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ON DENSITY OF GENERALIZED POLYNOMIALS

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ABSTRACT. We consider the density in $C[a,b]$ of generalized polynomials of the form $\sum_{j=1}^{n} c_j K(x, t_j)$. The main point of this note is that total positivity of $K(x,t)$ has little relationship to density: There is a symmetric, analytic, totally positive (in fact ETP (∞)) kernel K for which these generalized polynomials are not dense.

1. **Introduction and Statement of Results.** Let K: $[a,b] \times [c,d] \longrightarrow \mathbf{R}$ be continuous, and let $\mathcal{P}$ denote the set of all generalized polynomials of the form

$$\sum_{j=1}^{n} c_j K(x,\ t_j), \tag{1.1}$$

where $n \geqq 1$, $\{t_j\}_{j=1}^n \subset [c,d], \{c_j\}_{j=1}^n \subset \mathbf{R}$. The density of $\mathcal{P}$ in $C[a,b]$ (the functions continuous on $[a,b]$ with uniform norm) has been studied for many special kernels, for example, $K(x,t) := e^{xt}$, $K(x,t) := 1/(1-xt)$, $K(x,t) := x^t$. On suitable intervals, these all yield $\mathcal{P}$ that is dense in $C[a,b]$ [1]. When K has the form $K(x,t) = h(x-t)$, a classical theorem of Wiener [1] provides a complete answer to this question.

For totally positive K, the polynomials $\mathcal{P}$ often appear in approximation theory, and it seems of interest to study their density properties. As far as the authors could determine, this has not been considered in detail, though [4] contains some results of this type and perhaps it is implicitly investigated in numerical solution of certain types of integral equations [5]. Convergence of interpolatory polynomials of the form (1.1) was studied in [2], and similar questions for generalized rational functions in [3].

The main point of this note is that total positivity has little to do with density. First, let us recall:

DEFINITION 1.1. K is totally positive if for all $n \geqq 1, a \leqq s_1 < s_2 < \cdots < s_n \leqq b$; $c \leqq t_1 < t_3 < \cdots < t_n \leqq d$, we have

$$\det\,(K(s_i,\ t_j))_{i,j=1}^n > 0. \tag{1.2}$$

Suppose in addition that K has partial derivatives of all orders in $[a,b] \times [c,d]$. We say that K is ETP(∞) (extended totally positive of all orders) if for all $n \geqq 1$; $a \leqq s_1 \leqq$

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$s_2 \leqq \cdots \leqq s_n \leqq b; c \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq d$, we have

$$\det \left[\frac{\partial^{l_i+m_j}}{\partial s^{l_i} \partial t^{m_j}} K(s, t) \Big|_{s=s_i, t=t_j} \right]_{i,j=1}^{n} > 0. \tag{1.3}$$

Here for $1 \leqq i, j \leqq n$,

$$\begin{aligned} l_i &:= i - \min\{k : s_k = s_i\}, \\ m_j &:= j - \min\{k : t_k = t_j\}. \end{aligned} \tag{1.4}$$

The conditions (1.3) and (1.4) express the requirement that the determinant (1.2) remains positive, when some s_i or t_i coalesce, provided we replace the relevant rows or columns by suitable order partial derivatives.

Recall that $\mathcal{P}$ is the set of all generalized polynomials (1.1). Our main result is:

THEOREM 1.2. *Let $0 < a < b < \infty$ and let $\{\lambda_j\}_{j=0}^{\infty} \subset (0, \infty)$ be all distinct. Let $\{c_j\}_{j=0} \subset (0, \infty)$ and let*

$$K(x, t) := \sum_{j=0}^{\infty} c_j (xt)^{\lambda_j} \tag{1.5}$$

be convergent for x, t in an open interval containing $[a, b]$. Then K is ETP(∞), and the following are equivalent:

(a) $\mathcal{P}$ is dense in $C[a, b]$.
(b) We have

$$\sum_{j=0}^{\infty} \frac{1}{1 + \lambda_j} = \infty. \tag{1.6}$$

REMARKS. (i) Note that K is analytic in x and t, and also symmetric, that is $K(x, t) = K(t, x)$. In particular, when (1.6) is not satisfied, we obtain an ETP(∞) kernel for which $\mathcal{P}$ is not dense. (ii) We can obviously replace $C[a, b]$ by $L_p[a, b]$, and $p \geqq 1$.

