is isomorphic to $\mathfrak{A}_0{}^q$ for some $q \leqq n$ and since $\mathscr{C}(\mathfrak{A}^m) \cong 2^m$, any homomorphic image of $\mathfrak{A}^m$ is isomorphic to $\mathfrak{A}^p$ for some $p \leqq m$. Thus we need only show that $\mathscr{C}(\mathfrak{A}^m \times \mathfrak{A}_0{}^n) \cong \mathscr{C}(\mathfrak{A}^n) \times \mathscr{C}(\mathfrak{A}_0{}^m)$; i.e. that $\mathfrak{A}^m \times \mathfrak{A}_0{}^n$ has no skew congruences. Since $\mathfrak{A}^2$ has no skew congruences and $\mathfrak{A}^m \times \mathfrak{A}_0{}^n \cong \mathfrak{A} \times \ldots \times \mathfrak{A} \times \mathfrak{A}_0{}^n$, 5.3 tells us that $\mathfrak{A}^m \times \mathfrak{A}_0{}^n$ has a skew congruence if and only if $\mathfrak{A} \times \mathfrak{A}_0{}^n$ has a skew congruence.

So suppose $\mathfrak{A} \times \mathfrak{A}_0{}^n$ has a skew congruence. Now we invoke 5.4; since $\mathfrak{A}$ is simple, $\mathfrak{A}' = \mathfrak{A}$ while $(\mathfrak{A}_0{}^n)' = \mathfrak{A}_0{}^q$ for some $q \leqq n$. Next 5.5 tells us that dom (μ) is a congruence class of $\mathfrak{A}$; since $|\text{dom } (\mu)| > 1$ and $\mathfrak{A}$ is simple, this means that dom $(\mathfrak{A}) = A$. If $\mathfrak{A}$ is a squag then $\mu = \{(a, \mu(a)\}$ is a subalgebra of $\mathfrak{A} \times \mathfrak{A}_0{}^q$. If $\mathfrak{A}$ is a sloop, translate μ to the congruence class containing 1; this congruence class also satisfies all the conditions that μ does. Hence without loss of generality we may assume that μ is a subalgebra of $\mathfrak{A}$. But this means that μ is an isomorphism of $\mathfrak{A}$ into $\mathfrak{A}_0{}^q$, which is impossible. Hence $\mathfrak{A}^m \times \mathfrak{A}_0{}^n$ has no skew congruence and the theorem is proved.

LEMMA 5.7 [**6**]. *Let $\mathscr{K}$ be a variety generated by a finite algebra and having only finitely many finite subdirectly irreducible algebras. Then $\mathscr{K}$ has no infinite subdirectly irreducible algebras; i.e. if $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are all the finite subdirectly irreducible algebras in $\mathscr{K}$ then $\mathscr{K} = P_s(\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.*

THEOREM 5.8. $\mathscr{V}(\mathfrak{A}) = P_s(\{\mathfrak{A}, \mathfrak{A}_0\})$.

Proof. By 5.6, $\mathfrak{A}$ and $\mathfrak{A}_0$ are the only finite subdirectly irreducible algebras in

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$\mathscr{V}(\mathfrak{A})$ since every finite algebra in $\mathscr{V}(\mathfrak{A})$ is a member of $HSP_f(\mathfrak{A})$. Thus the theorem follows from 5.7.

THEOREM 5.9. *Let* $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ *be pairwise non-isomorphic finite simple planar squags or sloops. Then* $\mathscr{V}(\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}) = P_s(\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\})$.

Proof. The only additional fact we need is that $\mathfrak{A}_i \times \mathfrak{A}_j$ has no skew congruence for $i \neq j$. But as in the proof of 5.6, if $\mathfrak{A}_i \times \mathfrak{A}_j$ has a skew congruence then $\mathfrak{A}_i \cong \mathfrak{A}_j$.

