CERTAIN GROUPS OF ORTHONORMAL STEP FUNCTIONS

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1. Introduction. It was first pointed out by Fine **(2)**, that the Walsh functions are essentially the characters of a certain compact abelian group, namely the countable direct product of groups of order two. Later Chrestenson **(1)** considered characters of the direct product of cyclic groups of order α ($\alpha = 2, 3, \ldots$). In general, his results show that the analytic properties of these generalized Walsh functions are basically the same as those of the ordinary Walsh functions. That this is the case does not seem surprising, since the structures of the underlying groups are quite similar.

It is our purpose to show that a considerably different state of affairs prevails for characters of a direct product of finite cyclic groups whose orders are unbounded. We shall show by examples that for these functions, certain basic properties of the Walsh systems mentioned above no longer hold.

The functions we shall study as well as the Walsh systems are included in a very general class of step functions discussed recently by Ohkuma **(3)**. We shall quote some of his results. However, a number of his principal theorems are proved under special conditions which are not satisfied by the functions we shall consider.

We shall have occasion to cite a number of properties of the Walsh functions. Unless otherwise stated, the reference for all of these is the fundamental paper of Fine **(2)**.

2. Definitions. Let $\{n_1, n_2, \ldots\}$ be a sequence of integers, $n_i \geqslant 2$, and let $p_0 = 1, p_k = n_1 n_2 \ldots n_k$. Denote by $I(r, k)$ the interval $r/p_k \leqslant x < (r+1)/p_k$ and by $\chi_r^{(k)}(x)$ the characteristic function of the set $\bigcup I(j, k)$ where j runs over all integers $\equiv r \pmod{p_k}$. Let us define a system of functions $\{\phi_0(x), \phi_1(x), \ldots\}$ as follows.

$$\phi_{k-1}(x) = \sum_{r=0}^{p_k-1} \omega_k^{\,r} \chi_r^{(k)}(x), \qquad \omega_k = e^{2\pi i/n_k} \qquad (k = 1,2, \ldots). \tag{1}$$

By definition, $\phi_{k-1}(x)$ is a step function whose values run through the n_kth roots of unity. On the interval [0,1] there are p_k intervals of constancy each of length $1/p_k$.

Now let $\Psi(n_1, n_2, \ldots)$ be the set of all finite products of these functions. In particular, $\Psi(2,2,2, \ldots)$ is the system of Walsh functions and $\Psi(\alpha, \alpha, \alpha, \ldots)$ is the generalization of Chrestenson.

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To enumerate our system we use a scheme based on Paley's enumeration of the Walsh functions. Every positive integer N can be expressed uniquely in the form $r_0 p_0 + r_1 p_1 + \ldots + r_m p_m$ where $0 \leqslant r_i < n_{i+1}$. Set

(2) $$\psi_0(x) \equiv 1, \qquad \psi_N(x) = \phi_0^{r_0}(x)\phi_1^{r_1}(x) \ldots \phi_m^{r_m}(x).$$

It is clear that $\Psi(n_1, n_2, \ldots)$ is an orthonormal system. That it is complete can be seen in several ways. For instance, the reasoning used by Fine in **(2)** shows that it is the full set of characters of the countable direct product of cyclic groups of order n_i transferred to the unit interval in a measure-preserving manner. Or, one may invoke a general theorem of Ohkuma **(3)**.

3. Expansions of certain functions. In this section, we obtain the Fourier expansions of several functions with respect to the system $\Psi(n_1, n_2, \ldots)$.

THEOREM 1.

$$x - [x] \sim \tfrac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{p_k} \sum_{j=1}^{n_k - 1} \frac{1}{\omega_k^{-j} - 1} \phi_{k-1}{}^{j}(x).$$

Proof. For every real value of x, $x - [x]$ can be represented in the form

$$\sum_{k=1}^{\infty} \frac{c_k}{p_k}, \qquad 0 \leqslant c_k < n_k .$$

This representation is unique except when x is of the form m/p_k. Now let $\zeta_\nu^{(k)}(x)$ be the characteristic function of the set of all x for which $\phi_{k-1}(x) = \omega_k{}^{\nu}$. This set can also be described as $\bigcup I(r,k)$ where the union is taken over all integers $r \equiv \nu \pmod{n_k}$. Define

(3) $$c_k(x) = \sum_{\nu=0}^{n_k - 1} \nu \zeta_\nu^{(k)}(x) .$$

Then, for every value of x,

(4) $$x - [x] = \sum_{k=1}^{\infty} \frac{c_k(x)}{p_k} .$$

In the case that $x = m/p_k$, (4) gives the finite representation of $x - [x]$. In all cases, the series in (4) converges uniformly. We observe that

(5) $$\frac{1}{n_k} \sum_{j=0}^{n_k - 1} \omega_k^{-j\nu} \phi_{k-1}^{j}(x) = \begin{cases} 1, & \text{if } \phi_{k-1}(x) = \omega_k^{\nu} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the series (5) is the Fourier expansion of $\zeta_\nu^{(k)}(x)$. Substituting it in (3) yields

(6) $$c_k(x) = \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \phi_{k-1}{}^{j}(x) \sum_{\nu=0}^{n_k - 1} \nu (\omega_k{}^{-j})^{\nu} .$$