Theorem 1.2 is a consequence of a general necessary and sufficient condition involving the inner product for symmetric K,

$$(u, v) := \int_a^b \int_a^b u(t) K(x, t) v(x)\, dx\, dt, \quad u, v \in L_1[a, b] : \tag{1.7}$$

THEOREM 1.3. *Let $K : [a, b] \times [a, b] \longrightarrow \mathbf{R}$ be continuous, symmetric and satisfy*

$$(v, v) \geqq 0, \ v \in L_1[a, b]. \tag{1.8}$$

Then the following are equivalent: (a) $\mathcal{P}$ is dense in $C[a, b]$. (b) If $v \in L_1[a, b]$ satisfies $(v, v) = 0$, then $v = 0$ a.e. in $[a, b]$.

REMARKS. (i) We are not sure that Theorem 1.3 is new. (ii) One can replace $C[a, b]$ and $L_1[a, b]$ respectively by $L_p[a, b]$ and $L_q[a, b]$ for any $1 < p, q < \infty$ with $p^{-1} +$

$q^{-1} = 1$. (iii) Even without the non-negative definiteness of K in (1.8), (b) is sufficient to imply density of $\mathcal{P}$. (iv) An alternative formulation of Theorem 1.3 involves the integral equation

$$\int_a^b K(x,\ t)h(x)dx = 0,\ t \in [a,b],$$

having only the trivial solution $h = 0$ a.e. (v) Any symmetric totally positive kernel $K(x,\ t)$ can be seen to be non-negative definite in the sense (1.8).

There are two other classes whose density is naturally equivalent to that of $\mathcal{P}$ (compare [4, Theorem 10]). Let K: $[a,b] \times [c,d] \longrightarrow \mathbf{R}$ be continuous, and let $\mathcal{Q}$ denote the class of all functions of the form

$$g(x) := \int_c^d K(x,\ t)h(t)\,dt,\ x \in [a,b], \tag{1.9}$$

$h \in C[c,d]$. Furthermore, let $\mathcal{R}$ denote the class of all functions of the form

$$g(x) := \int_c^d K(x,\ t)\,d\mu(t),\ x \in [a,b], \tag{1.10}$$

where μ is a (signed) Borel measure on $[c,d]$ with

$$\int_c^d |d\mu|(t) < \infty. \tag{1.11}$$

THEOREM 1.4. *Let K: $[a,b] \times [c,d] \longrightarrow \mathbf{R}$ be continuous. The following are equivalent: (a) $\mathcal{P}$ is dense in $C[a,b]$. (b) $\mathcal{Q}$ is dense in $C[a,b]$. (c) $\mathcal{R}$ is dense in $C[a,b]$.*

An easy corollary of Theorem 1.4 is:

COROLLARY 1.5. *Let K: $[a,b] \times [c,d] \longrightarrow \mathbf{R}$ be continuous. Let $1 \leqq p \leqq \infty$, and let $\mathcal{T}_p$ denote the class of all functions of the form* (1.9), *where $h \in L_p[c,d]$. The following are equivalent: (a) $\mathcal{P}$ is dense in $C[a,b]$. (b) $\mathcal{T}_p$ is dense in $C[a,b]$.*

Finally, we note that (cf. [4]) when $K(x,t)$ is analytic in t, we can restrict t to lie in any infinite subset Δ of $[c,d]$: Let $\mathcal{P}(\Delta)$ denote the class of all polynomials of the form (1.1), with $\{t_j\}_{j=1}^n \subset \Delta$.