THEOREM 5.10. *The lattice of varieties of sloops (squags) contains as a cover preserving sublattice the lattice of all finite subsets of a countable set; the empty set corresponds to* $\mathscr{B}(\mathscr{I})$. *In particular,* $\mathscr{B}(\mathscr{I})$ *has infinitely many covers.*

6. Free algebras. Let $\mathscr{K}$ be a variety generated by finitely many finite simple planar squags or sloops. Thus $\mathscr{K}$ is locally finite (that is, finitely generated algebras in $\mathscr{K}$ are finite); so it is natural to ask what is $\mathfrak{F}_{\mathscr{K}}(n)$, the free algebra in $\mathscr{K}$ on n free generators and in particular what is $|\mathfrak{F}_{\mathscr{K}}(n)|$. As a special case let us assume that $\mathscr{K}$ is generated by the 7-element squag; the general case will follow immediately from the analysis of this special case.

Let $\mathfrak{T}$ denote the 7-element squag. We know that $\mathfrak{F}_{\mathscr{K}}(n) \cong \mathfrak{T}^{(\alpha_n)} \times I_3^{(\beta_n)}$ for some α_n, β_n. Now $\mathfrak{F}_{\mathscr{I}}(n) = I_3^{n-1}$; $\mathfrak{T}_{\mathscr{K}}(n)$ has a homomorphism onto $\mathfrak{F}_{\mathscr{I}}(n)$; $\mathscr{C}(\mathfrak{F}_{\mathscr{K}}(n)) \cong \mathscr{C}(\mathfrak{T}^{(\alpha_n)}) \times I_3^{(\beta_n)}$, and $\mathfrak{T}^{(\alpha_n)}$ has no homomorphism onto I_3. Hence $\beta_n = n - 1$.

Thus we must determine α_n. Recall that $\mathscr{C}(\mathfrak{T}^k) \cong 2^k$ so α_n is the number of congruences θ of $\mathfrak{F}_{\mathscr{K}}(n)$ for which $\mathfrak{F}_{\mathscr{K}}(n)/\theta \cong \mathfrak{T}$. Next consider maps δ, ϵ from $\{x_1, \ldots, x_n\}$, the set of free generators of $\mathfrak{F}_{\mathscr{K}}(n)$, into $\mathfrak{T}$, and let the corresponding congruences be $\theta(a_1, \ldots, a_n)$, $\theta(b_1, \ldots, b_n)$ where $a_i = \delta(x_i)$ and $b_i = \epsilon(x_i)$.

LEMMA 6.1. $\theta(a_1, \ldots, a_n) = \theta(b_1, \ldots, b_n)$ *if and only if the map* $a_i \to b_i$, $1 \leqq i \leqq n$, *induces an isomorphism from* $[a_1, \ldots, a_n]$ *onto* $[b_1, \ldots, b_n]$ *(*$[X]$ *is the subalgebra generated by* X*).*

Proof. Consider the homomorphism from $\mathfrak{F}_{\mathscr{K}}(n)$ into $\mathfrak{T}^2$ induced by $x_i \to (a_i, b_i)$. Then it is clear that $\theta(a_1, \ldots, a_n) = \theta(b_1, \ldots, b_n)$ if and only if $[a_1, \ldots, a_n] \cong [(a_1, b_1), \ldots, (a_n, b_n)] \cong [b_1, \ldots, b_n]$.

We are only interested in those δ whose image generates $\mathfrak{T}$. Write $\delta \sim \epsilon$ if and only if they have the same kernel (i.e. $\theta(a_1, \ldots, a_n) = \theta(b_1, \ldots, b_n)$). Thus $\delta \sim \epsilon$ if and only if they differ by an automorphism of $\mathfrak{T}$. Thus α_n is the number of $\sim$-classes whose members generate $\mathfrak{T}$. To count this number we make use of a special case of Polya's counting theory.

Let G be the automorphism group of $\mathfrak{T}$; it is well known that $|G| = 168$. Let G act on T^n in the obvious way (i.e. componentwise). Thus we can think of δ, $\epsilon \in T^n$ so that $\delta \sim \epsilon$ if and only if they are in the same orbit; we want to count the number of orbits of T^n under G all of whose elements generate $\mathfrak{T}$.

THEOREM 6.2. $\alpha_n = [(7^n - 7) - 7(3^n - 3)]/168$.