Now in general,

$$\sum_{\nu=0}^{N} \nu x^{\nu} = x \frac{d}{dx} \left(\frac{1 - x^{N+1}}{1 - x} \right) = x \frac{N x^{N+1} - (N+1) x^{N} + 1}{(1 - x)^2} .$$

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Putting $N = n_k - 1$, $x = \omega_k^{-j}$, and using the fact that $\omega_k^{n_k} = 1$,

$$(7)\qquad \sum_{\nu=0}^{n_k-1} \nu(\omega_k^{-j})^\nu = \frac{n_k(\omega_k^{-j} - 1)}{(1 - \omega_k^{-j})^2} = \frac{n_k}{\omega_k^{-j} - 1} \qquad (j \neq 0),$$

$$(8)\qquad \sum_{\nu=0}^{n_k-1} \nu(\omega_k^{-j})^\nu = \sum_{\nu=1}^{n_k-1} \nu = \frac{n_k(n_k - 1)}{2} \qquad (j = 0).$$

Substituting (7) and (8) in (6),

$$(9)\qquad c_k(x) = \frac{n_k - 1}{2} + \sum_{j=1}^{n_k-1} \frac{1}{\omega_k^{-j} - 1}\, \phi_{k-1}{}^j(x).$$

Finally, using (9), the series in (4) becomes

$$(10)\qquad \tfrac{1}{2}\sum_{k=1}^{\infty} \frac{n_k - 1}{p_k} + \sum_{k=1}^{\infty} \frac{1}{p_k} \sum_{j=1}^{n_k-1} \frac{1}{\omega_k^{-j} - 1}\, \phi_{k-1}{}^j(x).$$

Since the convergence in (4) is uniform, it follows from (9) that there is a sequence of partial sums of the series (10) converging uniformly to $x - [x]$. This implies that (10) is the Fourier series of $x - [x]$.

If we observe that

$$\sum_{k=1}^{\infty} \frac{n_k - 1}{p_k} = \sum_{k=1}^{\infty} \left(\frac{1}{p_{k-1}} - \frac{1}{p_k}\right) = \frac{1}{p_0} - \lim_{k\to\infty} \frac{1}{p_k} = \frac{1}{p_0} = 1,$$

then (10) is the series given in the statement of the theorem.

The next theorem involves a simple, but rather lengthy computation. For this reason, we introduce the following lemma which prepares a computation needed in the proof of the theorem. Define

$$s(\nu, k) = 1 + \omega_k + \omega_k^2 + \ldots + \omega_k^{\nu-1} - \nu\omega_k^\nu.$$

LEMMA 1.

$$\sum_{\nu=0}^{p_k-1} \omega_k^{-\nu m} s(\nu,k) = \begin{cases} \dfrac{2p_k}{1 - \omega_k} & (m = 0), \\[2ex] -\dfrac{p_k}{1 - \omega_k} - \dfrac{p_k(p_k - 1)}{2} & (m = 1), \\[2ex] \dfrac{p_k}{1 - \omega_k^{1-m}} & (m = 2,3,\ldots, n_k - 1). \end{cases}$$

Proof.

$$(11)\qquad \sum_{\nu=0}^{p_k-1} \omega_k^{-\nu m}(1 + \omega_k + \omega_k^2 + \ldots + \omega_k^{\nu-1}) = \sum_{\nu=0}^{p_k-1} \omega_k^{-\nu m} \frac{1 - \omega_k^\nu}{1 - \omega_k}$$

$$= \frac{1}{1 - \omega_k}\left[\sum_{\nu=0}^{p_k-1} \omega_k^{-\nu m} - \sum_{\nu=0}^{p_k-1} \omega_k^{\nu(1-m)}\right] = \begin{cases} \dfrac{p_k}{1 - \omega_k} & (m = 0), \\[2ex] -\dfrac{p_k}{1 - \omega_k} & (m = 1), \\[2ex] 0 \quad (m = 2,3,\ldots, n_k - 1). \end{cases}$$

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Applying formulas (7) and (8), with n_k replaced by p_k,

$$(12)\quad -\sum_{\nu=0}^{p_k-1} \omega_k^{-\nu m}(\nu\omega_k^{\nu}) = -\sum_{\nu=0}^{p_k-1} \nu(\omega_k^{1-m})^{\nu} = \begin{cases} -\dfrac{p_k(p_k-1)}{2} & (m=1), \\ \dfrac{p_k}{1-\omega_k^{1-m}} & (m = 0,2,3,\ldots, n_k - 1). \end{cases}$$

Adding (11) and (12) yields the lemma.