THEOREM 1.6. *Let K: $[a,b] \times [c,d] \longrightarrow \mathbf{R}$ be continuous and $K(x,t)$ be analytic in $t \in [c,d]$ for each fixed $x \in [a,b]$, while $\partial/\partial t\ K(x,t)$ is continuous for $x \in [a,b]$ and t in an open set containing $[c,d]$. Let Δ be an infinite subset of $[c,d]$. The following are equivalent: (a) $\mathcal{P}$ is dense in $C[a,b]$. (b) $\mathcal{P}(\Delta)$ is dense in $C[a,b]$.*

We prove Theorems 1.2 to 1.6 in Section 2.

2. **Proofs.** For $f \in C[a,b]$ and $\mathcal{T} \subset C[a,b]$, we define

$$\text{dist}\,(f,\ \mathcal{T}) := \inf_{P \in \mathcal{T}} \|f - P\|_{L_\infty[a,b]}. \tag{2.1}$$

LEMMA 2.1. *Let K: $[a,b] \times [a,b] \longrightarrow \mathbf{R}$ be continuous, symmetric, and $\mathcal{P}$ be the set of all generalized polynomials* (1.1)*, and let* $(\cdot,\cdot)$ *denote the inner product* (1.7). *For $f \in C[a,b]$,*

$$\begin{aligned} \text{dist}\,(f,\ \mathcal{P}) = \sup \Big\{ \int_a^b (fq)(t)dt : \int_a^b |q(t)|\,dt = 1 \text{ and} \\ (q,v) = 0 \text{ for all } v \in L_1[a,b] \Big\}. \end{aligned} \tag{2.2}$$

PROOF. If $\mathcal{S}$ is any dense linear subspace of $\mathcal{P}$ or $\bar{\mathcal{P}}$ (the closure of $\mathcal{P}$), it is clear that

$$\text{dist}\,(f,\ \mathcal{P}) = \text{dist}\,(f,\ \mathcal{S}) = \text{dist}\,(f,\ \bar{\mathcal{P}}). \tag{2.3}$$

Let $\mathcal{S}(= \mathcal{T}_1)$ be the class of functions g of the form

$$g(x) := \int_a^b K(x,t)v(t)dt,\ x \in [a,b], \tag{2.4}$$

some $v \in L_1[a,b]$. In view of the continuity of K, it is easy to see that $\mathcal{S} \subset \bar{\mathcal{P}}$. Furthermore, it is easy to see that any generalized polynomial $P \in \mathcal{P}$ can be approximated uniformly on $[a,b]$ by elements of $\mathcal{S}$. Hence (2.3) holds. Next, by the standard duality principle [1]

$$\begin{aligned} \text{dist}\,(f,\ \mathcal{S}) = \sup \Big\{ \int_a^b (fq)(t)dt : \int_a^b |q(t)|\,dt = 1, \text{ and} \\ \int_a^b (qg)(t)dt = 0 \text{ for all } g \in \mathcal{S} \Big\}. \end{aligned}$$

Since each $g \in \mathcal{S}$ has the form (2.4), we can write

$$\int_a^b (qg)(t)dt = (q,v).$$

Hence (2.2) follows. ∎

PROOF OF THEOREM 1.3. (a) $\Rightarrow$ (b). Suppose $q \in L_1[a,b]$ satisfies $(q,q) = 0$. We shall assume that

$$\eta := \int_a^b |q(t)|\,dt > 0, \tag{2.5}$$

and derive a contradiction to (a). We may normalize q so that $\eta = 1$. Now by symmetry and non-negativity of $(\cdot,\cdot)$, for any $v \in L_1[a,b]$ and $\lambda \in \mathbf{R}$, $0 \leqq (v + \lambda q,\ v + \lambda q) = (v,v) + 2\lambda(q,v)$. Dividing by $\lambda \neq 0$ and then letting $\lambda \longrightarrow \infty$ or $-\infty$, yields

