A PERMANENTAL INEQUALITY—THE CASE OF EQUALITY

MARVIN MARCUS AND HENRYK MINC

In **(3)** we proved that if A is a complex n-square normal matrix with characteristic roots $\alpha_1, \ldots, \alpha_n$, then

$$|\operatorname{per}(A)| \leqslant \frac{1}{n} \sum_{j=1}^{n} |\alpha_j|^n. \tag{1}$$

If A is positive semi-definite hermitian, the inequality (1) becomes

$$\operatorname{per}(A) \leqslant (1/n)\operatorname{tr}(A^n). \tag{2}$$

This inequality partially answers the problem of determining the maximum permanent of a positive semi-definite hermitian matrix with prescribed characteristic roots **(6)**. In **(1)**, Brualdi and Newman proved that (2) also holds when A is an n-square circulant with non-negative entries. In a recent conversation Dr. Newman raised the question of determining the cases of equality in (1). In the present note we answer this question. We are indebted to Dr. Newman for his helpful conversations.

Our main result is

THEOREM. *Let A be a normal matrix with characteristic roots $\alpha_1, \ldots, \alpha_n$. Then*

$$|\operatorname{per}(A)| \leqslant \frac{1}{n} \sum_{j=1}^{n} |\alpha_j|^n. \tag{1}$$

For $n > 3$, equality in (1) can occur if and only if A is a scalar multiple of a unitary matrix with exactly one non-zero entry in each row and column. For $n = 2$, equality in (1) holds if and only if either A is a scalar multiple of a unitary matrix with exactly one non-zero entry in each row and column or

$$A = U \operatorname{diag}(\alpha_1, \alpha_2) U^*$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \eta & \theta\eta \\ \zeta & -\theta\zeta \end{bmatrix}, \qquad |\eta| = |\zeta| = |\theta| = 1$$

and $\arg(\alpha_1 \bar{\alpha}_2) = k\pi$ for some integer k.

We use the following notation introduced in **(5)**. If $1 \leqslant k \leqslant n$, then $\Gamma_{k,n}$ denotes the totality of sequences $\gamma = (\gamma_1, \ldots, \gamma_k)$ of k integers chosen from

Received November 1, 1965. This research was supported by the Air Force Office of Scientific Research under Grants No. AFOSR-698-65 and AF-AFOSR-432-63.

$1, \ldots, n$; $G_{k,n}$ is the subset of $\Gamma_{k,n}$ consisting of all non-decreasing sequences, and $Q_{k,n}$ denotes the subset of $G_{k,n}$ consisting of all strictly increasing sequences. If $X = (x_{ij})$ is an $m \times n$ matrix and $\omega = (\omega_1, \ldots, \omega_h) \in G_{h,m}$, $\gamma = (\gamma_1, \ldots, \gamma_k) \in G_{k,n}$, then $X[\omega|\gamma]$ denotes the $h \times k$ matrix whose (i, j) entry is $x_{\omega_i \gamma_j}$. If γ is in $G_{k,n}$, then $m_t(\gamma)$ denotes the multiplicity of occurrence of the integer t in γ. For example, if $\gamma = (1, 1, 1, 4, 4, 6) \in G_{6,6}$, then $m_1(\gamma) = 3$, $m_4(\gamma) = 2$, $m_6(\gamma) = 1$, and $m_2(\gamma) = m_3(\gamma) = m_5(\gamma) = 0$. If γ is in $G_{k,n}$, we define

$$\nu(\gamma) = \prod_{t=1}^{n} m_t(\gamma)!.$$

The ith row and the jth column of an $m \times n$ matrix X are denoted by $X_{(i)}$ and $X^{(j)}$ respectively.

