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Approximation on Closed Sets by Analytic or Meromorphic Solutions of Elliptic Equations and Applications

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Abstract. Given a homogeneous elliptic partial differential operator L with constant complex coefficients and a class of functions (jet-distributions) which are defined on a (relatively) closed subset of a domain Ω in $\mathbf{R}^n$ and which belong locally to a Banach space V, we consider the problem of approximating in the norm of V the functions in this class by "analytic" and "meromorphic" solutions of the equation $Lu = 0$. We establish new Roth, Arakelyan (including tangential) and Carleman type theorems for a large class of Banach spaces V and operators L. Important applications to boundary value problems of solutions of homogeneous elliptic partial differential equations are obtained, including the solution of a generalized Dirichlet problem.

1 Introduction

Let L be a homogeneous elliptic partial differential operator with constant *complex* coefficients (such as powers of the Cauchy-Riemann operator $\bar{\partial}$ or the Laplacean Δ). In [2], given a Banach space $(V, \| \|)$ of functions (distributions) on $\mathbf{R}^n$, $n \geq 2$, we studied the problem of approximating, on a closed subset F of $\mathbf{R}^n$, the solutions of the equation $Lu = 0$ by global (L-analytic or L-meromorphic) solutions of the equation. Approximation theorems of Runge-type and Arakelyan-type were obtained whenever the operator L and the Banach space V satisfied certain conditions.

In this paper, we first generalize the results of [2] and [11] to Banach spaces of functions (distributions) defined on any domain Ω of $\mathbf{R}^n$ ($n \geq 2$). As already mentioned in [2], the only purpose of one of the important conditions on L and V ([2, Condition (4)]) was to obtain a "special maximum principle" ([2, Lemma 1]). Weakened assumptions of this lemma have now become our new Condition 4 (see Section 2 below), and consequently our proof has been slightly modified (and improved). For all operators L under consideration, our conditions are satisfied by a large class of classical (non-weighted) spaces.

Using results on the solution of the Dirichlet problem for strongly elliptic equations in bounded smooth domains, we find (see Proposition 2 below) that in this case our conditions are also satisfied by a wide class of spaces, for which an application of

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our theorems gives important new examples in the theory of tangential approximation (see Theorem 4(iii)).

Using Carleman-type approximation results (see Lemma 4 and Proposition 5), we obtain in Section 6 some very interesting examples of the possible boundary behaviour of solutions of homogeneous elliptic partial differential equations, analogous to those described in [5, Chapter IV, Section 5B] for functions holomorphic in a disc. First, given a domain Ω satisfying some mild conditions, we construct an L-analytic function f in Ω such that the limit of f and of all its derivatives along *any* path ending at the boundary of Ω does not exist (Theorem 5). To our knowledge, only very special cases of this result were known for the $\bar{\partial}$ equation in $\mathbf{R}^2$ and the Laplacean in $\mathbf{R}^n$, $n \geq 2$ (see [5, Chapter IV, Section 5], [6, Section 8]).

When the boundary of Ω is sufficiently smooth, we are also able to solve (see Theorem 6) a "weakened" Dirichlet problem where the boundary values of an L-analytic function, together with the boundary values of a fixed number of its derivatives are prescribed (almost everywhere on $\partial\Omega$) as we approach the boundary in the *normal* direction.

2 Definitions and Notation

For the reader's convenience, we summarize the definitions and main notation of [2]. Note that in [2], these were given only for $\mathbf{R}^n$, but here we extend them very naturally to general domains.

Let Ω be any fixed domain in $\mathbf{R}^n$, $n \geq 2$. We let $V = V(\Omega)$ stand for a Banach space, whose norm is denoted by $\| \ \|$, which contains $C_0^\infty(\Omega)$, the set of test functions in Ω and is contained in $\big(C_0^\infty(\Omega)\big)^*$, the space of distributions on Ω. We make some additional assumptions on V.

Conditions 1 and 2 We assume that V is a topological $C_0^\infty(\Omega)$-submodule of $\big(C_0^\infty(\Omega)\big)^*$, which means that for $f \in V$ and $\varphi \in C_0^\infty(\Omega)$, we have $\varphi f \in V$ with

$$\|\varphi f\| \leq C(\varphi)\|f\| \tag{1}$$

and

$$|\langle f, \varphi\rangle| \leq C(\varphi)\|f\|, \tag{2}$$

where $\langle f, \varphi\rangle$ denotes the action in Ω of the distribution f on the test function φ and $C(\varphi)$ is a constant independent of f. We note that this implies that the imbeddings $C_0^\infty(\Omega) \hookrightarrow V$ and $V \hookrightarrow \big(C_0^\infty(\Omega)\big)^*$ are continuous (see [2, Section 2.1]).

Given a closed subset F in Ω, let $I(F)$ be the closure in V of (the family of) those $f \in V$ whose support in Ω in the sense of distributions (which will be denoted by $\operatorname{supp}(f)$) is disjoint from F, and let $V(F) = V/I(F)$. The Banach space $V(F)$, endowed with the quotient norm, should be viewed as the natural (Whitney type) version of V on F (see [14, Chapter 6]). We will write $\|f\|_F$ for the norm of the equivalence class (jet) $f_{(F)} := f + I(F)$ in $V(F)$ of the distribution $f \in V$.

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For any open set D in Ω, let

$$V_{\mathrm{loc}}(D) = \left\{ f \in \left(C_0^\infty(D)\right)^* \;\middle|\; f\varphi \in V \text{ for each } \varphi \in C_0^\infty(D) \right\},$$

where φ and $f\varphi$ are extended to be identically zero in $\Omega \setminus D$. We endow $V_{\mathrm{loc}}(D)$ with the projective limit topology of the spaces $V(K)$ partially ordered by inclusion of the compact sets $K \subset D$. For a closed set F in Ω, define $V_{\mathrm{loc}}(F) = V_{\mathrm{loc}}(\Omega)/J(F)$, where $J(F)$ is the closure in $V_{\mathrm{loc}}(\Omega)$ of the family of those distributions in $V_{\mathrm{loc}}(\Omega)$ whose support is disjoint from F. The topology on $V_{\mathrm{loc}}(F)$ will be the quotient topology. Note that for compact sets K, the topological spaces $V(K)$ and $V_{\mathrm{loc}}(K)$ are identical.

For $f \in V_{\mathrm{loc}}(\Omega)$, we put $f_{(F),\mathrm{loc}} := f + J(F)$. If D is a neighbourhood of F in Ω, then each $h \in V_{\mathrm{loc}}(D)$ naturally defines an element (jet) $h_{(F),\mathrm{loc}}$ in $V_{\mathrm{loc}}(F)$ by taking $h_{(F),\mathrm{loc}}$ to be the closure in $V_{\mathrm{loc}}(\Omega)$ of the set of $f \in V_{\mathrm{loc}}(\Omega)$ such that $f = h$ (as distributions) in some neighbourhood (depending on f) of F. In particular, this works for each $h \in C^\infty(D) \subset V_{\mathrm{loc}}(D)$. For $f_{(F),\mathrm{loc}} \in V_{\mathrm{loc}}(F)$, we will write $f_{(F),\mathrm{loc}} \in V(F)$ (*or more briefly* $f \in V(F)$), if $V \cap f_{(F),\mathrm{loc}} \neq \varnothing$. We will then write $\|f_{(F),\mathrm{loc}}\|_F$, or equivalently $\|f\|_F$, to mean $\|g\|_F$, where $g \in V \cap f_{(F),\mathrm{loc}}$. Practically the same proof as in [2, Section 2.1] shows that $V \cap J(F) = I(F)$ holds for each closed set F in Ω, which means that $\|f_{(F),\mathrm{loc}}\|_F$ is well-defined.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_j \in \mathbf{Z}_+ (:= \{0, 1, 2, \dots\})$, we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$.

We denote by $B(a,\delta)$ (respectively $\bar{B}(a,\delta)$) the open (respectively closed) ball with center $a \in \mathbf{R}^n$ and radius $\delta > 0$. If $B = B(a,\delta)$ and $\theta > 0$ then $\theta B = B(a, \theta\delta)$ and $\theta\bar{B} = \bar{B}(a,\theta\delta)$.