We shall now find the Fourier series associated with the function

$$J_{k-1}(x) = \int_0^1 \phi_{k-1}(u)du.$$

THEOREM 2.

$$J_{k-1}(x) \sim \frac{1}{p_k(1-\omega_k)} + \phi_{k-1}(x)\left[\frac{1}{2p_k} - \frac{1}{p_k(1-\omega_k)}\right] + \sum_{\nu=k+1}^{\infty} \frac{1}{p_\nu} \sum_{j=1}^{n_\nu-1} \frac{1}{\omega_\nu^{-j}-1} \psi_{jp_{\nu-1}+p_{k-1}}(x).$$

Proof. If $x \in I(\nu, k)$,

$$\phi_{k-1}(x) = \omega_k^{\nu}; \int_0^x \phi_{k-1}(u)du = \int_0^{\nu/p_k} \phi_{k-1}(u)du + \omega_k^{\nu}\left(x - [x] - \frac{\nu}{p_k}\right).$$

Therefore, if $x \in I(\nu, k)$,

$$J_{k-1}(x) = \omega_k^{\nu}\left(x - [x] - \frac{\nu}{p_k}\right) + \frac{1}{p_k}\sum_{j=0}^{\nu-1} \omega_k^{j} = \omega_k^{\nu}(x - [x]) + \frac{1}{p_k} s(\nu,k).$$

Since for all x,

$$J_{k-1}(x) = \sum_{\nu=0}^{p_k-1} \chi_\nu^{(k)}(x)\, J_{k-1}(x),$$

$$(13)\quad J_{k-1}(x) = (x - [x])\sum_{\nu=0}^{p_k-1} \omega_k^{\nu}\chi_\nu^{(k)}(x) + \frac{1}{p_k}\sum_{\nu=0}^{p_k-1} s(\nu,k)\chi_\nu^{(k)}(x).$$

The first sum on the right side of (13) is $\phi_{k-1}(x)$ by definition.

Furthermore,

$$\chi_\nu^{(k)}(x) = \frac{1}{p_k}\sum_{j=0}^{p_k-1} \overline{\psi_j(\nu,k)}\, \psi_j(x),$$

where $\psi_j(\nu, k)$ is the value of $\psi_j(x)$ on $I(\nu, k)$. Substituting these facts into (13),

$$(14)\quad J_{k-1}(x) = (x - [x])\, \phi_{k-1}(x) + \frac{1}{p_k^2}\sum_{j=0}^{p_k-1} \psi_j(x) \sum_{\nu=0}^{p_k-1} \overline{\psi_j(\nu,k)}\, s(\nu,k).$$

We assert now that the Fourier expansion of $J_{k-1}(x)$ does not involve any of the functions $\psi_j(x)$ for which $1 \leqslant j < p_{k-1}$. To show that the corresponding Fourier coefficients vanish, we use the facts that $J_{k-1}(x)$ has period $1/p_{k-1}$, $\psi_j(x)$ is constant on intervals of the form $I(r, k-1)$, and

$$\int_0^1 \psi_j(x)dx = 0.$$

Consequently,

$$\int_0^1 J_{k-1}(x)\,\overline{\psi_j(x)}\,dx = \int_0^{1/p_{k-1}} J_{k-1}(x)dx \int_0^1 \overline{\psi_j(x)}dx = 0.$$

Now if $p_{k-1} \leqslant j < p_k$, then $\psi_j(x)$ can occur in the expansion of $J_{k-1}(x)$ only if $\psi_j(x)$ is one of the functions

$$\phi_{k-1}^m(x), \qquad (m = 1, 2, \ldots, n_k - 1).$$

If $\psi_j(x)$ is not of this type, it is of the form $\phi_{k-1}{}^m(x)\psi_r(x)$, where $1 \leqslant r < p_{k-1}$. But then $J_{k-1}(x)\phi_{k-1}{}^{-m}(x)$ has period $1/p_{k-1}$ and the same reasoning shows that

$$\int_0^1 J_{k-1}(x)\,\overline{\psi_j(x)}\,dx = \int_0^1 (J_{k-1}(x)\ \phi_{k-1}{}^{-m}(x))\ \overline{\psi_r(x)}dx = 0.$$

The expansion of $x - [x]$ is given by Theorem 1. We substitute it into (14). In view of the above discussion, however, we may drop all terms in $\psi_j(x)$ for $1 \leqslant j < p_k$ except those of the type $\phi_{k-1}{}^m(x)$. The result is:

$$\begin{aligned}(15)\qquad J_{k-1}(x) \sim \tfrac{1}{2}\phi_{k-1}(x) &+ \frac{1}{p_k}\sum_{j=1}^{n_k-1} \frac{1}{\omega_k^{-j}-1}\,\phi_{k-1}{}^{j+1}(x)\\ &+ \sum_{\nu=k+1}^{\infty} \frac{1}{p_\nu}\sum_{j=1}^{n_\nu-1} \frac{1}{\omega_\nu^{-j}-1}\,\phi_{\nu-1}{}^{j}(x)\ \phi_{k-1}(x)\\ &+ \frac{1}{p_k^2}\sum_{m=0}^{n_k-1} \phi_{k-1}{}^m(x)\sum_{\nu=0}^{p_k-1} \phi_{k-1}{}^{-m}(\nu,k)\ s(\nu,k).\end{aligned}$$

The first sum on the right side of (15) has a term in $\phi_{k-1}{}^{n_k}(x) \equiv 1$. Writing this term first and shifting the summation index,

$$(16)\quad \frac{1}{p_k}\sum_{j=1}^{n_k-1} \frac{1}{\omega_k^{-j}-1}\,\phi_{k-1}{}^{j+1}(x) = -\frac{1}{p_k(1-\omega_k)} - \frac{1}{p_k}\sum_{m=2}^{n_k-1} \frac{1}{1-\omega_k^{1-m}}\,\phi_{k-1}^m(x).$$