Proof of the theorem. Let $u_1, \ldots, u_n$ be an orthonormal set of characteristic vectors of A corresponding to $\alpha_1, \ldots, \alpha_n$. Let U be the unitary matrix whose jth column is u_j $(j = 1, \ldots, n)$. In order to analyse the case of equality, we outline the proof of inequality (1). It was proved in **(3)**, that

$$\operatorname{per}(A) = \sum_{\gamma \in G_{n,n}} \frac{c_\gamma}{\nu(\gamma)} \prod_{t=1}^{n} \alpha_t^{m_t(\gamma)},$$

where $c_\gamma = |\operatorname{per}(U[1, \ldots, n|\gamma])|^2$. We have also proved in the same paper that

$$\sum_{\gamma \in G_{n,n}} m_t(\gamma) \frac{c_\gamma}{\nu(\gamma)} = 1 \qquad (t = 1, \ldots, n),$$

and deduced that

$$|\operatorname{per}(A)| = \left| \sum_{\gamma \in G_{n,n}} \frac{c_\gamma}{\nu(\gamma)} \prod_{t=1}^{n} \alpha_t^{m_t(\gamma)} \right|$$

$$\leqslant \sum_{\gamma \in G_{n,n}} \frac{c_\gamma}{\nu(\gamma)} \prod_{t=1}^{n} |\alpha_t|^{m_t(\gamma)} \tag{3}$$

$$\leqslant \sum_{\gamma \in G_{n,n}} \frac{c_\gamma}{\nu(\gamma)} \left(\frac{\sum_{t=1}^{n} m_t(\gamma)|\alpha_t|}{n} \right)^n \tag{4}$$

$$\leqslant \sum_{\gamma \in G_{n,n}} \frac{c_\gamma}{\nu(\gamma)} \frac{1}{n} \sum_{t=1}^{n} m_t(\gamma)|\alpha_t|^n \tag{5}$$

$$= \frac{1}{n} \sum_{t=1}^{n} |\alpha_t|^n \left(\sum_{\gamma \in G_{n,n}} m_t(\gamma) \frac{c_\gamma}{\nu(\gamma)} \right)$$

$$= \frac{1}{n} \sum_{t=1}^{n} |\alpha_t|^n.$$

For each γ such that $c_\gamma \neq 0$, equality occurs in (4) if and only if

$$|\alpha_{\gamma_1}| = \ldots = |\alpha_{\gamma_n}|. \tag{6}$$

Suppose first that given any pair of integers i, j $(i \neq j)$ there exists a sequence γ containing i and j such that $c_\gamma \neq 0$. Then we can conclude from (6) that

$$|\alpha_1| = \ldots = |\alpha_n| = \alpha. \tag{7}$$

It follows that

$$A = \alpha W \tag{8}$$

where W is unitary. Then the right-hand side of (1) is equal to α^n while the left-hand side is

$$|\mathrm{per}(\alpha W)| = \alpha^p |\mathrm{per}(W)|.$$

Hence if equality holds in (1), then $|\mathrm{per}\ (W)| = 1$ and therefore the matrix W has exactly one non-zero entry in each row and column **(5)**. Suppose now that $n \geqslant 3$ and that there exists a pair i, j $(i \neq j)$ such that $c_\gamma = 0$ for every γ in $G_{n,n}$ containing i and j. We can assume without loss of generality that $i = 1$ and $j = 2$. Since

$$c_\gamma = |\mathrm{per}\ (U[1, \ldots, n|\gamma])|^2$$

and since the permanent is a symmetric function of the columns of the matrix, it follows that if γ is any sequence in $\Gamma_{n,n}$ of the form $\gamma = (1, 2, \gamma_3, \ldots, \gamma_n)$, then

$$\mathrm{per}\ (U[1, \ldots, n|\gamma]) = 0.$$

Let X be any matrix whose first two columns are u_1 and u_2. Then, since the permanent is a multilinear function of the columns **(4; 5)**, we see, by expressing each $X^{(t)}$ as a linear combination of $u_1, \ldots, u_n$, that

$$\begin{aligned} \mathrm{per}\ (X) &= \mathrm{per}\ (u_1, u_2, X^{(3)}, \ldots, X^{(n)}) \\ &= \mathrm{per}\left(u_1, u_2, \sum_{j=1}^{n} c_{3j} u_j, \ldots, \sum_{j=1}^{n} c_{nj} u_j\right) \\ &= \sum_{\omega \in \Gamma_{n-2,n}} \mathrm{per}\ (u_1, u_2, u_{\omega_1}, \ldots, u_{\omega_{n-2}}) \prod_{t=3}^{n} c_{t\omega_t} \\ &= 0. \end{aligned}$$