Throughout this paper we let $L(\xi) = \sum_{|\alpha|=r} a_\alpha \xi^\alpha$, $\xi \in \mathbf{R}^n$, be a fixed homogeneous polynomial of degree r ($r \geq 1$) with *complex* constant coefficients and which satisfies the ellipticity condition $L(\xi) \neq 0$, $\xi \neq 0$. We associate to L the homogeneous elliptic operator of order r

$$L = L(\partial) = \sum_{|\alpha|=r} a_\alpha \partial^\alpha.$$

Let D be an open set in $\mathbf{R}^n$ and denote by $L(D)$ the set of distributions f in D such that $Lf = 0$ in D in the sense of distributions. It is well known [7, Theorem 4.4.1] that $L(D) \hookrightarrow C^\infty(D)$. Therefore if $D \subset \Omega$, then $L(D) \subset V_{\mathrm{loc}}(D)$, and if $\{f_m\}$ is a sequence in $L(D)$ with $f_m \to f$ in $V_{\mathrm{loc}}(D)$ as $m \to \infty$, then $f \in L(D)$, since convergence in $V_{\mathrm{loc}}(D)$ is stronger than convergence in the sense of distributions, which preserves $L(D)$ [7, Theorem 4.4.2].

Functions from $L(D)$ will be called *L-analytic* in D. We shall also say that a distribution g in D is *L-meromorphic* in D if $\mathrm{supp}(Lg)$ is discrete in D and for each $a \in \mathrm{supp}(Lg)$ ($a \in D$) there exist h, which is L-analytic in a neighbourhood of a, $k \in \mathbf{Z}_+$ and $\lambda_\alpha \in \mathbf{C}$, $0 \leq |\alpha| \leq k$, such that

$$g(x) = h(x) + \sum_{|\alpha| \leq k} \lambda_\alpha \partial^\alpha \Phi(x-a)$$

in some neighbourhood of a, where Φ is a special fundamental solution of L as described in [7, Theorem 7.1.20]. The points $a \in \operatorname{supp}(Lg)$ will be called the *poles* of g.

We recall (see [3, p. 239] or [15, p. 163]) that there exists a $k > 1$ such that if T is a distribution with compact support contained in $B(a, \delta)$ and $f = \Phi * T$, then, for $|x - a| > k\delta$, we have the *Laurent-type expansion*:

$$f(x) = \langle T(y), \Phi(x - y)\rangle = \sum_{|\alpha|\geq 0} c_\alpha \partial^\alpha \Phi(x - a), \tag{3}$$

where $c_\alpha = (-1)^{|\alpha|}(\alpha!)^{-1}\langle T(y), (y - a)^\alpha\rangle$. The series converges in $C^\infty(\{|x - a| > k\delta\})$, which means that the series can be differentiated term by term and all such series converge uniformly on $\{|x - a| \geq k'\delta\}$, $k' > k$.

Let $\varphi \in C_0^\infty(\Omega)$. The *Vitushkin localisation operator* $\mathcal{V}_\varphi\colon \big(C_0^\infty(\Omega)\big)^* \to \big(C_0^\infty(\Omega)\big)^*$ associated to L and φ is defined as $\mathcal{V}_\varphi f = \big(\Phi * (\varphi L f)\big)\big|_\Omega$, where in the last equality $*$ denotes the convolution operator in $\mathbf{R}^n$.

Condition 3 We require that for each $\varphi \in C_0^\infty(\Omega)$, the operator $\mathcal{V}_\varphi$ be invariant on $V_{\mathrm{loc}}(\Omega)$, *i.e.* $\mathcal{V}_\varphi$ must send continuously $V_{\mathrm{loc}}(\Omega)$ into $V_{\mathrm{loc}}(\Omega)$. This means that if K is a compact subset of Ω and $\operatorname{supp}(\varphi) \subset K$, then for each $f \in V_{\mathrm{loc}}(\Omega)$ one has $\mathcal{V}_\varphi f \in V_{\mathrm{loc}}(\Omega)$ and

$$\|\mathcal{V}_\varphi f\|_K \leq C\|f\|_K, \tag{4}$$

where C is independent of f.

We make one more assumption on V in relation with L.

Condition 4 For each open ball B with $3\bar{B} \subset \Omega$, there exist $d > 0$ and $C > 0$ such that for each $h \in C^\infty(\mathbf{R}^n)$ satisfying $Lh = 0$ outside of B and $h(x) = O(|x|^{-d})$ as $|x| \to \infty$, one can find $v \in L(\Omega)$ with

$$(h - v) \in V \quad \text{and} \quad \|h - v\| \leq C\|h\|_{3\bar{B}}. \tag{5}$$

In this assumption, instead of the constant 3, one can take any fixed real number greater than 1.

3 Some Remarks on Conditions 1 to 4

All Conditions 1 to 4 are satisfied by classical (non-weighted) spaces on any domain Ω in $\mathbf{R}^n$, for example $BC^m(\Omega)$, $BC^{m+\mu}(\Omega)$, $\mathrm{VMO}(\Omega)$ and the Sobolev spaces $W_m^p(\Omega)$, $1 \leq p < \infty$. We shall give the definitions and prove this assertion only for the spaces $V = BC^m(\Omega)$ and $BC^{m+\mu}(\Omega)$.

For $m \in \mathbf{Z}_+$, let $BC^m(\Omega)$ be the space of all m-times continuously differentiable functions $f\colon \Omega \to \mathbf{C}$ with (finite) norm

$$\|f\|_{m,\Omega} = \max_{|\alpha|\leq m} \sup_{x\in\Omega} |\partial^\alpha f(x)|.$$

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If $m \in \mathbf{Z}_+$ and $0 < \mu < 1$, then

$$BC^{m+\mu}(\Omega) = \{f \in BC^m(\Omega) \mid \omega_\mu^m(f,\infty) < \infty \text{ and } \omega_\mu^m(f,\delta) \to 0 \text{ as } \delta \to 0\},$$

where $\omega_\mu^m(f,\delta) = \sup \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\mu}$, the supremum being taken over all multi-index α such that $|\alpha| = m$ and all $x, y \in \Omega$ with $0 < |x-y| < \delta$. The norm in this space is defined as

$$\|f\|_{m+\mu,\Omega} = \max\{\|f\|_{m,\Omega}, \omega_\mu^m(f,\infty)\}.$$

We shall omit the index Ω in the last norm whenever $\Omega = \mathbf{R}^n$. Finally, for any $\rho \geq 0$, we set $C^\rho(\Omega) = \big(BC^\rho(\Omega)\big)_{\text{loc}}$.

Proposition 1 *Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, and let $\rho \geq 0$. Then the pair $\big(L, V(\Omega)\big)$ with $V(\Omega) = BC^\rho(\Omega)$ satisfies Conditions 1, 2, 3 and satisfies Condition 4 with $v = 0$.*

Proof Conditions 1 and 2 are easily verified. Condition 3 is proved in [10, Corollary 5.6] in the case $\Omega = \mathbf{R}^n$ for all spaces mentioned above, since $C_0^\infty(\mathbf{R}^n)$ is locally dense in each of them. As Condition 3 is local, it holds for each pair $\big(L, V(\Omega)\big)$ under consideration.

To obtain Condition 4 with $v = 0$, we can easily use [2, Lemma 1] (see also [11, Lemma 2]). In fact, by this lemma, for each open ball B with $3\bar{B} \subset \Omega$, we even can find $d > 0$ and $C > 0$ such that if h satisfies the hypotheses of Condition 4 with this d, then

$$\|h\|_\rho \leq C\|h\|_{\rho,3\bar{B}}.$$

Since $\|h\|_{\rho,\Omega} \leq \|h\|_\rho$, the proof is complete. ∎

In [2, Corollary 1] (see also the brief discussion thereafter) and [11, Theorem 4] one sees how (whenever Conditions 1 to 3 are satisfied) Condition 4 can affect L-meromorphic and L-analytic approximation in the special case of weighted uniform holomorphic approximation ($n = 2$, $L = \bar{\partial}$).

We also wish to present here an example of a pair (L, V) satisfying Conditions 1, 2 and 4 (with $v = 0$), but not 3. Hence, this example eludes our method. The example seems new even without considering Condition 4.