$$(q,\ v) = 0 \text{ for all } v \in L_1[a,b]. \tag{2.6}$$

By Lemma 2.1, we have for all $f \in C[a,b]$,

$$\operatorname{dist}(f, \mathcal{P}) \geqq \int_a^b (fq)(t)\, dt.$$

But in view of (2.5), we can choose $f \in C[a,b]$ for which this last integral is positive. Then $f \notin \bar{\mathcal{P}}$, and we have a contradiction to (a). So necessarily $\eta = 0$, and $q = 0$ a.e.
(b) $\Rightarrow$ (a). In view of Lemma 2.1, it suffices to show that if $q \in L_1[a,b]$ and $(q,v) = 0$ for all $v \in L_1[a,b]$, then $q = 0$ a.e. But for such q, we have $(q,q) = 0$ and so $q = 0$ a.e., as required. ■

PROOF OF THEOREM 1.2. We can write for $x, t \in [a,b]$,

$$K(x, t) = \int_0^\infty e^{s\alpha} e^{s\beta}\, d\sigma(s),$$

where $\alpha := \log x$; $\beta := \log t \in [\log a, \log b]$, and $d\sigma(s)$ places a jump of c_j at $s = \lambda_j$, $j \geqq 0$. It follows that $K(x,t)$ is ETP(∞) in $[a,b]$ (see [6, p. 336]). Next, $K(x,t)$ is analytic for $x, t \in [a,b]$, symmetric and if $v \in L_1[a,b]$, then

$$(v, v) = \sum_{j=0}^{\infty} c_j \left[\int_a^b v(t) t^{\lambda_j} dt \right]^2 \geqq 0,$$

so K is non-negative definite. If $(v,v) = 0$, we deduce that (since $c_j > 0$),

$$\int_a^b v(t) t^{\lambda_j} dt = 0, \quad j \geqq 0.$$

Then Müntz' Theorem [1] shows that this implies $v = 0$ a.e. iff (1.6) holds. ■

PROOF OF THEOREM 1.4. The equivalence of the density of $\mathcal{P}$ and $\mathcal{Q}$ is easy, and was essentially proved in Lemma 2.1. Noting that K is continuous in $[a,b] \times [c,d]$, and that $\mathcal{Q} \subset \mathcal{R}$, it is easily seen that the density of $\mathcal{Q}$ and $\mathcal{R}$ are equivalent: By a "discretisation" argument, each g of the form (1.10) can be approximated uniformly on $[a,b]$ by generalized polynomials of the form (1.1) and hence by elements of $\mathcal{Q}$. ■

PROOF OF COROLLARY 1.5. This follows since $\mathcal{Q} \subset \mathcal{T}_p \subset \mathcal{R}$ for any $1 \leqq p \leqq \infty$. ■

PROOF OF THEOREM 1.6. (a) $\Rightarrow$ (b). Suppose that μ is a signed Borel measure on $[c,d]$ having finite total mass (that is, satisfying (1.11)). In view of duality, it suffices to show that if

$$\int_c^d P(x)\, d\mu(x) = 0$$

for all $P \in \mathcal{P}(\Delta)$, then $\mu \equiv 0$. Let

$$F(t) := \int_c^d K(x, t)\, d\mu(x),$$

and note that F is analytic for $t \in [c,d]$ by our assumptions on K. Since $F(t) = 0, t \in \Delta$, we obtain from the analyticity of F,

$$F(t) = 0, \ t \in [c,d],$$

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and hence

$$\int_c^d P(x)d\mu(x) = 0,$$

for all $P \in \mathcal{P}$. The density of $\mathcal{P}$ implies $\mu \equiv 0$. (b) $\Rightarrow$ (a). Since $\mathcal{P}(\Delta) \subset \mathcal{P}$, this is immediate. ∎

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