Proof. The 7 constant maps each generate a 1-element algebra; this leaves $7^n - 7$ maps generating non-trivial subalgebras of $\mathfrak{T}$. Of these, there are $7(3^n - 3)$ maps which generate I_3: there are 7 copies of I_3 in $\mathfrak{T}$ (i.e. the 7-element *STS* has 7 triples) and there are $3^n - 3$ maps into I_3 which generate I_3. Thus there are exactly $7^n - 7 - 7(3^n - 3)$ maps which generate $\mathfrak{T}$. If δ is such a map then the size of its orbit is 168. Thus α_n is as claimed.

THEOREM 6.3. *Let* $\mathfrak{A}$ *be a finite simple planar squag. Then* $\mathfrak{F}_{\mathfrak{A}}(n) \cong \mathfrak{A}^{\alpha_n(\mathfrak{A})} \times I_3^{n-1}$ *where* $\alpha_n(\mathfrak{A}) = [|A|^n - |A| - |A|(|A| - 1)(3^n - 3)/6]/|\mathrm{Aut}\ (\mathfrak{A})|$ *where* Aut $(\mathfrak{A})$ *is the automorphism group of* $\mathfrak{A}$.

Proof. The proof is the same as above; note that the number of 3-element subalgebras of $\mathfrak{A}$ is $|A|(|A| - 1)/6$.

THEOREM 6.4. *Let* $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ *be pairwise non-isomorphic finite planar squags; let* $\mathscr{K}$ *be the variety they generate. Then* $\mathfrak{F}_{\mathscr{K}}(n) \cong \mathfrak{A}_1^{\alpha_n(\mathfrak{A}_1)} \times \ldots \times \mathfrak{A}_m^{\alpha_n(\mathfrak{A}_m)} \times I_3^{n-1}$.

THEOREM 6.5. *Let* $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ *be pairwise non-isomorphic finite simple planar sloops and let* $\mathscr{K}$ *be the variety they generate. Then* $\mathfrak{F}_{\mathscr{K}}(n) \cong \mathfrak{A}_1^{\beta_n(\mathfrak{A}_1)} \times \ldots \times \mathfrak{A}_m^{\beta_n(\mathfrak{A}_m)} \times C_2^n$ *where*

$$\beta_n(\mathfrak{A}_i) = [|A_i| - 1 - (|A_i| - 1)(2^n - 1) - (|A_i| - 1) \times (|A_i| - 2)(4^n - 3(2^n - 1) - 1)/6]/|\mathrm{Aut}\ (\mathfrak{A}_i)|.$$

Proof. In counting the number of n-tuples in $\mathfrak{A}_i$ not generating $\mathfrak{A}_i$, there is one generating a 1-element algebra, there are $|A_i| - 1$ 2-element subalgebras each generated by $2^n - 1$ n-tuples, and there are $(|A_i| - 1)(|A_i| - 2)/6$ 4-element subalgebras each generated by $(4^n - 3(2^n - 1) - 1)$ n-tuples.

7. Projectivity, injectivity, and congruence extension. In this section we will be concerned with various properties of homomorphisms in varieties generated by finitely many finite planar squags or sloops.

Definition. An algebra $\mathfrak{A}$ contained in a class of algebras $\mathscr{K}$ is *projective* in $\mathscr{K}$ if for any $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathscr{K}$, any homomorphism $\varphi : \mathfrak{A} \to \mathfrak{A}_1$, and any onto homomorphism $\varphi_2 : \mathfrak{A}_2 \to \mathfrak{A}_1$ there is a homomorphism $\varphi : \mathfrak{A} \to \mathfrak{A}_2$ such that $\varphi_2\varphi = \varphi_1$.

Note that in any variety the free algebras are projective; it is well known that in the case of varieties the projective algebras are closely related to the free algebras.

Definition. An algebra $\mathfrak{A}$ is a *retract* of $\mathfrak{A}'$ if there is an onto homomorphism $\varphi_1 : \mathfrak{A}' \to \mathfrak{A}$ and a $1 - 1$ homomorphism $\varphi_2 : \mathfrak{A} \to \mathfrak{A}'$ with $\varphi_1\varphi_2$ being the identity map on $\mathfrak{A}$.