Take $L = \bar{\partial}$, $\Omega = \mathbf{R}^2 (= \mathbf{C})$, $B_1 = \{z \in \mathbf{C} \mid |z| < 1\}$ (the unit disk), and let

$$V = BC^0(\mathbf{R}^2) \cap BC^1(B_1) \quad \text{with norm} \quad \|f\| = \max\{\|f\|_0, \|f\|_{1,B_1}\}.$$

Conditions 1 and 2 are easily verified. Condition 4 (with $v = 0$, $d = 1$) follows from the maximum principle and from trivial estimates of derivatives (outside $2\bar{B}$) of a function, holomorphic outside $\bar{B}$ and vanishing at ∞. Finally, fixing any $\varphi \in C_0^\infty(3B_1)$ such that $\varphi(z) = \bar{z}$ on $2B_1$, one can check that there exists $f \in BC^0(\mathbf{R}^2)$, $f = 0$ in B_1, with $\mathcal{V}_\varphi f|_{B_1}$ not in $BC^1(B_1)$. In fact, in this case

$$\mathcal{V}_\varphi f(w) = f(w)\varphi(w) - \frac{1}{\pi}\int \frac{f(z)\bar{\partial}\varphi(z)}{w-z}\,dx_1\,dx_2 \quad z = x_1 + ix_2,$$

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so that one needs only to study the behavior (in B_1) of the function

$$\int_{2B_1 \setminus B_1} \frac{f(z)}{(w-z)^2}\, dx_1\, dx_2.$$

Easily, there is $g \in C(\mathbf{R}^2)$, $g \geq 0$, $\operatorname{supp}(g) \subset \{x_1 \geq 2|x_2|\} \cap B_1$, such that

$$\int \frac{g(z)}{|z|^2}\, dx_1\, dx_2 = +\infty.$$

It is enough to take $f(z) = g(z-1)$ and let $w \in (0,1)$ tend to 1. Indeed, set $1-w = \delta$. Then, it is enough to show that

$$\int \frac{g(z)}{(z+\delta)^2}\, dx_1\, dx_2$$

is unbounded as δ tends to zero. In fact,

$$\operatorname{Re}\left(\frac{1}{(z+\delta)^2}\right) \geq \frac{1}{2|z+\delta|^2}$$

on $\operatorname{supp}(g)$, and if the integrals

$$\int \frac{g(z)}{|z+\delta|^2}\, dx_1\, dx_2$$

were uniformly bounded for $\delta \in (0,1)$, then by Fatou's lemma, the integral with $\delta = 0$ would be convergent, which is not the case.

The following proposition provides us with another class of examples for which Conditions 1 to 4 are satisfied. These in turn will allow us to obtain in Section 4 new results on "tangential" approximation. Given m and q in $\mathbf{Z}_+$, with $q \leq m$, and a bounded domain Ω, set

$$BC_q^m(\Omega) = \{f \in BC^m(\Omega) \mid \text{ for each } \alpha, |\alpha| \leq q, \lim_{x \to \partial\Omega} \partial^\alpha f(x) = 0\},$$

which is a Banach space with the norm $\|f\|_{m,\Omega}$.

Proposition 2 *Let L be a strongly elliptic operator of order $r = 2\ell$, $\ell \in \mathbf{Z}_+$, $\ell \geq 1$ (see [1, p. 46]). Let $m, q \in \mathbf{Z}_+$, $m \geq \ell - 1$, $q \leq \ell - 1$. If Ω is bounded and $\partial\Omega$ is of class C^s, $s = \max\{2\ell, [n/2] + 1 + m\}$ (see [1, p. 128]), then the pair $\big(L, V = BC_q^m(\Omega)\big)$ satisfies Conditions 1 to 4.*

Proof Since $\big(BC_q^m(\Omega)\big)_{\text{loc}} = C^m(\Omega)$, Conditions 1, 2 and 3 are satisfied. Let us prove Condition 4. Fix any ball B, $3\bar{B} \subset \Omega$, and take any $h \in C^\infty(\mathbf{R}^n)$ with $Lh = 0$ outside B. Now, results on solvability and regularity of the classical Dirichlet problem applied to the operator L (see [1, Theorem 8.2 and Lemma 7.7, Theorem 9.8 and Lemma 9.1, Theorem 3.9]) show that under the hypotheses of Proposition 2, there

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exists $v_0 \in C^m(\bar\Omega) \cap L(\Omega)$ such that $u_0 = h - v_0$ satisfies $\partial^\alpha u_0|_{\partial\Omega} = 0$ for each α, $|\alpha| \le \ell - 1$ (so that $h - v_0 \in V$), and moreover

$$\|u_0\| \equiv \|u_0\|_{m,\Omega} \le C_1 \|h\|_{s,\Omega},$$

where C_1 is independent of h. We observe that we have not used here the property $Lh = 0$ in $\mathbf{R}^n \setminus B$. We also remark that our notations for m and $\|\cdot\|_{m,\Omega}$ are different from those of [1], and that the last inequality follows from [1, (9.23)] since

$$\|u_0\|_{L^2(\Omega)} \le \|u_0\|_{W^2_\ell(\Omega)} \le C_2 \|h\|_{W^2_\ell(\Omega)},$$

by [1, Theorems 8.1 and 8.2].

By [11, Lemmas 1 and 3], we can choose $d > 0$ and $C_3 > 0$ (independently of h) such that if additionally $h(x) = O(|x|^{-d})$ as $|x| \to \infty$, then (see also [2, Lemma 1])

$$h = \Phi * Lh, \quad \text{and} \quad \|h\|_{m,\Omega} \le \|h\|_{m,\mathbf{R}^n} \le C_3 \|h\|_{m,3B}.$$

Fix $\chi \in C_0^\infty(\frac{3}{2}B)$, $\chi = 1$ on B. Then for $x \in \mathbf{R}^n \setminus 2\bar B$, we get

$$h(x) = \int_B \Phi(x-y) Lh(y) \chi(y)\, dy = \int_B L\big(\chi(y)\Phi(x-y)\big)\, h(y)\, dy$$

and so since Ω is bounded,

$$\|h\|_{s,\Omega\setminus 2\bar B} \le C_4 \|h\|_{0,\frac{3}{2}B} \le C_4 \|h\|_{m,3B}.$$

We can now find a function $h_1 \in C^\infty(\mathbf{R}^n)$, $h_1 = h$ on $\mathbf{R}^n \setminus 2\bar B$ such that

$$\|h_1\|_{s,\Omega} \le C_5 \|h\|_{s,\Omega\setminus 2\bar B} \le C_6 \|h\|_{m,3B}.$$

Let now v_1 and $u_1 = h_1 - v_1$ satisfy the same properties as the functions v_0 and u_0 above, but taken with h_1 instead of h. Then

$$\|u_1\|_{m,\Omega} \le C_2 \|h_1\|_{s,\Omega} \le C_7 \|h\|_{m,3B}.$$

The function $v = v_1$ is as desired. In fact, since $\partial^\alpha u_1 = 0$ on $\partial\Omega$ for $|\alpha| \le \ell - 1$, then

$$\partial^\alpha (h - v)|_{\partial\Omega} = \partial^\alpha (h_1 - v_1)|_{\partial\Omega} = 0$$

for $|\alpha| \le \ell - 1$, so that $h - v \in V(\Omega)$. Finally

$$\|h - v\|_{m,\Omega} = \|h - h_1 + h_1 - v_1\|_{m,\Omega} \le \|h\|_{m,\Omega} + \|h_1\|_{m,\Omega} + \|u_1\|_{m,\Omega} \le C\|h\|_{m,3B},$$

since clearly

$$\|h_1\|_{m,\Omega} \le \|h_1\|_{s,\Omega} \le C_6 \|h\|_{m,3B}.$$

Note that the constants C_2 to C_7 and C are independent of h. This ends the proof. ∎

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Let Ω be any domain in $\mathbf{R}^n$. Denote by $\Omega^* = \Omega \cup \{*\}$ the one point compactification of Ω and by X° the interior of a set X. For $i \geq 1$, let

$$X_i = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \geq 1/i, |x| \leq i\}.$$

Then each X_i is a compact subset of Ω such that both $\Omega^* \setminus X_i$ and $\Omega^* \setminus X_i^\circ$ are connected and such that $X_i \subset X_{i+1}^\circ$.

In the next sections, we shall need frequently the following easy consequence of a very general version of Runge's theorem.

Proposition 3 *Assume $V = V(\Omega)$ satisfies Conditions 1 and 2. Then, given $i \geq 1$, $\varepsilon_i > 0$ and $f \in L(X_{i+1}^\circ)$, one can find $h_i \in L(\Omega)$ such that*

$$\|f - h_i\|_{X_i} \leq \varepsilon_i.$$

Proof By the generalization of Runge's theorem found in [7, Theorem 4.4.5], there exists a sequence $\{g_m\}_{m=1}^\infty \subset L(\Omega)$ such that $g_m \to f$ in $C^\infty(X_{i+1}^\circ)$ and hence $g_m \to f$ in $V(X_i)$ as $m \to \infty$, which gives the result if one takes $h_i = g_m$ for some m sufficiently large. ■

4 Approximation Theorems

As in [2, Section 3], a closed set F in Ω will be called a Roth-Keldysh-Lavrent'ev set in Ω, or more simply an Ω-RKL *set*, if $\Omega^* \setminus F$ is connected and locally connected. In this section, we formulate our main approximation results. They extend the analogous ones of [2] from $\mathbf{R}^n$ to general domains Ω. Using Proposition 2, concrete new applications to "tangential" approximation are also obtained (see Theorem 4(iii)). Note that Carleman-type approximation results will also be presented in Section 6 with interesting applications to the boundary behaviour of L-analytic functions.