Since $\phi_{k-1}{}^{-m}(\nu, k) = \omega_k{}^{-\nu m}$, the last sum in (15) can be simplified by means of Lemma 1. We obtain

$$\begin{aligned}(17)\qquad &\frac{1}{p_k^2}\sum_{m=0}^{n_k-1} \phi_{k-1}{}^m(x)\sum_{\nu=0}^{p_k-1} \omega_k^{-\nu m} s(\nu,k)\\ &= \frac{2}{p_k(1-\omega_k)} - \phi_{k-1}(x)\left[\frac{1}{p_k(1-\omega_k)} + \tfrac{1}{2} - \frac{1}{2p_k}\right]\\ &\qquad + \frac{1}{p_k}\sum_{m=2}^{n_k-1} \frac{1}{1-\omega_k^{1-m}}\,\phi_{k-1}^m(x).\end{aligned}$$

Substituting (16) and (17) into (15),

$$J_{k-1}(x) \sim \frac{1}{p_k(1-\omega_k)} + \phi_{k-1}(x)\left[\frac{1}{2p_k} - \frac{1}{p_k(1-\omega_k)}\right] + \sum_{\nu=k+1}^{\infty} \frac{1}{p_\nu} \sum_{j=1}^{n_\nu - 1} \frac{1}{\omega_\nu^{-j} - 1} \phi_{\nu-1}{}^j(x)\ \phi_{k-1}(x). \tag{18}$$

In our enumeration,

$$\phi_{\nu-1}{}^j(x)\ \phi_{k-1}(x) = \psi_{jp_{\nu-1}+p_{k-1}}(x).$$

Therefore, (18) is the desired expansion.

It is easy to obtain a similar expansion for the integral of the function $\phi_{k-1}{}^m(x)$. Basically, this amounts to replacing ω_k by $\omega_k{}^m$ in (18). We can now obtain the Fourier series associated with the integral of the function $\psi_j(x)$ for any value of j.

Given j, there exists k such that $p_{k-1} \leqslant j < p_k$. Then $j = mp_{k-1} + r$ where $1 \leqslant m \leqslant n_k - 1$ and $0 \leqslant r < p_{k-1}$ so that $\psi_j(x) = \phi_{k-1}{}^m(x)\psi_r(x)$. Now the integral of $\phi_{k-1}{}^m(x)$ has period $1/p_{k-1}$ whereas the period of $\psi_r(x)$ is a multiple of this number. Therefore,

$$\int_0^x \psi_j(u)du = \psi_r(x) \int_0^x \phi_{k-1}{}^m(u)du.$$

and we easily obtain the following expansion.

THEOREM 3. *Let $j = mp_{k-1} + r$ where $1 \leqslant m \leqslant n_{k-1}$ and $0 \leqslant r < p_{k-1}$. Then*

$$\int_0^x \psi_j(u)du \sim \frac{1}{p_k(1-\omega_k^m)}\psi_r(x) + \left[\frac{1}{2p_k} - \frac{1}{p_k(1-\omega_k{}^m)}\right]\psi_j(x) + \sum_{\nu=k+1}^{\infty} \frac{1}{p_\nu} \sum_{s=1}^{n_\nu - 1} \frac{1}{\omega_\nu^{-s} - 1} \psi_{sp_{\nu-1}+j}(x).$$

4. Fourier coefficients. The following theorems are of interest.

(A) If $f(x)$ is a continuous function with modulus of continuity $\omega(\delta)$ and if $p_{k-1} \leqslant j < p_k$, then $|a_j| \leqslant \frac{1}{2}\omega(1/p_{k-1})$ where a_j is the jth Fourier coefficient of $f(x)$ with respect to the system $\Psi(n_1, n_2, \ldots)$.

(B) If $f(x)$ has bounded total variation V, then

$$|\, a_j \,| \leqslant \frac{V}{p_k} \csc \pi/n_k$$

Theorem (A) is a particular case of a result of Ohkuma **(3)**; Theorem (B) is essentially a result of Chrestenson **(1)**. When the sequence $\{n_1, n_2, \ldots\}$ is bounded, there follow directly from (A) and (B) such facts as:

(a) If $f(x)$ satisfies a Lipschitz condition of order α, then its Fourier coefficients are $O(1/n^\alpha)$.

(b) If $f(x)$ is of bounded variation, its coefficients are $O(1/n)$.

However, if the sequence is unbounded, these assertions cannot be made. For instance, if $f(x)$ satisfies a Lipschitz condition of order 1, the most we can say about a_{p_k-1} is that it is less in absolute value than C/p_{k-1} where C is a constant. Consequently we can conclude only that $|a_j| < Cn_k/j$ when $p_{k-1} \leqslant j < p_k$. Since $\limsup n_k = \infty$, it is conceivable that $\limsup j|a_j| = \infty$. Furthermore, $\limsup V \csc \pi/n_k = \infty$, so it is also conceivable that $\limsup j|a_j| = \infty$ for functions of bounded variation. We shall show by means of examples that these possibilities actually do occur.