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LEMMA 7.1. *In a variety the projective algebras are exactly the retracts of the free algebras.*

THEOREM 7.2. *Let $\mathscr{K}$ be a variety generated by finitely many finite planar squags or sloops; then each finite algebra in $\mathscr{K}$ is projective.*

Definition. An algebra $\mathfrak{A} \in \mathscr{K}$ is (*weak*) *injective* in $\mathscr{K}$ if for any $\mathfrak{A}_1$, $\mathfrak{A}_2 \in \mathscr{K}$ with $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ and any $\varphi_1 : \mathfrak{A}_1 \to \mathfrak{A}$ (onto $\mathfrak{A}$) there is a $\varphi_2 : \mathfrak{A}_2 \to \mathfrak{A}$ such that φ_2 is an extension of φ_1.

THEOREM 7.3. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be finite simple planar squags or sloops and $\mathscr{K}$ the variety they generate; then $\mathscr{K}$ has no non-trivial injectives.*

Proof. Note that the 1-element algebra is injective in $\mathscr{K}$. Now $\mathfrak{A}_0$ (the 2-element sloop or the 3-element squag) is not injective in $\mathscr{K}$ because it is a subalgebra of $\mathfrak{A}_1$, but $\mathfrak{A}_1$ has no homomorphism onto $\mathfrak{A}_0$. Also $\mathfrak{A}_i$ is not injective in $\mathscr{K}$ since $(\mathfrak{A}_0)^2$ has more maximal congruences than $(\mathfrak{A}_i)^2$. Next let $\mathfrak{A}$ be a non-trivial injective in $\mathscr{K}$. It is well known that a retract of an injective algebra is also injective; hence if we can show that for some i between 0 and n, $\mathfrak{A}_i$ is a retract of $\mathfrak{A}$ we will have arrived at the desired contradiction. Since $\mathfrak{A}$ is non-trivial, we know by 5.9 that $\mathfrak{A}$ has a homomorphism onto some $\mathfrak{A}_i$ via the homomorphism φ_1. Since $\mathscr{K}$ is locally finite we may restrict our attention to a finitely generated subalgebra of $\mathfrak{A}$, say $\mathfrak{A}'$, such that φ_1 restricted to $\mathfrak{A}'$ is onto $\mathfrak{A}_i$. But now it is easy to see that $\mathfrak{A}_i$ is a retract of $\mathfrak{A}'$ and hence of $\mathfrak{A}$.

THEOREM 7.4. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be finite simple planar squags or sloops and $\mathscr{K}$ the variety they generate; then $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are weak injective in $\mathscr{K}$.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{A}'$ with $f : \mathfrak{A} \to \mathfrak{A}_i$ being onto. If $\mathfrak{A}'$ is finite then it is easy to find $f' : \mathfrak{A}' \to \mathfrak{A}_i$ extending f. For infinite $\mathfrak{A}'$, use the fact that finite subalgebras of $\mathfrak{A}'$ form a directed set whose union is $\mathfrak{A}'$. On this set form an inverse limit system of finite non-empty sets of homomorphisms extending f. The inverse limit is non-empty and any member induces an extension of f to $\mathfrak{A}'$.

THEOREM 7.5. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be pairwise non-isomorphic simple planar squags or sloops and $\mathscr{K}$ the variety they generate. If $\mathfrak{A}$ is weak injective in $\mathscr{K}$ then $\mathfrak{A} \cong \mathfrak{A}_1[\mathfrak{B}_1] \times \ldots \times \mathfrak{A}_n[\mathfrak{B}_n]$ where $\mathfrak{A}_i[\mathfrak{B}_i]$ is the extension of $\mathfrak{A}_i$ by the complete boolean algebra $\mathfrak{B}_i$.*

Proof. For a discussion of boolean extensions see [**7**]; a detailed proof of this theorem is similar to the proof of Theorem 7.5 of [**7**] and only the two necessary modifications are given here. The first is that if $\mathfrak{A}_i^*$ is the result of making each element of $\mathfrak{A}_i$ the value of a nullary operation, then $\mathfrak{A}_1^*, \ldots, \mathfrak{A}_n^*$ are independent primal algebras; this is because $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ belong to a congruence permutable variety and hence so do $\mathfrak{A}_1^*, \ldots, \mathfrak{A}_n^*$. The second is that the injectives in a variety generated by finitely many independent primal algebras are just the products of extensions of the primal algebras by complete boolean algebras.

