TRANSLATIONAL HULL AND SEMIGROUPS OF BINARY RELATIONS

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Introduction. Various semigroups of partial transformations (and more generally, semigroups of binary relations) on a set have been studied by a number of Soviet mathematicians; to mention only a few: Gluskin [2], Ljapin [4], Shutov [6], Zaretski [7], [8]. In their study the densely embedded ideal of a semigroup introduced by Ljapin [4] plays a central role. In fact, a concrete semigroup Q is described in several instances by its abstract characteristic, namely either by a set of postulates on an abstract semigroup or by a set of postulates (which are usually much simpler) on an abstract semigroup S which is a densely embedded ideal of a semigroup T isomorphic to Q. In many cases, the densely embedded ideal S is a completely 0-simple semigroup. The following theorem [3, 1.7.1] reduces the study of a semigroup Q with a weakly reductive densely embedded ideal S to the study of the translational hull of S:

THEOREM (Gluskin). *If S is a weakly reductive densely embedded ideal of a semigroup Q, then Q is isomorphic to the translational hull $\Omega(S)$ of S.*

The purpose of this work is to characterize the translational hull of some special classes of completely 0-simple semigroups by using the results obtained in [5]. We obtain as corollaries several results proved in the papers mentioned above (with different methods of proof); furthermore, the descriptions of the translational hull can be used, e.g., for constructing ideal extensions.

Notation and summary. In order to summarize our results, we find it expedient to first introduce the necessary notation. We follow the notation introduced in [5] (this paper will henceforth be referred to as TH) and use freely the results proved there (the construction of the translational hull of a completely 0-simple semigroup); otherwise the notation and terminology is that of Clifford and Preston [1].

Throughout the whole paper, I denotes a fixed non-empty set, G a fixed group whose identity is denoted by 1; a one-element group is also denoted by 1. For any set A, $|A|$ stands for the cardinality of A; $\mathbf{P}_\sigma$ denotes the family of all non-empty subsets A of I such that $|A| < \sigma$, where either $\sigma = 2$ or σ is an infinite cardinal. Further let

$$S_{\sigma\tau} = \mathscr{M}^0(G; \mathbf{P}_\sigma, \mathbf{P}_\tau; P),$$

where $P = (p_{BA})$ and

$$p_{BA} = \begin{cases} 1 & \text{if } A \cap B \neq \square, \\ 0 & \text{otherwise.} \end{cases}$$

We identify a one-element set with the element itself, so, e.g., we write I instead of $\mathbf{P}_2$, in which case $p_{Ba} = 1 \Leftrightarrow a \in B$. For any semigroup S, $\tilde{\Lambda}(S)$ $[\tilde{\mathrm{P}}(S)]$ denotes the semigroup of left [right] translations for which there exists a linked right [left] translation.

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It is easy to verify that $S_{\sigma\tau}$ is reductive, or, equivalently, that it satisfies condition (*c*) and its dual, of Theorem 8 of TH; hence by the corollary (and its dual) to the same theorem, we have

$$\tilde{\Lambda}(S_{\sigma\tau}) \cong \tilde{P}(S_{\sigma\tau}) \cong \Omega(S_{\sigma\tau}),$$

$$S_{\sigma\tau} \cong \Gamma(S_{\sigma\tau}) \cong \Delta(S_{\sigma\tau}) \cong \Pi(S_{\sigma\tau}),$$

where $\Gamma(S)$ and $\Delta(S)$ are respectively the semigroups of all inner left and all inner right translations of a semigroup S.

We are interested here in a description of the translational hull $\Omega(S_{\sigma\tau})$ of $S_{\sigma\tau}$; in view of the remarks just made, for convenience we will restrict our attention to either $\tilde{\Lambda}(S_{\sigma\tau})$ or $\tilde{P}(S_{\sigma\tau})$. In §1, we consider the case $\sigma = \tau = 2$; in §2, $\sigma = 2$, τ an infinite cardinal; and in §3, σ and τ are both infinite cardinals. In the first two sections, the principal results are expressed by means of partial transformations on a set, while in the third section, they are expressed by means of binary relations on a set.

For any non-empty set J and σ, τ having values as above, $W_{\sigma\tau}(J)$ [$W'_{\sigma\tau}(J)$] denotes the semigroup of all partial transformations α on the set J written as operators on the left [right] and such that rank $\alpha < \sigma$ and, for every $j \in \mathbf{d}\alpha$, $|\alpha^{-1}(\alpha j)| < \tau$ [$|(j\alpha)\alpha^{-1}| < \tau$]; $\alpha^{-1}k$ [$k\alpha^{-1}$] denotes the complete inverse image of k if $k \in \mathbf{r}\alpha$; J will be one of the sets I, $\mathbf{P}_\sigma$, $\mathbf{P}_\tau$. If $\sigma > |I|$, we write $W_{\omega\tau}(J)$ instead of $W_{\sigma\tau}(J)$; similarly for $W'_{\sigma\tau}(J)$ and for τ. We write $W(J)$ instead of $W_{\omega\omega}(J)$ which conforms with the notation introduced in TH; similarly for $W'_{\omega\omega}(J)$.