THEOREM 4. *If $\{n_1, n_2, \ldots\}$ is unbounded, then there exist functions of bounded variation whose Fourier coefficients with respect to the system $\Psi(n_1, n_2, \ldots)$ are not $O(1/n)$.*

Proof. Take the function $f(x) = x - [x]$ whose expansion is given by Theorem 1. Let a_j denote the jth Fourier coefficient of $f(x)$. For any k, $\phi_{k-1}{}^{n_k-1}(x)$ is the $(n_k - 1)p_{k-1}$th or the $(p_k - p_{k-1})$th function in our enumeration of the system Ψ. By Theorem 1,

$$a_{p_k - p_{k-1}} = \frac{1}{p_k(\omega_k - 1)} = \frac{1}{p_k} \frac{1}{e^{2\pi i/n_k} - 1}.$$

$$(p_k - p_{k-1}) \mid a_{p_k - p_{k-1}} \mid = \frac{p_k - p_{k-1}}{2p_k} \csc \pi/n_k .$$

$$\limsup\, (p_k - p_{k-1}) \mid a_{p_k - p_{k-1}} \mid = \limsup \tfrac{1}{2} \left(1 - \frac{1}{n_k}\right) \csc \pi/n_k = \infty .$$

Hence, $\limsup j|a_j| = \infty$.

THEOREM 5. *If $\{n_1, n_2, \ldots\}$ is unbounded there exist absolutely continuous functions, in fact functions satisfying a Lipschitz condition of order* 1, *whose Fourier coefficients with respect to the system $\Psi(n_1, n_2, \ldots)$ are not $O(1/n)$.*

Proof. Take the function $J_{k-1}(x)$. Formula (14) in the proof of Theorem 2 shows that except for its first p_k terms, the Fourier series of $J_{k-1}(x)$ agrees with that of $(x - [x])\phi_{k-1}(x)$. Except for a shift, the Fourier coefficients of the latter function are the same as those of $x - [x]$. Consequently the argument used in the proof of Theorem 4 again shows that $\limsup j|a_j| = \infty$.

The following elementary facts will be useful. We list them as a lemma and sketch the proof.

LEMMA 2. *If $n_1 < n_2 < n_3 < \ldots$, then as $k \to \infty$,*

$$\frac{1}{n_k}\sum_{j=1}^{n_k-1} \frac{1}{|\omega_k^{-j} - 1|} \sim \frac{1}{n_k}\left|\sum_{j=1}^{[n_k/4]} \frac{1}{\omega_k^{-j} - 1}\right| \sim \log n_k .$$

Proof. Since $|\omega_k^{-j} - 1|^{-1} = |e^{-2\pi ij/n_k} - 1|^{-1} = \frac{1}{2} \csc \pi j/n_k$, the first sum is clearly asymptotic to

$$\frac{1}{n_k}\sum_{j=1}^{[n_k/2]} \csc \pi j/n_k .$$

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Here, $\csc \pi j/n_k$ may be replaced by $n_k/\pi j$, since the ratio of $\csc x$ to $1/x$ is bounded for $0 \leqslant x \leqslant \frac{1}{2}\pi$. Therefore, the first sum is asymptotic to

$$\frac{1}{n_k}\sum_{j=1}^{[n_k/2]} \frac{n_k}{\pi j} \sim \log [n_k/2] \sim \log n_k .$$

To prove the second assertion, observe that

$$\frac{1}{n_k}\left|\sum_{j=1}^{[n_k/4]} \frac{1}{\omega_k^{-j} - 1}\right| = \frac{1}{n_k}\left|\tfrac{1}{2}\sum_{j=1}^{[n_k/4]} (-1 + i \cot \pi j/n_k)\right| \sim \frac{1}{n_k}\sum_{j=1}^{[n_k/4]} \cot \pi j/n_k .$$

Arguing as above, $\cot \pi j/n_k$ may be replaced by $n_k/\pi j$ and the result follows in the same way.

Fine has proved that the Walsh-Fourier coefficients of an absolutely continuous function are not $O(1/n)$ unless the function is a constant. We shall modify his argument to show that there is a class of unbounded sequences $\{n_i\}$ such that the Fourier coefficients of a non-constant absolutely continuous function with respect to $\Psi(n_1, n_2, \ldots)$ are not even $O(1/n)$.

We shall consider sequences $\{n_i\}$ satisfying:

$$(19) \qquad n_1 < n_2 < n_3 < \ldots ; \sum_{\nu=k+1}^{\infty} \frac{1}{p_\nu} \sum_{s=1}^{n_{\nu-1}} \frac{1}{|\omega_\nu^{-s} - 1|} = O(1/p_{k-1}) .$$

In view of Lemma 2, the latter condition is equivalent to

$$\sum_{\nu=k+1}^{\infty} \frac{\log n_\nu}{p_{\nu-1}} = O(1/p_{k-1}) .$$

This is a restriction on the rate of increase of the sequence. Still, it does admit sequences growing as rapidly as

$$2, 2^2, 2^{2^2}, \ldots ; n_{k+1} = 2^{n_k} .$$

THEOREM 6. *If $\{n_1, n_2, \ldots\}$ satisfies (19) then the Fourier coefficients of a non-constant absolutely continuous function with respect to the system $\Psi(n_1, n_2, \ldots)$ are not $O(1/n)$.*

Proof. Suppose that $F(x)$ is absolutely continuous and that $f(x)$ is its derivative. Let $\{a_j\}$ and $\{b_j\}$ be the Fourier coefficients of $F(x)$ and $f(x)$ respectively. We assume $F(x)$ is not a constant so that $f(x) \not\equiv 0$. Therefore one of the Fourier coefficients of $f(x)$, say b_r, does not vanish. We shall show that if $i_k = p_k - p_{k-1} + r$ then

$$i_k|a_{i_k}| \backsim n_k.$$

Hence, $\limsup j|a_j| = \infty$.