In other words, the permanent of any matrix whose first two columns are u_1 and u_2 must vanish. Clearly, by choosing appropriate X and using the Laplace expansion of per (X) on the first two columns, we can conclude that, for any ω in $Q_{2,n}$,

$$\mathrm{per}\ (U[\omega|1, 2]) = 0. \tag{9}$$

We next show that the $n \times 2$ matrix $Y = (y_{ij}) = U[1, \ldots, n|1, 2]$ has exactly two non-zero rows. Since U is unitary, it is clear that Y has at least two non-zero rows. Suppose it has three non-zero rows: $Y_{(i)}$, $Y_{(j)}$, and $Y_{(k)}$. Now, by (9) with (i, k) and (j, k) for ω, we conclude that

$$y_{i1} y_{k2} + y_{i2} y_{k1} = 0, \qquad y_{j1} y_{k2} + y_{j2} y_{k1} = 0. \tag{10}$$

Since $Y_{(k)}$ is assumed to be different from 0, the equations (10) imply that $Y_{(i)}$ and $Y_{(j)}$ are linearly dependent. In other words, if there were at least three non-zero rows, then any two would be linearly dependent. But the rank of Y is 2, since U is unitary, and thus Y must have exactly two non-zero rows, say $Y_{(1)}$ and $Y_{(2)}$. It follows that

$$U = Z_2 \dotplus Z_{n-2}$$

where "$\dotplus$" indicates direct sum and Z_2 is a 2-square unitary matrix. Then

$$\begin{aligned} (11) \qquad A &= U \operatorname{diag}(\alpha_1, \ldots, \alpha_n) U^* \\ &= Z_2 \operatorname{diag}(\alpha_1, \alpha_2) Z_2^* \dotplus Z_{n-2} \operatorname{diag}(\alpha_3, \ldots, \alpha_n) Z_{n-2}^*. \end{aligned}$$

Now, applying the inequality (1), proved in **(3)**, to (11), we obtain

$$\begin{aligned} (12) \qquad \frac{|\alpha_1|^n + \ldots + |\alpha_n|^n}{n} &= \operatorname{per}(A) \\ &= |\operatorname{per}(Z_2 \operatorname{diag}(\alpha_1, \alpha_2) Z_2^*)| \\ &\qquad \times |\operatorname{per}(Z_{n-2} \operatorname{diag}(\alpha_3, \ldots, \alpha_n) Z_{n-2}^*)| \\ &\leqslant \frac{|\alpha_1|^2 + |\alpha_2|^2}{2} \frac{|\alpha_3|^{n-2} + \ldots + |\alpha_n|^{n-2}}{n-2}. \end{aligned}$$

By a well-known inequality **(5**, p. 105**)**,

$$(13) \qquad \tfrac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2) \leqslant \{\tfrac{1}{2}(|\alpha_1|^n + |\alpha_2|^n)\}^{2/n}$$

with equality for $n \geqslant 3$ if and only if $|\alpha_1| = |\alpha_2|$, and

$$(14) \qquad \frac{|\alpha_3|^{n-2} + \ldots + |\alpha_n|^{n-2}}{n-2} \leqslant \left(\frac{|\alpha_3|^n + \ldots + |\alpha_n|^n}{n-2}\right)^{(n-2)/n}$$

with equality if and only if $|\alpha_3| = \ldots = |\alpha_n|$. Thus, multiplying (13) and (14) and using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} (15) \qquad \frac{|\alpha_1|^2 + |\alpha_2|^2}{2} \frac{|\alpha_3|^{n-2} + \ldots + |\alpha_n|^{n-2}}{n-2} &\leqslant \left(\frac{|\alpha_1|^n + |\alpha_2|^n}{2}\right)^{2/n} \\ &\qquad \times \left(\frac{|\alpha_3|^n + \ldots + |\alpha_n|^n}{n-2}\right)^{(n-2)/n} \\ &\leqslant \frac{2}{n} \frac{|\alpha_1|^n + |\alpha_2|^n}{2} + \frac{n-2}{n} \\ &\qquad \times \frac{|\alpha_3|^n + \ldots + |\alpha_n|^n}{n-2} \\ &= \frac{|\alpha_1|^n + \ldots + |\alpha_n|^n}{n} \end{aligned}$$

with equality if and only if

$$\frac{|\alpha_1|^n + |\alpha_2|^n}{2} = \frac{|\alpha_3|^n + \ldots + |\alpha_n|^n}{n-2}.$$

It follows from (12) and (15) that both (12) and (15) are equalities. But then also (13) and (14) are equalities. The conditions for equality to hold in (13), (14), and (15) immediately imply (7). The proof of necessity for $n \geqslant 3$ is complete.