If D is a semigroup with zero 0, we denote by $D \times_0 G$ (G a fixed group as above) the Rees difference semigroup $(D \times G)/(0 \times G)$ (that is, the Cartesian product of D and G modulo the ideal $0 \times G$). To simplify the notation, we identify $\Lambda(S_{\sigma\tau})$ with $L(\mathbf{P}_\sigma, G)$ and $P(S_{\sigma\tau})$ with $R(\mathbf{P}_\tau, G)$; thus left translations are written as (α, ϕ), right translations as (β, ψ).

1. $\sigma = \tau = 2$. The semigroup S_{22} is a Brandt semigroup, that is, $S_{22} = \mathcal{M}^0(G; I, I; \Delta)$, where Δ is the identity matrix.

THEOREM 1.

$$\tilde{\Lambda}(S_{22}) = \{(\alpha, \phi) \in \Lambda(S_{22}) \mid \alpha \in W_{\omega 2}(I)\} \cup 0;$$

$$\tilde{P}(S_{22}) = \{(\beta, \psi) \in \mathbf{P}(S_{22}) \mid \beta \in W'_{\omega 2}(I)\} \cup 0.$$

Proof. Let $(\alpha, \phi) \in \tilde{\Lambda}(S_{22})$; then (α, ϕ) is linked to some $(\beta, \psi) \in \tilde{P}(S_{22})$. By Theorem 3 of TH, we have

$$i \in \mathbf{d}\alpha,\ p_{j(\alpha i)} \neq 0 \iff j \in \mathbf{d}\beta,\ p_{(j\beta)i} \neq 0,$$

which implies that

$$i \in \mathbf{d}\alpha,\ j = \alpha i \iff j \in \mathbf{d}\beta,\ j\beta = i,$$

and thus $\beta = \alpha^{-1}$, that is, $\alpha \in W_{\omega 2}(I)$. Conversely, if $(\alpha, \phi) \in \Lambda(S_{22})$ and $\alpha \in W_{\omega 2}(I)$, then, letting $j\psi = \phi(\alpha^{-1}j)$ for all $j \in \mathbf{r}\alpha$, in view of Theorem 3 of TH we easily see that (α, ϕ) and (α^{-1}, ψ) are linked. Hence $(\alpha, \phi) \in \tilde{\Lambda}(S_{22})$; the case of $\tilde{P}(S_{22})$ follows by symmetry.

THEOREM 2. *The following statements hold*:

(*a*) $S_{22} \cong W_{22}(I) \times_0 G$;

(*b*) *the set of idempotents of* $\tilde{\Lambda}(S_{22}) = \{(\iota_A, \phi_A) \mid \Box \neq A \subseteq I\} \cup 0$;

(*c*) $\tilde{\Lambda}(S_{22})$ *has* $2^{|I|}$ *idempotents*;

(*d*) $\tilde{\Lambda}(S_{22})$ *is an inverse semigroup.*

Proof. (*a*) It is easy to verify that the function $\mathbf{e}$ defined on S_{22} by

$$\mathbf{e}(g; i, j) = (\alpha, g), \qquad \mathbf{e}0 = 0,$$

where $\alpha j = i$, is an isomorphism of S_{22} onto $W_{22}(I) \times_0 G$.

(*b*) Recall that ι_A is the partial transformation on I defined by $\iota_A i = i$ for all $i \in A$, while ϕ_A maps A onto 1; this statement follows easily from Proposition 3 of TH (and can also be established directly).

(*c*) The mapping $(\iota_A, \phi_A) \to A$, $0 \to \Box$, is a one-to-one mapping of the set of idempotents of $\tilde{\Lambda}(S_{22})$ onto the set of all subsets of I.

(*d*) Since

$$(\iota_A, \phi_A)(\iota_B, \phi_B) = \begin{Bmatrix} (\iota_{A \cap B}, \phi_{A \cap B}) & \text{if} \quad A \cap B \neq \Box \\ 0 & \text{otherwise} \end{Bmatrix},$$

idempotents of $\tilde{\Lambda}(S_{22})$ commute. If $(\alpha, \phi) \in \tilde{\Lambda}(S_{22})$, then $\alpha \in W_{\omega 2}(I)$; let $\phi' i = (\phi \alpha^{-1} i)^{-1}$ for all $i \in \mathbf{r}\alpha$. One verifies easily that

$$(\alpha, \phi)(\alpha^{-1}, \phi')(\alpha, \phi) = (\alpha, \phi),$$

and $\tilde{\Lambda}(S_{22})$ is also regular.

The semigroup P satisfying conditions (1)–(5) in **[4]** is isomorphic to $T = \mathscr{M}^0(1; I, I; \Delta)$; it is stated there that $P \cong W_{22}(I)$ (in our notation) and that $W_{22}(I)$ is a densely embedded ideal of $W_{\omega 2}(I)$. By Gluskin's theorem and $\Omega(S_{22}) \cong \tilde{\Lambda}(S_{22})$, this case follows from above for $|G| = 1$, namely

COROLLARY. *For* $T = \mathscr{M}^0(1; I, I; \Delta)$, *we have*

$$T \cong W_{22}(I), \quad \tilde{\Lambda}(T) \cong W_{\omega 2}(I).$$

See also [**2**, 2.5.1].