Set

$$H_j(x) = \int_0^x \psi_j(u)du .$$

Since $H_j(0) = H_j(1) = 0$, we obtain, on integrating by parts,

$$(20)\qquad a_{i_k} = \int_0^1 F(x)\,\overline{\psi_{i_k}(x)}dx = \Big[F(x)\,\overline{H_{i_k}(x)}\Big]_0^1 - \int_0^1 f(x)\,\overline{H_{i_k}(x)}dx$$
$$= -\int_0^1 f(x)\,\overline{H_{i_k}(x)}dx.$$

Since the Fourier series given by Theorem 3 has a subsequence of partial sums converging uniformly to

$$H_{i_k}(x),$$

we may substitute it in (20) and integrate termwise.

$$-a_{i_k} = \frac{1}{p_k(1-\omega_k)}\int_0^1 f(x)\overline{\psi_r(x)}dx + \left[\frac{1}{2p_k} - \frac{1}{p_k(1-\omega_k)}\right]\int_0^1 f(x)\overline{\psi_{i_k}(x)}dx$$
$$+\sum_{\nu=k+1}^{\infty}\frac{1}{p_\nu}\sum_{s=1}^{n_\nu-1}\frac{1}{\omega_\nu^s - 1}\int_0^1 f(x)\overline{\psi_{sp_{\nu-1}+i_k}(x)}dx.$$

$$(21)\qquad -a_{i_k} = \frac{1}{p_k(1-\omega_k)}\,b_r + \left[\frac{1}{2p_k} - \frac{1}{p_k(1-\omega_k)}\right]b_{i_k}$$
$$+\sum_{\nu=k+1}^{\infty}\frac{1}{p_\nu}\sum_{s=1}^{n_\nu-1}\frac{1}{\omega_\nu^s - 1}\,b_{sp_{\nu-1}+i_k}.$$

Now

$$\max_{j>p_k}|b_j| \to 0, \qquad k\to\infty.$$

Therefore, by assumption (19), the absolute value of the sum in (21) is $O(1/p_{k-1}) = O(n_k/p_k)$. Multiplying equation (21) by $-i_k$,

$$(22)\qquad i_k a_{i_k} = -\frac{i_k}{p_k}\frac{b_r}{1-\omega_k} - \frac{i_k}{p_k}\left(\frac{1}{2} - \frac{1}{1-\omega_k}\right)b_{i_k} + o(i_k n_k/p_k).$$

Since

$$\lim_{k\to\infty}\frac{i_k}{p_k} = \lim_{k\to\infty}\frac{p_k - p_{k-1} + r}{p_k} = 1;\ \frac{1}{1-\omega_k}\sim n_k,$$

the absolute value of the first term in (22) is asymptotic to n_k. The second term is $o(n_k)$ since $b_{i_k}\to 0$. Hence,

$$i_k\,|a_{i_k}| \sim n_k, \qquad k\to\infty.$$

5. Summability. The trigonometric Fourier series of a continuous function $f(x)$ is uniformly $(C,1)$ summable to $f(x)$. The analogous statement is true for the Walsh functions. In fact, a theorem of Ohkuma **(3)** implies it is true in any system $\Psi(n_1, n_2, \ldots)$ provided $\{n_i\}$ is a bounded sequence. We shall show that if $\{n_i\}$ is unbounded, the situation is quite different.

THEOREM 7. *Let* $\{n_1, n_2, \ldots\}$ *be unbounded. Then, given any real value* a, *there exists a continuous function whose Fourier series with respect to the system* $\Psi(n_1, n_2, \ldots)$ *is not* $(C,1)$ *summable at* $x = a$.

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Proof. Denote by $D_k(x, u)$ the kth Dirichlet kernel of the system.

$$D_k(x,u) = \sum_{j=0}^{k-1} \psi_j(x)\overline{\psi_j(u)}. \tag{23}$$

It will be useful to note that if $x \in I(r, k)$, then

$$D_{p_k}(x,u) = \prod_{\nu=0}^{k-1} \left\{ \sum_{j=0}^{n_{\nu+1}-1} (\phi_\nu(x)\phi_\nu^{-1}(u))^j \right\} = p_k \chi_r^{(k)}(u). \tag{24}$$

Denote by $K_n(x, u)$ the nth Cesaro kernel:

$$K_n(x, u) = \frac{1}{n}\sum_{j=1}^{n} D_j(x, u). \tag{25}$$

If the sequence

$$\left\{ \int_0^1 |K_n(a, u)|du \right\}$$

is unbounded, then by a well-known theorem of Haar, there exists a continuous function whose Fourier series is not $(C, 1)$ summable at $x = a$.