We first obtain sufficient conditions for approximation of Runge-type on closed sets.

Theorem 1 *Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$. Let $\big(L, V(\Omega)\big)$ be a pair satisfying Conditions 1 to 4, F be a (relatively) closed subset of Ω, and f be L-analytic in some neighbourhood of F in Ω. Then, for each $\varepsilon > 0$, there exists an L-meromorphic function g on Ω with poles off F such that $(f_{(F),\mathrm{loc}} - g_{(F),\mathrm{loc}}) \in V(F)$ and*

$$\|f - g\|_F < \varepsilon.$$

Moreover, if F is an Ω-RKL set, then g can be chosen in $L(\Omega)$.

The next theorem deals with approximation of a single function and shows that the problem is essentially local.

Theorem 2 *Let Ω be a domain in $\mathbf{R}^n$ $(n \geq 2)$, $\big(L, V(\Omega)\big)$ be a pair satisfying Conditions 1 to 4, F be a (relatively) closed subset of Ω, and $f \in V_{\mathrm{loc}}(\Omega)$. Then the following are equivalent:*

(i) *for each positive number ε, there exists an L-meromorphic function g in Ω with poles off F such that $(f_{(F),\mathrm{loc}} - g_{(F),\mathrm{loc}}) \in V(F)$ and $\|f - g\|_F < \varepsilon$;*

(ii) *for each ball B, $\bar{B} \subset \Omega$ and positive number ε, there exists g such that $Lg = 0$ on some neighbourhood of $F \cap \bar{B}$ and $\|f - g\|_{F\cap\bar{B}} < \varepsilon$;*

(iii) *the previous property is satisfied by each ball from some locally finite family of balls $\{B'_j\}$ covering F, where $\overline{B'_j} \subset \Omega$ for each j.*

For any subset X of $\mathbf{R}^n$, we let $L(X)$ stand for the collection of all functions f defined and L-analytic in some neighbourhood (depending on f) of X. For a closed set F in Ω we denote by $M_{LV}(F)$ (respectively $E_{LV}(F)$) the space of all $f_{(F),\mathrm{loc}} \in V_{\mathrm{loc}}(F)$ which satisfy the following property: for each $\varepsilon > 0$ there exists an L-meromorphic function g in Ω with poles outside of F (respectively a function $g \in L(\Omega)$) such that $f - g \in V(F)$ and $\|f - g\|_F < \varepsilon$. We also introduce the space $V_L(F) = V_{\mathrm{loc}}(F) \cap L(F^\circ)$. Whenever Conditions 1 to 4 hold, we have that by Theorem 1, $M_{LV}(F)$ is the closure in $V_{\mathrm{loc}}(F)$ of the space $\{h_{(F),\mathrm{loc}} \in V_{\mathrm{loc}}(F) \mid h \in L(F)\}$. Moreover, if F is an Ω-RKL set, then $M_{LV}(F) = E_{LV}(F)$.

We now study the necessity of being a Ω-RKL set for approximation by L-analytic functions.

Let K be a compact set in Ω. Denote by $\hat{K}$ the union of K and all the (connected) components of $\Omega \setminus K$ which are pre-compact in Ω. Obviously, the property $\hat{K} = K$ means precisely that $\Omega^* \setminus K$ is connected, so that K is a Ω-RKL set.

Define

$$N(K) = N_{LV}(K) = \{a \in \hat{K} \setminus K \mid (\Phi_a)_{(K)} \notin E_{LV}(K)\},$$

where $\Phi_a(x) = \Phi(x - a)$.

Condition N We shall say that a pair $\big(L, V(\Omega)\big)$ satisfies Condition N ("nonremovability of holes") if $N(K) \neq \varnothing$ for each compact set K with "holes", *i.e.* such that $K \neq \hat{K}$.

Remark 1 The same proof as in [2, Proposition 2] shows that $\big(L, V(\Omega)\big)$ satisfies Condition N whenever all of the following conditions hold:

(1) $\big(L, V(\Omega)\big)$ satisfies Conditions 1 and 2;

(2) $n = 2$ or $n \geq 3$ and L has the following symbol:

$$L(\xi) = P_2(\xi)Q_{r-2}(\xi), \quad \xi \in \mathbf{R}^n,$$

where P_2 is some homogeneous (elliptic) polynomial of order two with real coefficients (so that P_2 has constant sign in $\mathbf{R}^n \setminus \{0\}$), and Q_{r-2} is some homogeneous polynomial of order $r - 2$;

(3) $\mathrm{Ord}(V) \geq r - 1$.

For the definition of $\mathrm{Ord}(V)$ when Ω is $\mathbf{R}^n$, see [2, Section 4.3]. Replacing $\mathbf{R}^n$ by Ω everywhere in that definition, we get the corresponding definition of $\mathrm{Ord}\big(V(\Omega)\big)$ for an arbitrary domain Ω.

One can also find in [2, Section 4.2] some informative examples concerning Condition N.

Theorem 3 *If $\big(L, V(\Omega)\big)$ satisfies Conditions* 1 *to* 4, *then the following statements are equivalent:*

(i) *for each (relatively) closed set $F \subset \Omega$ one has*

$$M_{LV}(F) = E_{LV}(F) \iff \{F \textit{ is a } \Omega\text{-RKL } \textit{set}\};$$

(ii) *for each compact set $K \subset \Omega$,*

$$M_{LV}(K) = E_{LV}(K) \iff \{\Omega^* \setminus K \textit{ is connected}\};$$

(iii) *the pair $\big(L, V(\Omega)\big)$ satisfies Condition N.*

Remark 2 Our proof of (ii) ⇒ (iii) in fact shows that if for *some* compact set K in Ω there is a function $f \in L(K)$ which is not in $E_{LV}(K)$, then the same is true for some Φ_a, $a \in \hat{K} \setminus K$.

From Theorems 2 and 3, it is not difficult to obtain the corresponding approximation (reduction) theorems for *classes* of functions (jets), analogous to that of [2, Proposition 1]. In this direction, we present only the following result which extends [2, Theorem 4]. Note that (iii) is a result on tangential approximation.

Theorem 4 *Let L (of order r) satisfy property (2) of Remark 1, Ω be an arbitrary domain in $\mathbf{R}^n$ and F be a closed subset of Ω.*

(i) *For $V = BC^\rho\big(\Omega\big)$, where $\rho \in (r-1, r)$ (see Section 3), the equality $V_L(F) = M_{LV}(F)$ holds if and only if there exists a constant $A \in (0, +\infty)$ such that for each ball B in Ω*
$$M_*^{n-r+\rho}(B \setminus F^\circ) \le AM^{n-r+\rho}(B \setminus F).$$
(ii) *For $V = BC^m(\Omega)$ ($m = r, r+1, \dots$) or $V = BC^\rho(\Omega)$ ($\rho > r$, $\rho \notin \mathbf{Z}$) the equality $V_L(F) = M_{LV}(F)$ holds if and only if F° is dense in F.*
(iii) *Let L, Ω and $V = BC^m_q(\Omega)$ be as in Proposition 2, and additionally suppose that $m \ge r$. Then the equality $V_L(F) = M_{LV}(F)$ holds if and only if F° is dense in F.*
(iv) *For each space $V(\Omega)$, which is mentioned in (i), (ii) or (iii), the equality $V_L(F) = E_{LV}(F)$ holds if and only if $V_L(F) = M_{LV}(F)$ and (at the same time) F is a Ω-RKL set.*

Here $M^{n-r+\rho}(\cdot)$ and $M_*^{n-r+\rho}(\cdot)$ are the Hausdorff and lower Hausdorff *contents* of order $n - r + \rho$ respectively (*cf.* [15]).

5 Proofs of Theorems 1, 2, 3 and 4

Fix a pair $\big(L, V(\Omega)\big)$ satisfying Conditions 1 to 4, and let $k = k(L) > 1$ be the constant which appears in (3).