2. $\sigma = 2$, τ **is an infinite cardinal.** We are dealing with $S_{2\tau} = \mathscr{M}^0(G; I, \mathbf{P}_\tau; P)$, where $P = (p_{Ai})$,

$$p_{Ai} = \begin{Bmatrix} 1 & \text{if} \quad i \in A \\ 0 & \text{otherwise} \end{Bmatrix}, \quad |A| < \tau.$$

Let $\mathscr{C}$ be the set of all $\beta \in W'(\mathbf{P}_\omega)$ which can be constructed as follows. Let A be a nonempty subset of I, and let

$$\mathbf{d}\beta = \{B \in \mathbf{P}_\tau \mid B \cap A \neq \Box\}; \tag{1}$$

with each $i \in A$ associate a non-empty set $B_i \in \mathbf{P}_\tau$ so that $B_i \cap B_j = \square$ if $i \neq j$. For $B \in \mathbf{d}\beta$, let

$$B\beta = \bigcup_{i \in B} B_i. \tag{2}$$

Then $\beta \in W'(\mathbf{P}_\tau)$; for $|B\beta| = |\bigcup_{i \in B} B_i| < \tau\tau = \tau$ since $B, B_i \in \mathbf{P}_\tau$.

THEOREM 3.

$$\tilde{\Lambda}(S_{2\tau}) = \{(\alpha, \phi) \in \Lambda(S_{2\tau}) \mid \alpha \in W_{\omega\tau}(I), \phi = \text{constant}\} \cup 0;$$

$$\tilde{P}(S_{2\tau}) = \{(\beta, \psi) \in P(S_{2\tau}) \mid \beta \in \mathscr{C}, \psi = \text{constant}\} \cup 0.$$

Proof. Let $((\alpha, \phi), (\beta, \psi)) \in \Omega(S_{2\tau})$; then, by Theorem 3 of TH, we have

$$i \in \mathbf{d}\alpha, \alpha i \in B \Leftrightarrow B \in \mathbf{d}\beta, i \in B\beta \qquad (i \in I, B \in \mathbf{P}_\tau), \tag{3}$$

$$\phi i = B\psi \quad \text{if} \quad i \in \mathbf{d}\alpha, \quad \alpha i \in B \qquad (i \in I, B \in \mathbf{P}_\tau). \tag{4}$$

Let $i, j \in \mathbf{d}\alpha$ and let $B = \{\alpha i, \alpha j\}$. Then by (3), $B \in \mathbf{d}\beta$ and $i, j \in B\beta$, since $|B| \leqq 2 < \tau$. By (4), $i, j \in B\beta$ implies that $\phi i = \phi j = B\psi$; hence $\phi = \text{constant}$. Further, let $A, B \in \mathbf{d}\beta$ and let $i \in A\beta, j \in B\beta$. Then (3) and (4) imply that $\phi i = B\psi$, $\phi j = C\psi$. Since $\phi = \text{constant}$, it follows that $B\psi = C\psi$ and thus $\psi = \text{constant}$.

Let $i \in \mathbf{d}\alpha$. If $j \in (\alpha i)\beta$, then, by (3), $\alpha j = \alpha i$ and hence $j \in \alpha^{-1}(\alpha i)$. Conversely, if $j \in \alpha^{-1}(\alpha i)$, then $\alpha j = \alpha i$, which again by (3) implies that $j \in (\alpha i)\beta$. Consequently

$$(\alpha i)\beta = \alpha^{-1}(\alpha i) \qquad (i \in \mathbf{d}\alpha); \tag{5}$$

since $\beta \in W'(\mathbf{P}_\tau)$, we must have $(\alpha i)\beta \in \mathbf{P}_\tau$, whence $\alpha^{-1}(\alpha i) \in \mathbf{P}_\tau$. But then $\alpha \in W_{\omega\tau}(I)$.

We show next that $\beta \in \mathscr{C}$. Let $A = \mathbf{r}\alpha$. If $i \in \mathbf{r}\alpha \cap B$, then $i = \alpha j \in B$ for some $j \in \mathbf{d}\alpha$, which by (3) implies that $B \in \mathbf{d}\beta$. Conversely, if $B \in \mathbf{d}\beta$, then, for any $i \in B\beta$, we obtain $i \in \mathbf{d}\alpha$, $\alpha i \in B$ by (3) so that $\mathbf{r}\alpha \cap B \neq \square$. Hence (1) holds, since $B \in W'(\mathbf{P}_\tau)$. For every $i \in \mathbf{r}\alpha$, let $B_i = i\beta$; again, since $\beta \in W'(\mathbf{P}_\tau)$, $B_i \in \mathbf{P}_\tau$ (note that $i \in \mathbf{r}\alpha$ implies that $i \in \mathbf{d}\beta$ by (1)). If $B_i \cap B_j \neq \square$, then there is $k \in i\beta \cap j\beta$, which by (3) implies that $k \in \mathbf{d}\alpha$, $\alpha k = i$, $\alpha k = j$; thus $i = j$ as required. To prove (2), we must show that, if $B \in \mathbf{d}\beta$, then

$$j \in B\beta \Leftrightarrow j \in i\beta \quad \text{for some} \quad i \in B.$$

Suppose that $B \in \mathbf{d}\beta$. If $j \in B\beta$, then (3) implies that $j \in \mathbf{d}\alpha$, $\alpha j \in B$. Let $i = \alpha j$; then by (5), $i\beta = (\alpha j)\beta = \alpha^{-1}(\alpha j)$, whence $j \in \alpha^{-1}(\alpha j) = i\beta$. Consequently $j \in i\beta$ and $i \in B$. Conversely, suppose that $j \in i\beta$, $i \in B$. By (3), we obtain $\alpha j = i \in B$, and again by (3), $j \in B\beta$. This establishes (2).