Definition. An algebra $\mathfrak{A}$ has the *congruence extension property* (c.e.p.) if for every $\mathfrak{B} \subseteq \mathfrak{A}$ and $\theta' \in \mathscr{C}(\mathfrak{B})$ there is a $\theta \in \mathscr{C}(\mathfrak{A})$ with $\theta|_{\mathfrak{B}} = \theta'$ (equivalently, the restriction mapping from $\mathscr{C}(\mathfrak{A})$ to $\mathscr{C}(\mathfrak{B})$ is onto). A variety has c.e.p. if every algebra in the variety has c.e.p.

THEOREM 7.6. *Let $\mathfrak{A}$ be a finite simple sloop or squag with $|\mathfrak{A}| > 3$. Then any variety containing $\mathfrak{A}$ does not have* c.e.p.

Proof. By 4.7, $\mathfrak{A}$ is functionally complete so that $|\mathscr{C}(\mathfrak{A}^2)| = 4$. On the other hand, $\mathfrak{A}_0 \subseteq \mathfrak{A}$ and $|\mathscr{C}((\mathfrak{A}_0)^2)| > 4$. Thus $\mathfrak{A}^2$ does not have c.e.p. so that no variety containing $\mathfrak{A}$ has c.e.p.

8. Comments and problems. 1) Theorem 4.2 is not true for infinite *STS*s. The following example was contructed by Eric Mendelsohn and the author: Let $\mathfrak{S}_1$ be the 1-element *STS*. Given $\mathfrak{S}_i$, $\mathfrak{L}_i$ is the corresponding loop and $\mathfrak{Q}_i$ the corresponding quasigroup. If $\mathfrak{S}_i$ is defined and i is odd then $\mathfrak{S}_{i+1}$ is the *STS* corresponding to the 3-generated free algebra over $\mathfrak{L}_i$; if i is even then $\mathfrak{S}_{i+1}$ is the *STS* corresponding to the 3-generated free algebra over $\mathfrak{Q}_i$. Thus $\mathfrak{S}_2$ is the 3-element *STS* and $\mathfrak{S}_3$ is the 9-element *STS*. Notice that each $\mathfrak{S}_i$ is finite so that we can apply 4.2.

CLAIM. *If i is odd (even) then $\mathfrak{L}_{i+1}$ ($\mathfrak{Q}_{i+1}$) is a direct product of simple algebras.*

Proof. Let i be odd (even). Now $\mathfrak{L}_{i+1}$ ($\mathfrak{Q}_{i+1}$) is a subdirect product of subalgebras of $\mathfrak{L}_i$ ($\mathfrak{Q}_i$). As $\mathfrak{Q}_i$ ($\mathfrak{L}_i$) is free, $\mathfrak{L}_i(\mathfrak{Q}_i)$ is simple. If $\mathfrak{Q} \subseteq \mathfrak{Q}_i$ ($\mathfrak{L} \subseteq \mathfrak{L}_i$) is not simple then the corresponding subalgebra of $\mathfrak{L}_i(\mathfrak{Q}_i)$, namely $\mathfrak{L}(\mathfrak{Q})$, is simple by 4.2. Thus assume that $\mathfrak{Q}(\mathfrak{L})$ is simple. But by induction, $\mathfrak{Q}_i(\mathfrak{L}_i)$ is a direct product of simple algebras, each of which is a subalgebra of $\mathfrak{Q}_{i-1}(\mathfrak{L}_{i-1})$. Hence $\mathfrak{L} \subseteq \mathfrak{L}_{i-1}(\mathfrak{Q} \subseteq \mathfrak{Q}_{i-1})$ so by induction $\mathfrak{L}(\mathfrak{Q})$ is semi-simple. Thus $\mathfrak{L}_{i+1}(\mathfrak{Q}_{i+1})$ is a subdirect product of semi-simple algebras and hence is a direct product of simple algebras as claimed.