Lemma 1 *Let $B = B(a,\delta)$ be a ball in Ω with $6k\bar{B} \subset \Omega$ and T be a distribution with* $\operatorname{supp}(T) \subset B$. *Set $h = \Phi * T$ and let*

$$h_m = \sum_{0 \le |\alpha| \le m} c_\alpha \partial^\alpha \Phi(x-a)$$

be the partial sums of the Laurent series expansion of h outside $k\bar{B}$ (see (3)). Then there exists $M \in \mathbf{Z}_+$ such that for all $m \ge M$, one can find $v_m \in L(\Omega)$ such that $h - h_m - v_m \in V(\Omega \setminus 2kB)$ and

$$\|h - h_m - v_m\|_{\Omega \setminus 2kB} \to 0 \quad \text{as } m \to \infty.$$

Proof First recall that $h_m \to h$ in $C^\infty(\Omega \setminus k\bar{B})$. Let $\psi \in C^\infty(\mathbf{R}^n)$ such that

$$\psi = \begin{cases} 0 & \text{in a neighbourhood of } k\bar{B} \\ 1 & \text{in a neighbourhood of } \mathbf{R}^n \setminus 2kB. \end{cases}$$

Take d from Condition 4 for the ball $2kB$ and the pair (L, V). Since we have that $\psi h_m \to \psi h$ in $C^\infty(\Omega)$, there exists $M \in \mathbf{Z}_+$ such that for $m \ge M$, one has

$$h_m^* \equiv \psi(h - h_m) = O(|x|^{-d}) \quad \text{as } |x| \to \infty.$$

Using Condition 4 when $m \ge M$, we can find $v_m \in L(\Omega)$ such that $(h_m^* - v_m) \in V$ and

$$\|h_m^* - v_m\| \le C\|h_m^*\|_{6k\bar{B}} \to 0 \quad \text{as } m \to \infty.$$

By definition, $(h - h_m - v_m) \in V(\Omega \setminus 2kB)$ and

$$\|h - h_m - v_m\|_{\Omega \setminus 2kB} \le \|h_m^* - v_m\| \to 0 \quad \text{as } m \to \infty.$$

The lemma is proved. ■

Proof of Theorem 1 The proof relies on a localization technique. Let f be a function L-analytic on some neighbourhood U of F in Ω and U_1 be a neighbourhood of F, with $\bar{U}_1 \subset U$. We extend f to a function (also denoted by f) in $C^\infty(\Omega)$ so that f is still L-analytic in a neighbourhood of $\bar{U}_1$. We can find a family of couples $\{B(a_j,\delta_j), \varphi_j\}_{j=1}^\infty$ where the family of balls $\{B_j = B(a_j,\delta_j\}$ is locally finite in Ω, $6k\bar{B}_j \subset \Omega \setminus F$, each $\varphi_j \in C_0^\infty(B_j)$, with $0 \le \varphi_j \le 1$ and $\sum_{j=1}^\infty \varphi_j = 1$ on some neighbourhood U_2 of $\Omega \setminus U_1$.

Let $f_j = \mathcal{V}_{\varphi_j} f = \Phi * (\varphi_j L f)$. Each f_j is in $C^\infty(\mathbf{R}^n)$. Let $\{X_i\}$, $i \ge 1$, be the sequence of compact sets described before Proposition 3. Put $J_i = \{j \mid B_j \cap X_{i+1} \neq \varnothing\}$. Note that $L(f - \sum_{j \in J_1} f_j) = Lf - \sum_{j \in J_1} \varphi_j L f = Lf(1 - \sum_{j \in J_1} \varphi_j) = 0$ (*i.e.* $f - \sum_{j \in J_1} f_j$ is L-analytic) in X_2°. By Proposition 3, one can find $P_1 \in L(\Omega)$ such that

$$\Big\| f - \Big(\sum_{J_1} f_j\Big) - P_1 \Big\|_{X_1} < \frac{1}{2}.$$

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Now, since $f-(\sum_{J_1} f_j)-P_1-(\sum_{J_2\setminus J_1} f_j)$ is L-analytic in X_3°, there exists $P_2 \in L(\Omega)$ such that

$$\Big\| f-\Big(\sum_{J_1} f_j\Big)-P_1-\Big(\sum_{J_2\setminus J_1} f_j\Big)-P_2\Big\|_{X_2} < \frac{1}{2^2}.$$

Inductively, we can thus find $P_i \in L(\Omega)$ such that

$$\Big\| f-\Big(\sum_{J_1} f_j\Big)-P_1-\Big(\sum_{J_2\setminus J_1} f_j\Big)-P_2-\cdots-\Big(\sum_{J_i\setminus J_{i-1}} f_j\Big)-P_i\Big\|_{X_i} < \frac{1}{2^i}.$$

so that, setting $J_0=\varnothing$, the equality

$$f=\sum_{i=1}^{\infty}\Big(\sum_{J_i\setminus J_{i-1}} f_j+P_i\Big)$$

holds in $V_{\mathrm{loc}}(\Omega)$.

Now, from (3), each f_j has a Laurent series expansion

$$f_j(x)=\sum_{|\alpha|\geq 0} c_\alpha^j \partial^\alpha \Phi(x-a_j)$$

valid outside $k\bar{B}_j$, and thus on a neighbourhood of F. Using Lemma 1, given any $\eta_j>0$, there exists $m_j \in \mathbf{Z}_+$ and $\nu_j \in L(\Omega)$ such that if

$$g_j(x)=\sum_{|\alpha|=0}^{m_j} c_\alpha^j \partial^\alpha \Phi(x-a_j),$$

then $(f_j-g_j-\nu_j)\in V(\Omega\setminus 2kB_j)$ and $\|f_j-g_j-\nu_j\|_{\Omega\setminus 2kB_j}<\eta_j$.

Put $F_1=\Omega\setminus\cup_j(2kB_j)$; then $F\subset F_1^\circ$ and, for all j, $(f_j-g_j-\nu_j)\in V(F_1)$, $\|f_j-g_j-\nu_j\|_{F_1}<\eta_j$. Fix $\varepsilon>0$ and choose the sequence $\{\eta_j\}$, $\eta_j>0$, such that $\sum_j \eta_j<\varepsilon$. Define

$$g=\sum_{i=1}^{\infty}\Big(\sum_{J_i\setminus J_{i-1}}(g_j+\nu_j)+P_i\Big).$$

Since for each $m\geq 1$ the series

$$\sum_{i=m+1}^{+\infty}\Big(\sum_{J_i\setminus J_{i-1}}(g_j+\nu_j)+P_i\Big)$$

converges in $V(X_m)$, g is L-meromorphic in Ω with "poles" only (possibly) at a_j, $j=1,2,\ldots$. Moreover $g\in V_{\mathrm{loc}}(F_1)$ and

$$(f-g)_{(F_1),\mathrm{loc}}=\sum_{i=1}^{\infty}\Big(\sum_{J_i\setminus J_{i-1}}(f_j-g_j-\nu_j)_{(F_1),\mathrm{loc}}\Big).$$

But then $f - g \in V(F)$ and

$$\|f - g\|_F < \varepsilon,$$

since $(f - g)_{(F),\mathrm{loc}}$ can be defined by the element

$$\sum_{i=1}^{\infty}\Big(\sum_{J_i \setminus J_{i-1}} \Psi_j\Big),$$

where $\Psi_j \in V$ are such that $(\Psi_j)_{(\Omega\setminus 2kB_j)} = (f_j - g_j - \nu_j)_{(\Omega\setminus 2kB_j)}$ and $\|\Psi_j\| \le \eta_j$. This proves the first part of Theorem 1.

Now assume that F is a RKL-set in Ω, *i.e.* $\Omega^*\setminus F$ is connected and locally connected. It suffices to show that there exists a function $h \in L(\Omega)$ such that

$$\|g - h\|_F < \varepsilon.$$

Let $\{a_j\}_{j\ge 1}$ be the sequence of "poles" of g in Ω. Each $a_j \in \Omega \setminus F$ and the sequence has no limit points in Ω. Since $\Omega^* \setminus F$ is connected and locally connected at the "point" $*$, we can find paths σ_j from a_j to $*$, $\sigma_j \subset \Omega \setminus F$, such that the family of curves $\{\sigma_j\}$ is locally finite in Ω.

For a fixed j, we can find sequences $\{a_{j_m}\}_{m=0}^{\infty} \subset \sigma_j$ and $\{r_{j_m}\}_{m=0}^{\infty} \subset (0,1)$ such that $a_{j_0} = a_j$, $a_{j_m} \to *$ as $m \to \infty$, $|a_{j_m} - a_{j_{m+1}}| < r_{j_{m+1}}$, $B_{j_m} = B(a_{j_m}, 7kr_{j_m}) \subset \Omega\setminus F$. Additionally we can require that the family of balls $\{B_{j_m}\}$ is locally finite in Ω. If $G_j = \bigcup_{m=0}^{\infty} B_{j_m}$ then $\bar{G}_j \cap F = \varnothing$ and $\{G_j\}$ is also locally finite in Ω.