We have proved so far that $\tilde{\Lambda}(S_{2\tau})$ and $\tilde{P}(S_{2\tau})$ are contained in the respective sets stated in the theorem. We now turn to the converse.

Let $\alpha \in W_{\omega\tau}(I)$, $\alpha \neq 0$, $\phi : \mathbf{d}\alpha \to g \in G$; then $(\alpha, \phi) \in \Lambda(S_{2\tau})$. Let

$$\mathbf{d}\beta = \{B \in \mathbf{P}_\tau \mid \alpha i \in B \quad \text{for some} \quad i \in \mathbf{d}\alpha\}, \tag{6}$$

$$B\beta = \{i \in \mathbf{d}\alpha \mid \alpha i \in B\} \quad \text{if} \quad B \in \mathbf{d}\beta, \tag{7}$$

$$\psi : \mathbf{d}\beta \to g. \tag{8}$$

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Since $\alpha \in W_{\omega\tau}(I)$, for every $i \in \mathbf{d}\alpha$, $|\alpha^{-1}(\alpha i)| < \tau$; if $B \in \mathbf{d}\beta$, then $|B| < \tau$ by (6). Thus, by (7), $|B\beta| < \tau\tau = \tau$ since τ is an infinite cardinal, and $\beta \in W'(\mathbf{P}_\tau)$. Consequently $(\beta, \psi) \in \mathrm{P}(S_{2\tau})$; (3) follows immediately from (6) and (7), while (8) trivially implies (4). Thus (α, ϕ) and (β, ψ) are linked and therefore $(\alpha, \phi) \in \tilde{\Lambda}(S_{2\tau})$.

Finally let $\beta \in \mathscr{C}, \psi : \mathbf{d}\beta \to g \in G$; then $(\beta, \psi) \in \mathrm{P}(S_{2\tau})$. Let

$$\mathbf{d}\alpha = \bigcup_{j \in \mathbf{d}\beta} j\beta, \tag{9}$$

$$\alpha i = j \quad \text{if} \quad i \in j\beta, \tag{10}$$

$$\phi : \mathbf{d}\alpha \to g. \tag{11}$$

If $i \in j\beta \cap k\beta$, then, by (2), $j\beta = B_j$, $k\beta = B_k$, and thus $B_j \cap B_k \neq \square$ which implies that $j = k$, that is, α is single-valued and $(\alpha, \phi) \in \Lambda(S_{2\tau})$. Suppose that $B \in \mathbf{P}_\tau$. If $i \in \mathbf{d}\alpha$, $\alpha i \in B$, then let $j = \alpha i$; by (10), $i \in j\beta$. It follows from (1) that $A = \{j \in I \mid j \in \mathbf{d}\beta\}$. Hence $j \in \mathbf{d}\beta \cap B$ implies that $B \in \mathbf{d}\beta$. Further, $i \in j\beta = B_j$, $j \in B$ implies by (2) that $i \in B\beta$. Conversely, suppose that $B \in \mathbf{d}\beta$, $i \in B\beta$. Then, by (2), $i \in B_j = j\beta$ for some $j \in I$, whence, by (9) and (10), $i \in \mathbf{d}\alpha$, $\alpha i = j$. Moreover, $i \in j\beta \cap B\beta$ so that (2) implies that $j \in B$, that is, $\alpha i \in B$. Consequently (3) holds; (4) follows trivially from (11). Therefore (α, ϕ) and (β, ψ) are linked, which proves that $(\beta, \psi) \in \tilde{\mathrm{P}}(S_{2\tau})$.

THEOREM 4. *The following statements hold*:

(*a*) $S_{2\tau} \cong W_{2\tau}(I) \times_0 G$;

(*b*) $\tilde{\Lambda}(S_{2\tau}) \cong W_{\omega\tau}(I) \times_0 G$;

(*c*) $\tilde{\Lambda}(S_{2\tau})$ *is a regular semigroup.*

Proof. (*a*) It is easy to verify that the function $\mathbf{f}$ defined on $S_{2\tau}$ by

$$\mathbf{f}(g; i, A) = (\alpha, g), \qquad \mathbf{f}0 = 0,$$

where $\alpha : A \to i$, is an isomorphism of $S_{2\tau}$ onto $W_{2\tau}(I) \times_0 G$.

(*b*) By Theorem 3, if $(\alpha, \phi) \in \tilde{\Lambda}(S_{2\tau})$, then $\alpha \in W_{\omega\tau}(I)$, $\phi =$ constant. It then follows easily that the function $\mathbf{g}$ defined on $\tilde{\Lambda}(S_{2\tau})$ by

$$\mathbf{g}(\alpha, \phi) = (\alpha, g), \qquad \mathbf{g}0 = 0,$$

where $\phi : \mathbf{d}\alpha \to g$, is an isomorphism of $\tilde{\Lambda}(S_{2\tau})$ onto $W_{\omega\tau}(I) \times_0 G$.