This means that $\mathfrak{S}_i \subseteq \mathfrak{S}_{i+1}$. Thus we can form the direct limit $\mathfrak{S}_\omega$ with corresponding sloop $\mathfrak{L}_\omega$ and squag $\mathfrak{Q}_\omega$. Note that $\mathfrak{S}_\omega$ is the direct limit of $\mathfrak{S}_1 \subseteq \mathfrak{S}_3 \subseteq \mathfrak{S}_5 \subseteq \ldots$ and $\mathfrak{S}_2 \subseteq \mathfrak{S}_4 \subseteq \mathfrak{S}_6 \ldots$ Thus $\mathfrak{L}_\omega$ is the direct limit of $\mathfrak{L}_2 \subseteq \mathfrak{L}_4 \subseteq \mathfrak{L}_6 \subseteq \ldots$; since $\mathfrak{L}_{2i+2}$ has a homomorphism onto $\mathfrak{L}_{2i}$, $\mathfrak{L}_\omega$ has a homomorphism onto $\mathfrak{L}_{2i}$ for all i. Therefore $\mathfrak{L}_\omega$ is not simple. Similarly, $\mathfrak{Q}_\omega$ is not simple.

2) Does there exist a proper subclass of *STS*s which is both a variety of sloops and a variety of squags? Note that by reasoning similar to point 1 above, there is a class of finite *STS*s which is a proper subclass of the finite *STS*s and which is closed under H, S, and finite products both as sloops and squags.

3) There are other varieties generated by finite squags and sloops which cover the atom in their lattice of subvarieties. First there are the semi-planar *STS*s: A squag (sloop) $\mathfrak{A}$ is *semi-planar* if every triangle either generates $\mathfrak{A}$ or the 9 (7) element squag (sloop) and if $\mathfrak{A}$ is simple. Undoubtedly there are

infinitely many such. Then there are those like the semi-planar algebras but which are subdirectly irreducible and not simple; moreover, any proper homomorphic image must be contained in $\mathscr{I}(\mathscr{B})$. These can be shown to have bounded cardinality and hence there are only finitely many of them. Finally there is the 81-element squag of M. Hall [**1**] which has the property that every triangle generates the 9-element squag but is itself not in $\mathscr{I}$. Note that there is no such corresponding sloop since $\mathscr{B}$ is defined by 3-variable identities whereas $\mathscr{I}$ needs a 4-variable identity.

4) Find $2^{\aleph_0}$ varieties of squags and sloops.

5) One of the consequences of a variety $\mathscr{V}$ having c.e.p. is that for any $\mathscr{K} \subseteq \mathscr{V}$, $HS(\mathscr{K}) = SH(\mathscr{K})$. An interesting open question is: is there a variety $\mathscr{V}$ such that $HS(\mathscr{K}) = SH(\mathscr{K})$ for all $\mathscr{K} \subseteq \mathscr{V}$ but not having c.e.p.? If $\mathfrak{A}$ is a simple planar sloop or squag then $\mathscr{V}(\mathfrak{A})$ seems to be a good candidate for a yes answer. By 7.6, $\mathscr{V}(\mathfrak{A})$ does not have c.e.p. On the other hand, for any finite $\mathfrak{B} \in \mathscr{V}(\mathfrak{A})$ it is easily seen that $HS(\mathfrak{B}) = SH(\mathfrak{B})$.

6) Find a finite squag $\mathfrak{A}$ with a subsquag $\mathfrak{B}$ such that $|\mathfrak{A}| = 3|\mathfrak{B}|$ and such that $\mathfrak{B}$ is not normal in $\mathfrak{A}$.

7) Finally, the author thanks the Technische Hochschule, Darmstadt and especially the Arbeitsgruppe Algemeine Algebra under the direction of Rudolf Wille for the opportunity to develop the ideas of this paper during his stay in Darmstadt.

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