Let us first consider the case when $a = 0$. This is a convenient value of a because $\psi_j(0) = 1$ for all j. For brevity, let

$$D_n(u) = D_n(0, u), \; K_n(u) = K_n(0, u), \; K_n = \int_0^1 |K_n(u)|du.$$

We shall show that the sequence $\{K_n\}$ is unbounded.

$$\begin{aligned} p_{r+1} K_{p_{r+1}}(u) &= \sum_{j=1}^{p_{r+1}} D_j(u) = \sum_{j=1}^{p_r} D_j(u) + \sum_{p_r+1}^{p_{r+1}} D_j(u) \\ &= p_r K_{p_r}(u) + \sum_{\nu=1}^{n_{r+1}-1} \sum_{j=1}^{p_r} D_{\nu p_r + j}(u). \end{aligned} \tag{26}$$

Our enumeration of $\Psi(n_1, n_2, \ldots\}$ is such that

$$\begin{aligned} \sum_{j=1}^{p_r} D_{\nu p_r + j}(u) &= \sum_{j=1}^{p_r} \{(1 + \phi_r^{-1}(u) + \phi_r^{-2}(u) + \ldots + \phi_r^{-(\nu-1)}(u))\, D_{p_r}(u) \\ &\qquad + \phi_r^{-\nu}(u)\, D_j(u)\} \\ &= p_r D_{p_r}(u)\, ((1 + \phi_r^{-1}(u) + \ldots + \phi_r^{-(\nu-1)}(u)) + \phi_r^{-\nu}(u)\, p_r K_{p_r}(u). \end{aligned} \tag{27}$$

Combining (26) and (27),

$$\begin{aligned} p_{r+1} K_{p_{r+1}}(u) &= p_r D_{p_r}(u) \sum_{\nu=1}^{n_{r+1}-1} (1 + \phi_r^{-1}(u) + \ldots + \phi_r^{-(\nu-1)}(u)) \\ &\quad + p_r K_{p_r}(u) \sum_{\nu=0}^{n_{r+1}-1} \phi_r^{-\nu}(u). \end{aligned} \tag{28}$$

An easy computation yields

$$K_{p_{r+1}}(u) = \begin{cases} \dfrac{1}{1 - \phi_r^{-1}(u)} D_{p_r}(u) & (\phi_r(u) \neq 1), \\[2ex] K_{p_r}(u) + \dfrac{n_{r+1} - 1}{2} D_{p_r}(u) & (\phi_r(u) = 1). \end{cases} \tag{29}$$

By means of (24) and the recursion relation (29), it is possible to obtain an exact formula for $K_{p_r}(u)$. The reasoning involves a number of cases, all of which are quite simple. We shall simply list the results.

$$(30)\quad K_{p_r}(u) = \begin{cases} \dfrac{p_r + 1}{2}, & u \in I\,(0, r); \\ \dfrac{p_{s-1}}{1 - \omega_s^{-j}}, & u \in I\,(j\,p_r/p_s, r) \qquad (j = 1, 2, \ldots, n_s - 1; \\ 0, & \text{otherwise.} \qquad\qquad s = 1, 2, \ldots, r); \end{cases}$$

It follows from (30) that

$$(31)\quad \int_0^1 |K_{p_r}(u)|d\,u = \frac{1}{p_r}\left\{\frac{p_r+1}{2} + \sum_{s=1}^{r}\sum_{j=1}^{n_s-1}\frac{p_{s-1}}{|1-\omega_s^{-j}|}\right\} > \frac{1}{p_r}\sum_{j=1}^{n_r-1}\frac{p_{r-1}}{|1-\omega_r^{-j}|}$$
$$= \frac{1}{n_r}\sum_{j=1}^{n_r-1}\frac{1}{|1-\omega_r^{-j}|}.$$

Therefore, by Lemma 2, the sequence $\{K_{p_r}\}$ is unbounded, increasing at least as rapidly as $\{C \log n_r\}$ where C is a constant. Consequently there exists a continuous function whose Fourier series is not $(C, 1)$ summable at the point $x = 0$.

The functions

$$\int_0^1 |K_r(x, u)|du$$

are actually independent of x, hence constant. This is because $\Psi(n_1, n_2, \ldots)$ is essentially the set of characters of a compact group whose Haar measure coincides with the Lebesgue measure on the unit interval. (The argument given by Fine for the Walsh functions (2) carries over directly.) It follows that our proof is valid not only for $a = 0$, but for any value of a. This proves the theorem.

We show next that the situation with respect to summability can be even worse if the sequence $\{n_i\}$ increases rapidly enough. Such a sequence will be one such that

$$(32)\qquad n_1 < n_2 < n_3 < \ldots;\ \frac{1}{p_k}\sum_{r=1}^{[n_k/4]}\frac{1}{\omega_k^{-r} - 1} \to \infty.$$

By Lemma 2, the latter condition is equivalent to

$$\frac{\log n_k}{p_{k-1}} \to \infty$$

and is satisfied if

$$n_k = e^{a_k p_{k-1}},$$

where $\{a_k\}$ is any sequence tending to infinity.