(*c*) For a given $(\alpha, \phi) \in \Lambda(S_{2\tau})$, (α', ϕ') constructed in the proof of Proposition 2 of TH, has the properties that α' is one-to-one and, if $\phi =$ constant, so is ϕ'. Thus $\alpha' \in W_{\omega\tau}(I)$ and consequently, by Theorem 3, $(\alpha, \phi) \in \tilde{\Lambda}(S_{2\tau})$ implies that $(\alpha', \phi') \in \tilde{\Lambda}(S_{2\tau})$. Since $(\alpha, \phi)(\alpha', \phi')(\alpha, \phi) = (\alpha, \phi)$, $\tilde{\Lambda}(S_{2\tau})$ is regular.

For the case $|G| = 1$, $\tau > |I|$, cf. 4.6, 4.7, 4.7.1 of [2], and for properties of $W_{2\omega}(I)$ (in the notation there $W_0(\Omega)$), see 4.7.2 of [2]; more generally we get

COROLLARY. *For* $T_\tau = \mathscr{M}^0(1; I, \mathbf{P}_\tau; P)$, *we have*

$$T_\tau \cong W_{2\tau}(I); \quad \tilde{\Lambda}(T_\tau) \cong W_{\omega\tau}(I).$$

The case $\tau = \aleph_0$ was considered in [6]; the semigroup V_I^1 there is our $W_{2\tau}(I)$ (with $\Omega = I$); by the corollary, $W_{2\tau}(I) \cong T_\tau$. It is asserted there (9°) that V_I^1 is a densely embedded ideal of the semigroup V_I of all almost identical partial transformations on the set I (a partial transformation α is almost identical if $\alpha i \neq i$ for at most a finite number of i). According to the corollary, $T_\tau \cong W_{2\tau}(I)$ and, since $\Omega(T_\tau) \cong \tilde{\Lambda}(T_\tau)$, also $\Omega(T_\tau) \cong W_{\omega\tau}(I)$. Consequently $W_{2\tau}(I)$ is a densely embedded ideal of $W_{\omega\tau}(I)$ by Gluskin's theorem. However, if I is infinite, V_I is strictly contained in $W_{\omega\tau}(I)$ (recall that $W_{\omega\tau}(I)$ is the semigroup of all partial transformations α on I for which $|\alpha^{-1}(\alpha i)| < \tau = \aleph_0$, that is $\alpha^{-1}(\alpha i)$ is finite, for every $i \in \mathbf{d}\alpha$), so that $W_{2\tau}(I) = V_I^1$ cannot be a densely embedded ideal of V_I. (If I is finite, $V_I = W_{\omega\tau}(I)$.)

3. σ, τ are infinite cardinals. We consider here $S_{\sigma\tau} = \mathscr{M}^0(G; \mathbf{P}_\sigma, \mathbf{P}_\tau; P)$, where $P = (p_{BA})$ and

$$p_{BA} = \begin{cases} 1 & \text{if } A \cap B \neq \Box \\ 0 & \text{otherwise} \end{cases}, \quad |A| < \sigma, \quad |B| < \tau.$$

Let $\mathscr{D}$ be the set of all $\beta \in W'(\mathbf{P}_\omega)$ which can be constructed as follows. Let $\mathscr{E}$ be a non-empty subset of $\mathbf{P}_\sigma$ such that, for every $i \in I$, $|\{E \in \mathscr{E} \mid i \in E\}| < \tau$. Let

$$\mathbf{d}\beta = \{B \in \mathbf{P}_\tau \mid B \cap E \neq \Box \quad \text{for some} \quad E \in \mathscr{E}\}; \tag{12}$$

with each $E \in \mathscr{E}$ associate an element $k_E \in I$ such that $k_E \neq k_{E'}$ if $E \neq E'$. For $B \in \mathbf{d}\beta$, let

$$B\beta = \{k_E \mid B \cap E \neq \Box\}. \tag{13}$$

If $i \in B$ and $i \in \bigcap E_j$, where $E_j \in \mathscr{E}$ and $j \in J$, then by hypothesis $|J| < \tau$; since also $|B| < \tau$, we obtain $|B\beta| < \tau\tau = \tau$, τ being an infinite cardinal. Thus $B\beta \in \mathbf{P}_\tau$ which proves that $\beta \in W'(\mathbf{P}_\tau)$. It is of interest to compare the construction of the family $\mathscr{D}$ with that of $\mathscr{C}$ in the preceding section, and the proof of the next theorem with that of Theorem 3. We consider in the next theorem only $\tilde{\mathrm{P}}(S_{\sigma\tau})$; the case of $\tilde{\Lambda}(S_{\sigma\tau})$ is symmetric.

THEOREM 5. $\tilde{\mathrm{P}}(S_{\sigma\tau}) = \{(\beta, \psi) \in \mathrm{P}(S_{\sigma\tau}) \mid \beta \in \mathscr{D},\ \psi = \text{constant}\} \cup 0$.