Set $h_0 = g$. We construct a sequence of functions h_j such that h_j is L-meromorphic on Ω, h_j has the same poles (and singular parts) as h_{j-1} except at a_j where h_j is L-analytic, and such that

$$\|h_{j-1} - h_j\|_{\Omega\setminus G_j} < \frac{\varepsilon}{2^j}.$$

If such a sequence exists, then $h = \lim_{j\to\infty} h_j$ is in $L(\Omega)$. Indeed, by construction (since $\{G_j\}$ is locally finite), we have $G_j \to \{*\}$ as $j \to \infty$, and thus $\{h_j\}$ is a Cauchy sequence in $V(X_i)$ for each i. Moreover convergence in $V_{\mathrm{loc}}(\Omega)$ preserves L-analyticity. Finally we would have

$$\|g - h\|_F < \varepsilon,$$

as desired.

To construct the functions h_j ($h_0 = g$), assume that h_ℓ has been constructed for $\ell \le j - 1$. Let s_0 be the singular part of h_{j-1} at $a_j = a_{j_0}$. By Lemma 1 (applied to $h = s_0$ and $a = a_{j_1}$), we can find an L-meromorphic function s_1 in Ω whose only singularity is at a_{j_1} and such that

$$\|s_0 - s_1\|_{\Omega\setminus B_{j_1}} < \Big(\frac{1}{2}\Big)\frac{\varepsilon}{2^j}.$$

By induction, construct an L-meromorphic function s_m whose only singularity is at a_{j_m} and such that

$$\|s_{m-1} - s_m\|_{\Omega\setminus B_{j_m}} < \Big(\frac{1}{2^m}\Big)\frac{\varepsilon}{2^j}.$$

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Finally set

$$h_j = h_{j-1} + \sum_{m=1}^{\infty}(s_m - s_{m-1}).$$

The function h_j has the desired properties. ∎

The proofs of Theorems 2, 3 and 4 are also very similar to the proofs of the corresponding Theorems in [2], and will not be reproduced here. We simply note that $\mathbf{R}^n$ is to be replaced everywhere by Ω, $\{\infty\}$ (of $\mathbf{R}^n_\infty$) by $\{*\}$, P_{LV} by E_{LV}, "bounded" by "precompact in Ω" and so on. Our Theorem 1 above replaces the corresponding Theorem 1 of [2]. Balls also sometimes need to be replaced by sets having appropriate properties. For example, in the proof of Theorem 1 above, balls $\bar{B}(0,i)$ were replaced by sets X_i, where, for each i, $\Omega^* \setminus X_i$ was connected. In Theorem 3, the balls $B(0,R)$, $B(0,2R)$ and $B(0,3R)$ need to be replaced respectively by Ω-precompact domains U, U_1 and U_2 such that ∂U is *smooth*, $\bar{U} \subset U_1 \subset \overline{U_1} \subset U_2$ and the union of all Ω-precompact components of $(\Omega \setminus F) \setminus \bar{U}$ is not precompact in Ω. The existence of such domains follows from the existence of an exhaustion of Ω by smooth *domains* (which are precompact in Ω) and the assumption made in the proof that $\Omega^* \setminus F$ is not locally connected (see also [5, Chapter IV, Section 2 B]). Of course, the corresponding conditions on D_m and a_m need to be changed accordingly. Moreover, the following version of [2, Lemma 5] is needed in Theorem 3.

Lemma 2 *For each open sets U_1, U_2 such that $\bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset \Omega$, there exists a positive constant A (depending only on the space V and the sets U_1 and U_2) such that for any compact set K and for each $f_{(K)} \in V(K)$ one has*

$$\|f\|_K \leq A(\|f\|_{K\cap\bar{U}_2} + \|f\|_{K\setminus U_1}).$$

We leave the details to the reader.

6 Boundary Behaviour of L-Analytic Functions

Let $\mathcal{L}^n_r$ stand for the class of all homogeneous elliptic operators of order r in $\mathbf{R}^n$ ($n \geq 2$, $r \geq 1$) with constant complex coefficients (see Section 2 above).

In this section, given $L \in \mathcal{L}^n_r$ and a domain Ω satisfying some mild conditions, we will construct in Ω solutions of the equation $Lu = 0$ having some prescribed boundary behaviour.

6.1 No Limits at the Boundary

Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, $\Omega \neq \mathbf{R}^n$, and let $b \in \partial\Omega$. We shall say that a (continuous) path $\gamma\colon [0,1] \to \mathbf{R}^n$ is *admissible for Ω with end point b* if $\gamma\colon [0,1) \to \Omega$ and $\gamma(1) = b$. Given a continuous function f in Ω, denote by $\mathcal{C}_\gamma(f)$ the *cluster set* of f along γ at b, that is:

$$\mathcal{C}_\gamma(f) = \Big\{ w \in \mathbf{C}^* \mid \text{there exists a sequence } \{t_n\} \subset [0,1) \\ \text{such that } t_n \to 1 \text{ and } f\big(\gamma(t_n)\big) \to w \text{ as } n \to \infty \Big\}.$$

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Theorem 5 *Let $L \in \mathcal{L}_r^n$, and let $\Omega \subset \mathbf{R}^n$, $\Omega \neq \mathbf{R}^n$, be a domain such that its boundary $\partial\Omega$ has no (connected) components that consist of a single point. Then there exists $g \in L(\Omega)$ with the property that for each $b \in \partial\Omega$, for each admissible path γ for Ω ending at b and for each $\alpha \in \mathbf{Z}_+^n$, one has*

$$\mathcal{C}_\gamma(\partial^\alpha g) = \mathbf{C}^*.$$

The following proposition and remark show that, at least for $L = \Delta$ in $\mathbf{R}^n$ and $L = \partial/\partial\bar{z}$ in $\mathbf{R}^2$, our theorem is close to being sharp.

Proposition 4 *If Ω is a domain in $\mathbf{R}^n$ such that $\partial\Omega$ has an isolated point $b \in \mathbf{R}^n \cup \{\infty\}$, then for each function f harmonic in Ω or (if $n = 2$) for each function f holomorphic in Ω, there exists an admissible path γ for Ω ending at b such that $\mathcal{C}_\gamma(f)$ is a single point in $\mathbf{C}^*$.*

Remark 3 It follows from Proposition 4 that for each $\alpha \in \mathbf{Z}_+^n$ there exists an admissible path γ_α for Ω ending at b such that $\mathcal{C}_{\gamma_\alpha}(\partial^\alpha f)$ is just a single point in $\mathbf{C}^*$ since the point b is also an isolated singularity of the harmonic (or holomorphic) function $\partial^\alpha f$.

Proof of Proposition 4 It is well known that if f is bounded at b (that is in some punctured neighbourhood of b), then f has a removable singularity at b and that consequently the proposition holds for every admissible path.

If f is unbounded at b, then the result follows from a generalization of a theorem of Iversen due to B. Fuglede (see [4, Corollary 1]). ∎

Lemma 3 *Let $L \in \mathcal{L}_r^n$. For each $\beta \in \mathbf{Z}_+^n$ there exists a homogeneous polynomial $P_\beta \in L(\mathbf{R}^n)$ of degree $|\beta|$ with $\partial^\beta P_\beta \equiv 1$.*

Proof The lemma is obvious if $|\beta| < r$. So let us assume that $|\beta| \geq r$. We claim that $\partial^\beta \Phi \not\equiv 0$ on $\mathbf{R}^n \setminus \{0\}$, where Φ is a special fundamental solution for L as before (see Section 2).