Conversely, suppose that $A = DP$ where P is a permutation matrix and $D = \operatorname{diag}(z_1, \ldots, z_n)$, $|z_j| = \alpha$, $j = 1, \ldots, n$. Then

$$|\text{per }(A)| = \alpha^n.$$

Now, it is clear that every characteristic root α_j of A has modulus α. Hence

$$\frac{1}{n}\sum_{j=1}^{n} |\alpha_j|^n = \alpha^n$$

and equality holds in (1).

If $n = 2$ and equality holds in (1), then either $c_\gamma \neq 0$ for $\gamma = (1, 2)$, in which case it follows as in the proof of the general case ($n \geqslant 3$), that A is of the form (8), or $c_\gamma = 0$ for $\gamma = (1, 2)$, i.e., per $(U) = 0$. Let

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

and suppose that

$$u_{11} u_{22} + u_{12} u_{21} = 0. \tag{16}$$

Now, U is unitary and therefore (16) implies that none of u_{11}, u_{12}, u_{21}, u_{22} is 0. Hence

$$\frac{u_{12}}{u_{11}} = -\frac{u_{22}}{u_{21}}.$$

Thus

$$U = \begin{bmatrix} u_{11} & \theta u_{11} \\ u_{21} & -\theta u_{21} \end{bmatrix}$$

where $\theta = u_{21}/u_{11} = -u_{22}/u_{21} \neq 0$. We again use the fact that U is unitary to obtain

$$\theta|u_{11}|^2 - \theta|u_{21}|^2 = u_{11}\,\bar{u}_{21} - u_{11}\,\bar{u}_{21}\,|\theta|^2 = 0.$$

Hence $|\theta| = 1$ and $|u_{11}| = |u_{21}| = 1/\sqrt{2}$. Thus setting $u_{11} = \eta/\sqrt{2}$ and $u_{21} = \zeta/\sqrt{2}$, where $|\eta| = |\zeta| = 1$, we see that U must be of the form

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} \eta & \theta\eta \\ \zeta & -\theta\zeta \end{bmatrix}.$$

We compute

$$A = U \operatorname{diag}(\alpha_1, \alpha_2) U^* = \frac{1}{2}\begin{bmatrix} \alpha_1 + \alpha_2 & (\alpha_1 - \alpha_2)\eta\bar{\zeta} \\ (\alpha_1 - \alpha_2)\bar{\eta}\zeta & \alpha_1 + \alpha_2 \end{bmatrix}$$

and therefore

$$\text{per }(A) = \tfrac{1}{2}(\alpha_1^2 + \alpha_2^2).$$

If equality holds in (1), i.e., if

$$\tfrac{1}{2}(\alpha_1^2 + \alpha_2^2)| = \tfrac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)|,$$

then the arguments of α_1^2 and α_2^2 differ by an integer multiple of 2π and hence $\arg(\alpha_1 \bar{\alpha}_2) = k\pi$. We check by direct computation that the conditions of the theorem for $n = 2$ are sufficient.

References

1. R. A. Brualdi and M. Newman, *Some theorems on the permanent* (to appear).
2. Marvin Marcus and Henryk Minc, *Inequalities for general matrix functions*, Bull. Amer. Math. Soc., *70* (1964), 308–313.
3. ——— *Generalized matrix functions*, Trans. Amer. Math. Soc. (to appear).
4. ——— *Permanents*, Amer. Math. Monthly, *72* (1965), 577–591.
5. ——— *A survey of matrix theory and matrix inequalities* (Boston, Mass., 1964).
6. Marvin Marcus and Morris Newman, *Inequalities for the permanent function*, Ann. Math. (2), *75* (1962), 47–62.

University of California, Santa Barbara