Proof. Let $((\alpha, \phi), (\beta, \psi)) \in \Omega(S_{\sigma\tau})$; then, by Theorem 3 of TH, we have

$$A \in \mathbf{d}\alpha,\ \alpha A \cap B \neq \Box \Leftrightarrow B \in \mathbf{d}\beta,\ A \cap B\beta \neq \Box \qquad (A \in \mathbf{P}_\sigma,\ B \in \mathbf{P}_\tau), \tag{14}$$

$$\phi A = B\psi \quad \text{if} \quad A \in \mathbf{d}\alpha,\ \alpha A \cap B \neq \Box \qquad (A \in \mathbf{P}_\sigma,\ B \in \mathbf{P}_\tau). \tag{15}$$

Let $B, B' \in \mathbf{d}\beta$ and let $A \in \mathbf{P}_\sigma$ be any set intersecting both $B\beta$ and $B'\beta$. By (14) and (15), $B\psi = B'\psi = \phi A$, which implies that $\psi = \text{constant}$.

We show next that $\beta \in \mathscr{D}$. Let $\mathscr{E} = \{\alpha i \mid i \in \mathbf{d}\alpha\}$ (note that $\mathscr{E}$ is not necessarily the range of α). Since $(\alpha, \phi) \in \Lambda(S_{\sigma\tau})$, it follows that $|\alpha i| < \sigma$ for every $i \in \mathbf{d}\alpha$, so that $\mathscr{E}$ is a non-empty subset of $\mathbf{P}_\sigma$. Further, if $i \in \bigcap \alpha j$, where j ranges over a subset J of $\mathbf{d}\alpha$, then, for every $j \in J$, $i \in \mathbf{d}\beta$ and $j \in i\beta$ by (14). Since $i\beta \in \mathbf{P}_\tau$, we obtain $|J| < \tau$, that is, $|\{E \in \mathscr{E} \mid i \in E\}| < \tau$. If $B \cap \alpha i \neq \Box$ for some $i \in \mathbf{d}\alpha$, then by (14), $B \in \mathbf{d}\beta$. Conversely, if $B \in \mathbf{d}\beta$, let $i \in B\beta$ be arbitrary. Again by (14), we get $i \in \mathbf{d}\alpha$ and $B \cap \alpha i \neq \Box$. Hence (12) holds, since $\beta \in W'(\mathbf{P}_\tau)$. For each $E \in \mathscr{E}$, let $k_E = i$ if $E = \alpha i$. If $\alpha i \neq \alpha j$, then $i \neq j$ and thus $k_E \neq k_{E'}$ if $E \neq E'$. To prove (13), we must show that, if $B \in \mathbf{d}\beta$, then

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$$j \in B\beta \Leftrightarrow j \in \mathbf{d}\alpha, B \cap \alpha j \neq \Box;$$

but this follows directly from (14). Therefore $\beta \in \mathscr{D}$.

Conversely, suppose that $\beta \in \mathscr{D}$, $\psi : \mathbf{d}\beta \to g \in G$; then $(\beta, \psi) \in \mathrm{P}(S_{\sigma\tau})$. Let

$$\mathbf{d}\alpha = \{A \in \mathbf{P}_\sigma \mid k_E \in A \quad \text{for some} \quad E \in \mathscr{E}\}, \tag{16}$$

$$\alpha A = \bigcup_{k_E \in A} E \quad \text{if} \quad A \in \mathbf{d}\alpha, \tag{17}$$

$$\phi : \mathbf{d}\alpha \to g. \tag{18}$$

Since $\{k_E \mid k_E \in A\} \subseteq A$ and $|E| < \sigma$, for every $A \in \mathbf{P}_\sigma$ we have, by (17), $|\alpha A| < \sigma\sigma = \sigma$, so that $\alpha A \in \mathbf{P}_\sigma$; consequently $\alpha \in W(\mathbf{P}_\sigma)$ and $(\alpha, \phi) \in \Lambda(S_{\sigma\tau})$. We obtain, for $A \in \mathbf{P}_\sigma$, $B \in \mathbf{P}_\tau$,

$$A \in \mathbf{d}\alpha, \alpha A \cap B \neq \Box$$

$$\Leftrightarrow A \in \mathbf{d}\alpha, E \cap B \neq \Box \quad \text{for some} \quad E \in \mathscr{E} \quad \text{such that} \quad k_E \in A$$

$$\Leftrightarrow B \in \mathbf{d}\beta, A \cap B\beta \neq \Box$$

by (16), (17), (12), (13), which proves (14); (15) follows trivially from (18). Therefore, by Theorem 3 of TH, (α, ϕ) and (β, ψ) are linked, which proves that $(\beta, \psi) \in \tilde{\mathrm{P}}(S_{\sigma\tau})$.

Let

$$\mathscr{R}_{\sigma\tau}(I) = \{A \times B \subseteq I \times I \mid \ |A| < \sigma, \ |B| < \tau\},$$

$$\mathscr{B}_{\sigma\tau}(I) = \{C \subseteq I \times I \mid \ |C \cap (I \times i)| < \sigma, \ |C \cap (i \times I)| < \tau \quad \text{for every} \quad i \in I\}$$

under the usual multiplication of binary relations, that is,

$$i\, \gamma\delta\, j \Leftrightarrow i\, \gamma\, k, k\, \delta\, j \quad \text{for some} \quad k \in I$$

(we write $i\,\gamma\, k$ rather than $(i,k) \in \gamma$). It is easy to see that both $\mathscr{R}_{\sigma\tau}(I)$ and $\mathscr{B}_{\sigma\tau}(I)$ are semigroups, in fact $\mathscr{R}_{\sigma\tau}(I)$ is an ideal of $\mathscr{B}_{\sigma\tau}(I)$ (the zero of $\mathscr{B}_{\sigma\tau}(I)$ is the empty set denoted by 0) (see [7, 2.3]).