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THEOREM 8. *If* $\{n_1, n_2, \ldots\}$ *satisfies* (32), *then there exist functions in* Lip 1 *whose Fourier series with respect to* $\Psi(n_1, n_2, \ldots)$ *are not* $(C, 1)$ *summable on a set* S *which is non-denumerable and includes all rationals of the form* r/p_m.

Proof. It will suffice to prove that the assertion is true for the function $x - [x]$. For, as pointed out in the proof of Theorem 5, the expansion of $J_{k-1}(x)$ is essentially the same as that of $x - [x]$.

The latter expansion is given by Theorem 1. Let $s_j(x)$, $\sigma_j(x)$ denote its jth partial sum and $(C, 1)$ mean respectively. Since the series has gaps, $s_j(x)$ will be constant over blocks of consecutive values of j. More precisely,

$$(33) \quad s_j(x) = s_{p_{k-1}}(x) + \frac{1}{p_k}\sum_{r=1}^{\nu} \frac{1}{\omega_k^{-r} - 1}\, \phi_{k-1}^{j}(x),$$
$$\nu p_{k-1} < j \leqslant (\nu + 1)p_{k-1};\ 1 \leqslant \nu < n_k.$$

Now suppose $\phi_{k-1}(x) = 1$. It is easy to obtain an expression for $\sigma_j(x)$ from (33) when $j = mp_{k-1}$.

$$(34) \quad mp_{k-1}\,\sigma_{mp_{k-1}}(x) = \sum_{j=1}^{mp_{k-1}} s_j(x) = \sum_{j=1}^{p_{k-1}} s_j(x) + \sum_{j=p_{k-1}+1}^{mp_{k-1}} s_j(x)$$
$$= p_{k-1}\,\sigma_{p_{k-1}}(x) + p_{k-1}(m-1)s_{p_{k-1}}(x) + \frac{p_{k-1}}{p_k}\sum_{r=1}^{m-1}(m-r)\frac{1}{\omega_k^{-r} - 1}.$$

Therefore,

$$(35) \quad \sigma_{mp_{k-1}}(x) = \frac{1}{m}\,\sigma_{p_{k-1}}(x) + \left(1 - \frac{1}{m}\right)s_{p_{k-1}}(x) + \frac{1}{p_k}\sum_{r=1}^{m-1}\left(1 - \frac{r}{m}\right)\frac{1}{\omega_k^{-r} - 1}.$$

We may assume $\{\sigma_{p_{k-1}}(x)\}$ is bounded. Otherwise x is already a point where summability fails. Furthermore, $\{s_{p_{k-1}}(x)\}$ is also bounded. A general theorem of Ohkuma **(3)** says that the p_kth partial sums of the Fourier series of an integrable function converge to the function at all points of continuity. (For $x = 0$, this argument does not apply but a direct verification is easy.) Consequently,

$$(36) \quad \sigma_{mp_{k-1}}(x) = O(1) + \frac{1}{p_k}\sum_{r=1}^{m-1}\left(1 - \frac{r}{m}\right)\frac{1}{\omega_k^{-r} - 1}$$
$$= O(1) + \frac{1}{p_k}\sum_{r=1}^{m-1}\frac{1}{\omega_k^{-r} - 1} - \frac{1}{p_k}\sum_{r=1}^{m-1}\frac{r}{m}\frac{1}{\omega_k^{-r} - 1}.$$

Take $m = [\frac{1}{4}n_k] + 1$. Since $|\omega_k^{-r} - 1|^{-1} = \frac{1}{2}\csc \pi r/n_k < Cn_k/r$ for $0 < r \leqslant [\frac{1}{4}n_k] + 1$, the absolute value of the last sum in (36) is dominated by

$$\frac{1}{p_k}\sum_{r=1}^{m}\frac{r}{m}\frac{Cn_k}{r} = \frac{Cn_k}{p_k} = \frac{C}{p_{k-1}} = o(1).$$

Thus if $\phi_{k-1}(x) = 1$, and $j_k = ([\frac{1}{4}n_k] + 1)p_{k-1}$,

$$(37) \quad \sigma_{j_k}(x) = O(1) + \frac{1}{p_k}\sum_{r=1}^{[n_k/4]}\frac{1}{\omega_k^{-r} - 1}.$$

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If for a given x, relation (37) holds for infinitely many values of k, assumption (32) shows that the sequence

$$\{\sigma_{j_k}(x)\}$$

is unbounded. But there is a non-denumerable set S including all rationals r/p_m of such values of x. Let

$$x = \sum_{k=1}^{\infty} \frac{c_k}{p_k}; \quad o \leqslant c_k < n_k.$$

Since $\phi_{k-1}(x) = \omega_k^{c_k}$, $x \in S$ if and only if infinitely many of the c_k are zero. This completes the proof.

It is known that the Walsh-Fourier series of a function of bounded variation converges at each dyadic rational and at each point of continuity of the function. Since the rationals r/p_m are the analogues of the dyadic rationals, Theorem 8 demonstrates a significant difference between the analytic properties of the Walsh system and those of $\Psi(n_1, n_2, \ldots)$ when $\{n_i\}$ is unbounded.

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