Assuming the claim, fix a point $a \neq 0$ where $\partial^\beta \Phi(a) \neq 0$. By Taylor's formula, we have

$$\Phi(x) = \sum_{k=0}^{\infty} Q_k(x)$$

where

$$Q_k(x) = \sum_{|\alpha|=k} \frac{\partial^\alpha \Phi(a)}{\alpha!}(x-a)^\alpha$$

belongs to $L(\mathbf{R}^n)$ (see [2, Section 2.4]). It suffices to take

$$P_\beta = \frac{Q_{|\beta|}}{\partial^\beta \Phi(a)}.$$

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To prove the claim, note that by [15, Lemma 1.1], one has in fact that

$$\partial^\beta \Phi(x) = \sum_{|\alpha|=|\beta|-r} c_\alpha \partial^\alpha \delta(x) + K(x),$$

where $\delta(\cdot)$ is the Dirac delta function, $c_\alpha \in \mathbf{C}$ and K is a Calderón-Zygmund $(n + |\beta| - r)$-dimensional kernel. Assuming that $\partial^\beta \Phi(x) = 0$ for all $x \neq 0$, then $K(x) \equiv 0$. Thus

$$(-i)^r \xi^\beta \tilde{\Phi}(\xi) = \sum_{|\alpha|=|\beta|-r} c_\alpha \xi^\alpha,$$

where $\tilde{\Phi}$ denotes the Fourier transform of Φ. On the other hand, since $L\Phi = \delta(\cdot)$, one has

$$(-i)^r L(\xi)\tilde{\Phi}(\xi) \equiv 1.$$

It follows that $\xi^\beta = A(\xi)L(\xi)$, where A is a polynomial. Choose $\eta = (\eta_1, \dots, \eta_n)$ with $\eta_j > 0$, $j = 1, \dots, n$, and fix $(\xi_2, \dots, \xi_n) = (\eta_2, \dots, \eta_n)$. We have, for all ξ_1 (after division by $\eta_2^{\beta_2} \cdots \eta_n^{\beta_n}$):

$$\xi_1^{\beta_1} = A_1(\xi_1)L_1(\xi_1),$$

where $L_1(\xi_1) = L(\xi_1, \eta_2, \dots, \eta_n)$ and $A_1(\xi_1)$ are also polynomials. The polynomial $L_1(\xi_1)$ has no zeros (on $\mathbf{R}$) and divides $\xi_1^{\beta_1}$, so that it is constant. Similarly, we can show that L is constant on each line through η which is parallel to a coordinate axis. Since this is true for each point η in the open cone $\{\eta \mid \eta_j > 0, j = 1, \dots, n\}$, we conclude that the polynomial $L(\xi)$ is constant in this cone and hence identically constant. Thus $L = L(0) = 0$, since L is homogeneous of order $r \geq 1$. This contradicts the ellipticity hypothesis, proves the claim and ends the proof of the lemma. ■

Proof of Theorem 5 Following the idea of the proof of [5, Chapter IV, Section 5, Theorem 4], we will construct a set of Carleman approximation which must be intersected infinitely often by every admissible path.

By Whitney's approximation theorem [9, Theorem 1.6.5], we can find a real analytic function Ψ on Ω such that for each $x \in \Omega$ one has

$$\frac{1}{2}\min\left(\operatorname{dist}(x, \partial\Omega), \frac{1}{|x|}\right) \leq \Psi(x) \leq 2\min\left(\operatorname{dist}(x, \partial\Omega), \frac{1}{|x|}\right) \tag{6}$$

From Sard's theorem [9, Theorem 1.4.6], we can find a sequence $\{\rho_j\}_{j=0}^{\infty}$, $\rho_j \searrow 0$ as $j \to \infty$ such that the level sets $R_j = \{x \in \Omega \mid \Psi(x) = \rho_j\}$ do not contain any critical point of Ψ, *i.e.* $\nabla\Psi \neq 0$ on R_j and R_j consists of only finitely many C^∞-smooth (in fact real analytic) hypersurfaces. Let $\Omega_j = \{x \in \Omega \mid \Psi(x) > \rho_j\}$. We additionally require (as we can) that $(\overline{\Omega_j})^\wedge \subset \Omega_{j+1}$. We define $E_j = \partial\big((\overline{\Omega_j})^\wedge\big)$ and note that E_j also consists of finitely many C^∞-smooth closed hypersurfaces which we denote $E_{j\nu}$, $1 \leq \nu \leq k_j$.

For positive but small enough δ_j, the δ_j-neighbourhood Ω_j' of $(\overline{\Omega_j})^\wedge$ is C^∞-smooth, $(\overline{\Omega_j'})^\wedge = \overline{\Omega_j'}$ and $E_j' = \partial\Omega_j'$ has the same number k_j of components $E_{j\nu}'$

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as E_j. The sequence $\{\delta_j\}$ is also chosen to satisfy $\delta_j \searrow 0$ as $j \to \infty$, $\overline{\Omega'_j} \subset \Omega_{j+1}$, $\operatorname{dist}(E'_j, E_{j+1}) \geq 2\delta_j$ and $\delta_j < \min_\nu(\operatorname{diam} E_{j\nu})/10$. Choose $a_{j\nu} \in E_{j\nu}$ and $a'_{j\nu} \in E'_{j\nu}$ such that

$$|a_{j\nu} - a'_{j\nu}| \geq \frac{\operatorname{diam}(E_{j\nu})}{2}. \tag{7}$$

Now let

$$K_j = \overline{\Omega'_j},$$

$$F_j = \bigcup_{\nu=1}^{k_j} \left\{ \left(E_{j\nu} \setminus B(a_{j\nu}, \delta_j)\right) \cup \left(E'_{j\nu} \setminus B(a'_{j\nu}, \delta_j)\right) \right\},$$

and define

$$F = \bigcup_{j=0}^{\infty} F_j.$$

For each j, we can find disjoint closed η_j-neighbourhoods G_j of F_j (with $0 < \eta_j < \delta_j/4$) such that $G_{j+1} \cap K_j = \varnothing$ and $\Omega^* \setminus (G_{j+1} \cup K_j)$ is connected.

Finally we define the function f, L-analytic in some neighbourhood of the set $G = \bigcup_{j=0}^{\infty} G_j$ as follows. For each $\beta \in \mathbf{Z}_+^n$, we can find $I_\beta \subset \mathbf{Z}_+$ such that $\bigcup_{\beta \in \mathbf{Z}_+^n} I_\beta = \mathbf{Z}_+$, each I_β contains infinitely many elements and $I_\beta \cap I_{\beta'} = \varnothing$ for $\beta \neq \beta'$. Let $\{\lambda_i^\beta\}_{i \in I_\beta}$ be a fixed sequence in $\mathbf{C}$ such that $\mathbf{C}^*$ is the set of its limit points. Now fix $j \in \mathbf{Z}_+$. Then j is in position i_j in I_β for some (unique) $\beta \in \mathbf{Z}_+^n$. Let $P_\beta \in L(\mathbf{R}^n)$ be a polynomial of degree $|\beta|$ with $\partial^\beta P_\beta \equiv 1$ (see Lemma 3), and let U_j be pairwise disjoint (open) neighbourhoods of G_j such that $\overline{U_{j+1}} \cap K_j = \varnothing$ for all j. Then define f on U_j as

$$f(x) = \lambda_{i_j}^\beta P_\beta(x).$$

We will need the following "Carleman-type" approximation lemma.

Lemma 4 *Let f and G be as above. Then for any sequence $\{\varepsilon_j\}_{j=0}^\infty$, $\varepsilon_j \searrow 0$ as $j \to \infty$, there exists $g \in L(\Omega)$ such that*

$$\|f - g\|_{0,G_j} \leq \varepsilon_j \tag{8}$$

where $\|\cdot\|_{0,E}$, as before, denotes the uniform norm on E.

Assuming the lemma, fix a sequence $\{\tau_j\}_{j=0}^\infty$, $\tau_j \searrow 0$ as $j \to \infty$. Now choose a sequence $\{\varepsilon_j\}$, $\varepsilon_j \searrow 0$ as $j \to \infty$ such that if (8) is satisfied for a function $g \in L(\Omega)$, then

$$\|\partial^\beta g - \lambda_{i_j}^\beta\|_{0,F_j} < \tau_j, \quad j \in I_\beta. \tag{9}$$

This can be done by choosing ε_j small enough, since $\partial^\beta f = \lambda_{i_j}^\beta$ on F_j.

The function g has the desired properties. Indeed, let γ be an admissible path for Ω with end point $b \in \partial\Omega$. Then we claim that $[\gamma] = \gamma([0,1])$ must intersect all F_j, except possibly finitely many of them. Combining the claim with (9) and the choice of $\{\lambda_i^\beta\}$ proves the theorem.

To prove the claim, assume that $[\gamma]$ does not intersect infinitely many F_j, say $\{F_{j_m}\}_{m=1}^\infty$ with $j_m \nearrow \infty$ as $m \to \infty$. It then follows that there exists an m_0 such that for each $m > m_0$, one can find $\nu = \nu(j_m)$ such that $[\gamma]$ intersects $B(a_{j_m\nu}, \delta_j)$ and $B(a'_{j_m\nu}, \delta_j)$ and where each $E_{j_m\nu}$ is either the outer boundary (in $\mathbf{R}^n$) of $(\overline{\Omega_{j_m}})^\wedge$ or $E_{j_m\nu}$ surrounds the point b. Notice that by (7),

$$|a_{j_m\nu} - a'_{j_m\nu}| \geq \frac{\operatorname{diam}(E_{j_m\nu})}{2} \geq 5\delta_j,$$

and thus, from the continuity of γ at b, we must have that $\operatorname{diam}(E_{j_m\nu}) \to 0$ as $j_m \to \infty$. But this is impossible. In fact, if $E_{j_m\nu}$ is the boundary of the unbounded component of $(\overline{\Omega_{j_m}})^\wedge$, then $\operatorname{diam}(E_{j_m\nu}) = \operatorname{diam}(\Omega_{j_m})$ which grows with m, so that all but a finite number of $E_{j_m\nu}$ must be "inner" components of the boundary of $(\overline{\Omega_{j_m}})^\wedge$ which surround the component of the boundary of Ω containing b. But our assumption on the boundary of Ω also makes this impossible. This proves the claim and completes the proof of Theorem 5. ∎

Proof of Lemma 4 Lemma 4 is a consequence of a rather general theorem of A. Sinclair [13, Theorem 1], but we include the following relatively simple proof for the reader's convenience.