THEOREM 6. *The following statements hold*:

(*a*) $S_{\sigma\tau} \cong \mathscr{R}_{\sigma\tau}(I) \times_0 G$;

(*b*) $\tilde{\mathrm{P}}(S_{\sigma\tau}) \cong \mathscr{B}_{\sigma\tau}(I) \times_0 G$.

Proof. (*a*) It is easy to verify that the function $\mathbf{h}$ defined on $S_{\sigma\tau}$ by

$$\mathbf{h}(g; A, B) = (A \times B, g), \quad \mathbf{h}0 = 0,$$

is an isomorphism of $S_{\sigma\tau}$ onto $\mathscr{R}_{\sigma\tau}(I) \times_0 G$.

(*b*) Let $(\beta, \psi) \in \tilde{\mathrm{P}}(S_{\sigma\tau})$; by Theorem 5, $\psi = \text{constant}$. We define the function $\mathbf{i}$ on $\tilde{\mathrm{P}}(S_{\sigma\tau})$ by

$$(\beta, \psi)\mathbf{i} = (\bar{\beta}, g), \quad 0\mathbf{i} = 0, \tag{19}$$

where $\bar{\beta}$ is the binary relation on I defined by

$$i\, \bar{\beta}\, j \Leftrightarrow i \in \mathbf{d}\beta, \ j \in i\beta \qquad (i, j \in I), \tag{20}$$

and $\psi: \mathbf{d}\beta \to g$. We now show that $\mathbf{i}$ is the required isomorphism.

Let $(\beta,\psi)\in\tilde{\mathrm{P}}(S_{\sigma\tau})$; by Theorem 8 of TH, there is a unique left translation (α,ϕ) of $S_{\sigma\tau}$ linked to (β,ψ). By (14) and (20), we have

$$i\,\bar{\beta}\,j \Leftrightarrow j\in\mathbf{d}\alpha,\ i\in\alpha j \qquad (i,j\in I). \tag{21}$$

We obtain by (21)

$$\bar{\beta}\langle i\rangle = \{j\in I \mid j\,\bar{\beta}\,i\} = \begin{Bmatrix} \alpha i & \text{if} \quad i\in\mathbf{d}\alpha \\ \Box & \text{otherwise} \end{Bmatrix},$$

so that $|\bar{\beta}\langle i\rangle| < \sigma$ since $\alpha\in W(\mathbf{P}_\sigma)$; further by (20)

$$\langle i\rangle\bar{\beta} = \{j\in I \mid i\,\bar{\beta}\,j\} = \begin{Bmatrix} i\beta & \text{if} \quad i\in\mathbf{d}\beta \\ \Box & \text{otherwise} \end{Bmatrix},$$

so that $|\langle i\rangle\bar{\beta}| < \tau$ since $\beta\in W'(\mathbf{P}_\tau)$. Consequently $\bar{\beta}\in\mathscr{B}_{\sigma\tau}(I)$ which shows that $\mathbf{i}$ maps $\mathbf{P}(S_{\sigma\tau})$ into $\mathscr{B}_{\sigma\tau}(I)\times_0 G$.

Let $((\alpha,\phi),(\beta,\psi)),\ ((\alpha',\phi'),(\beta',\psi'))\in\Omega(S_{\sigma\tau})$, where $\psi:\mathbf{d}\beta\to g$, $\psi':\mathbf{d}\beta'\to g'$. Then

$$(\beta,\psi)(\beta',\psi') = \begin{Bmatrix} (\beta\beta',\psi'') & \text{if} \quad \beta\beta'\neq 0 \\ 0 & \text{otherwise} \end{Bmatrix}, \tag{22}$$

where

$$B\psi'' = (B\psi)(B\beta\psi') = gg' \qquad (B\in\mathbf{d}(\beta\beta')). \tag{23}$$

Furthermore, we obtain

$$(\bar{\beta},g)(\bar{\beta}',g') = \begin{Bmatrix} (\bar{\beta}\bar{\beta}',gg') & \text{if} \quad \bar{\beta}\bar{\beta}'\neq 0 \\ 0 & \text{otherwise} \end{Bmatrix}, \tag{24}$$

and, using (20), (21), and (14),

$$\begin{aligned} i\,\bar{\beta}\bar{\beta}'\,j &\Leftrightarrow i\,\bar{\beta}\,k,\ k\,\bar{\beta}'\,j \quad \text{for some} \quad k\in I \\ &\Leftrightarrow i\in\mathbf{d}\beta,\ k\in i\beta,\ j\in\mathbf{d}\alpha',\ k\in\alpha' j \quad \text{for some} \quad k\in I \\ &\Leftrightarrow i\in\mathbf{d}\beta,\ j\in\mathbf{d}\alpha',\ i\beta\cap\alpha' j\neq\Box \\ &\Leftrightarrow j\in\mathbf{d}\alpha',\ \alpha' j\in\mathbf{d}\alpha,\ i\in\alpha(\alpha' j) \\ &\Leftrightarrow j\in\mathbf{d}(\alpha\alpha'),\ i\in\alpha\alpha' j \\ &\Leftrightarrow i\,\overline{\beta\beta'}\,j, \end{aligned}$$

which together with (22), (23), and (24) proves that $\mathbf{i}$ is a homomorphism.

If $(\bar{\beta},g) = (\bar{\beta}',g')$, then $\bar{\beta} = \bar{\beta}'$ and $g = g'$, whence

$$i\in\mathbf{d}\beta,\ j\in i\beta \Leftrightarrow i\,\bar{\beta}\,j \Leftrightarrow i\,\bar{\beta}'\,j \Leftrightarrow i\in\mathbf{d}\beta',\ j\in i\beta'.$$

It follows that, if $i\in\mathbf{d}\beta$, then there is $j\in i\beta$ and thus $i\in\mathbf{d}\beta'$, that is, $\mathbf{d}\beta\subseteq\mathbf{d}\beta'$; by symmetry we conclude that $\mathbf{d}\beta = \mathbf{d}\beta'$. Further, if $i\in\mathbf{d}\beta$, then $j\in i\beta\Leftrightarrow j\in i\beta'$, so that $i\beta = i\beta'$. Consequently $\beta = \beta'$ and $\mathbf{i}$ is one-to-one.

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It remains to show that $\mathbf{i}$ is onto. Let $(\gamma, g) \in \mathscr{B}_{\sigma\tau}(I) \times_0 G$ and let

$$\mathbf{d}\alpha = \{A \in \mathbf{P}_\sigma \mid (I \times A) \cap \gamma \neq \Box\},$$
$$\alpha A = \{i \in I \mid i\,\gamma\,j \text{ for some } j \in A\} \quad \text{if} \quad A \in \mathbf{d}\alpha,$$
$$\phi : A \to g,$$
$$\mathbf{d}\beta = \{B \in \mathbf{P}_\tau \mid (B \times I) \cap \gamma \neq \Box\},$$
$$B\beta = \{j \in I \mid i\,\gamma\,j \text{ for some } i \in B\} \quad \text{if} \quad B \in \mathbf{d}\beta,$$
$$\psi : B \to g.$$

For $A \in \mathbf{d}\alpha$, we have $|\alpha A| < \sigma\sigma = \sigma$, since $\gamma \in \mathscr{B}_{\sigma\tau}(I)$ and $A \in \mathbf{P}_\sigma$. Consequently $\alpha \in W(\mathbf{P}_\sigma)$ and thus $(\alpha, \phi) \in \Lambda(S_{\sigma\tau})$; similarly $(\beta, \psi) \in \mathrm{P}(S_{\sigma\tau})$. For $A \in \mathbf{P}_\sigma$, $B \in \mathbf{P}_\tau$, we obtain

$$\begin{aligned} A \in \mathbf{d}\alpha,\ \alpha A \cap B \neq \Box &\Leftrightarrow A \in \mathbf{d}\alpha,\ i \in \alpha A \cap B \text{ for some } i \in I \\ &\Leftrightarrow i\,\gamma\,j,\ i \in B \text{ for some } i \in I,\ j \in A \\ &\Leftrightarrow B \in \mathbf{d}\beta,\ j \in B\beta \text{ for some } j \in A \\ &\Leftrightarrow B \in \mathbf{d}\beta,\ A \cap B\beta \neq \Box, \end{aligned}$$

which proves (14); since (15) is trivially satisfied, we conclude that (α, ϕ) and (β, ψ) are linked. Thus $(\beta, \psi) \in \tilde{\mathrm{P}}(S_{\sigma\tau})$. Clearly $i \in \mathbf{d}\beta, j \in i\beta \Leftrightarrow i\,\gamma\,j$, which by (20) implies that $\bar{\beta} = \gamma$. But then $(\beta, \psi)\mathbf{i} = (\gamma, g)$, which proves that $\mathbf{i}$ is onto. Therefore $\mathbf{i}$ is an isomorphism of $\tilde{\mathrm{P}}(S_{\sigma\tau})$ onto $\mathscr{B}_{\sigma\tau}(I) \times_0 G$.

The case $|G| = 1$, $\sigma, \tau > |I|$ was considered (via Gluskin's theorem) in [7] (elements of $\mathscr{R}_{\sigma\tau}$ with $\sigma, \tau > |I|$, are called there "rectangular binary relations"); more generally we obtain

COROLLARY. *For $T_{\sigma\tau} = \mathscr{M}^0(1; \mathbf{P}_\sigma, \mathbf{P}_\tau; P)$, we have*

$$T_{\sigma\tau} \cong \mathscr{R}_{\sigma\tau}(I); \quad \tilde{\mathrm{P}}(T_{\sigma\tau}) \cong \mathscr{B}_{\sigma\tau}(I).$$

It follows from 1.8 and 3.2 of [8], that the semigroup $\mathscr{B}$ of all binary relations on the set I need not be regular. In view of Theorem 6, it then follows that the translational hull of a completely 0-simple semigroup (even of a very special kind) need not be regular; however, we have seen in Theorem 2 that $\Omega(S_{22})$ (that is, the translational hull of an inverse completely 0-simple semigroup) is an inverse semigroup, and in Theorem 4 that $\Omega(S_{2\tau})$ is a regular semigroup. It follows from Proposition 1 of TH that $\Lambda(S_{\sigma\tau})$, $\mathrm{P}(S_{\sigma\tau})$ are regular semigroups for any σ, τ.

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