Let $\{\varepsilon'_k\}_{k=0}^\infty$ be the sequence of positive numbers satisfying $\varepsilon_j = \sum_{k\geq j} \varepsilon'_k$. Since G_0 is an Ω-RKL set and $f \in L(U_0)$, then by Theorem 1, one can find $g_0 \in L(\Omega)$ with

$$\|f - g_0\|_{0,G_0} \leq \varepsilon'_0.$$

Let U'_j be a neighbourhood of K_j such that $U'_j \cap U_{j+1} = \varnothing$. Define

$$f_1(x) = \begin{cases} g_0(x), & x \in U'_0 \\ f(x), & x \in U_1. \end{cases}$$

Since $K_0 \cup G_1$ is a RKL-*set* in Ω and $f_1 \in L(U'_0 \cup U_1)$, we can find $g_1 \in L(\Omega)$ such that

$$\|f_1 - g_1\|_{0,K_0\cup G_1} \leq \varepsilon'_1.$$

Inductively, for $j \geq 1$, we define

$$f_{j+1}(x) = \begin{cases} g_j(x), & x \in U'_j \\ f(x), & x \in U_{j+1}, \end{cases}$$

and choose $g_{j+1} \in L(\Omega)$ such that

$$\|f_{j+1} - g_{j+1}\|_{0,K_j\cup G_{j+1}} \leq \varepsilon'_{j+1}.$$

Since $K_j \nearrow \Omega$, we have that

$$g = \lim_{j\to\infty} g_j \quad \bigl(\in L(\Omega)\bigr)$$

satisfies the lemma. ∎

6.2 A Dirichlet Problem

Our next example is in some sense in the opposite direction of the first one. Given a (smooth) domain Ω, we would like to prescribe (almost everywhere on $\partial\Omega$) the boundary values of an L-analytic function in Ω, together with the boundary values of a fixed number of its derivatives, as we approach the boundary of Ω in the normal direction (a "weakened" Dirichlet problem).

We first prove an abstract Carleman-type approximation theorem when F is without interior.

Proposition 5 *Let $L \in \mathcal{L}_r^n$, Ω be a domain in $\mathbf{R}^n$ and let $V = V(\Omega)$ be a Banach space such that the pair (L, V) satisfies Conditions 1 and 2. Let F be a closed subset of Ω with $F^\circ = \varnothing$ and assume that there exists an exhaustion of Ω by compact sets K_j (that is, $K_0 = \varnothing$, $K_j \subset K_{j+1}^\circ$ and $\bigcup_{j=0}^{\infty} K_j = \Omega$) which is "compatible" with F in the sense that for each $j \geq 0$, one has*

$$V_L\big(K_j \cup (K_{j+2} \cap F)\big) = E_{LV}\big(K_j \cup (K_{j+2} \cap F)\big). \tag{10}$$

Then for each sequence $\{\varepsilon_j\}_{j=0}^{\infty}$, $\varepsilon_j \searrow 0$ as $j \to \infty$ and for each $f \in V_{\text{loc}}(F)$, one can find $g \in L(\Omega)$ such that, for all $j \geq 0$,

$$\|f - g\|_{F \setminus K_j^\circ} < \varepsilon_j.$$

Proof Fix $\{\delta_j\}_{j=0}^{\infty} \subset (0, \infty)$, with $\sum_{j=0}^{\infty} \delta_j < \infty$. Let $g_0 = f$. For each $j \geq 1$, we shall find $g_j \in V_{\text{loc}}(\Omega) \cap L(K_j)$ such that

$$\|g_{j-1} - g_j\|_{K_{j-1}} < \delta_{j-1}, \tag{11}$$

and

$$\|g_{j-1} - g_j\|_{F \setminus K_k^\circ} < \frac{\varepsilon_k}{2^j} \quad \text{for each } k \geq 0. \tag{12}$$

Letting $g = \lim_{j\to\infty} g_j = g_0 + \sum_{j=1}^{\infty}(g_j - g_{j-1})$ will give the result.

First, for each $j \geq 1$, fix $\varphi_j \in C_0^\infty(K_{j+1}^\circ)$, $0 \leq \varphi_j \leq 1$ and $\varphi_j \equiv 1$ on some neighbourhood of K_j. We now proceed by induction on j. By (10) with $j = 0$, we can find $h_1 \in L(\Omega)$ such that

$$\|g_0 - h_1\|_{K_2 \cap F} < \mu_1,$$

where $\mu_1 \in (0, \infty)$ will be specified below. Let

$$g_1 = h_1\varphi_1 + g_0(1 - \varphi_1).$$

Then $g_1 \in V_{\text{loc}}(\Omega) \cap L(K_1)$, and it follow from Condition 1 that

$$\|g_0 - g_1\|_F = \|(g_0 - h_1)\varphi_1\|_F \leq C(\varphi_1)\|g_0 - h_1\|_{K_2 \cap F} < C(\varphi_1)\mu_1$$

and

$$\|g_0 - g_1\|_{F\setminus K_2^\circ} = 0.$$

Consequently, (11) and (12) hold for $j = 1$ if $C(\varphi_1)\mu_1 \le \varepsilon_1/2$. Note that (11) is an empty condition at this stage since K_0 is the empty set.

Suppose now that we have found $g_0, \dots, g_J$ such that (11) and (12) hold for $1 \le j \le J$. By (10) with $j = J$, one can finds $h_{J+1} \in L(\Omega)$ such that

$$\|g_J - h_{J+1}\|_{K_J \cup (K_{J+2} \cap F)} < \mu_{J+1}, \tag{13}$$

where μ_{J+1} is a small positive constant to be chosen later. Let

$$g_{J+1} = h_{J+1}\varphi_{J+1} + g_J(1 - \varphi_{J+1}).$$

Then

$$\|g_J - g_{J+1}\|_{K_J} = \|(g_J - h_{J+1})\varphi_{J+1}\|_{K_J} = \|g_J - h_{J+1}\|_{K_J} < \mu_{J+1},$$

which gives (11) (with $j = J + 1$) whenever $\mu_{J+1} \le \delta_J$. Since $\|g_J - g_{J+1}\|_{F\setminus K_{J+2}} = 0$, it is enough, in order to get (12), to require that

$$\|g_J - g_{J+1}\|_F < \frac{\varepsilon_{J+1}}{2^{J+1}}.$$

But this follows from (13) and Condition 1 if μ_{J+1} is small enough. Indeed,

$$\begin{aligned}\|g_J - g_{J+1}\|_F = \|(g_J - h_{J+1})\varphi_{J+1}\|_F &\le C(\varphi_{J+1})\|g_J - h_{J+1}\|_{F\cap K_{J+2}}\\ &< C(\varphi_{J+1})\mu_{J+1},\end{aligned}$$

and thus it suffices to take $\mu_{J+1} = \min\Big(\delta_J, \varepsilon_{J+1}/\big(2^{J+1}C(\varphi_{J+1})\big)\Big)$. This completes the proof. ■

We shall also need the following lemma.

Lemma 5 *For $0 < d < 1$, denote by $Q'_d = [-d, d]_{y_1} \times [-d, d]_{y_2} \times \cdots \times [-d, d]_{y_{n-1}}$ the $n - 1$ dimensional closed cube centered at zero in $\mathbf{R}^{n-1}$ and let $Q_d = Q'_d \times [0, 2d]_{y_n}$. Let $s \in \mathbf{Z}_+$ be fixed. Given $h_0, \dots, h_s \in C(Q'_d)$, there exists a function $H \in C^\infty\big(Q_d \setminus (Q'_d \times \{0\})\big) \cap C(Q_d)$ such that, if $y' = (y_1, y_2, \dots, y_{n-1})$, then*

$$\frac{\partial^k H}{\partial y_n^k}(y', y_n) \to h_k(y') \tag{14}$$

uniformly on Q'_d as $y_n \to 0$, $0 \le k \le s$.

Remark 4 We first note that (14) and the mean-value theorem implies that the one-sided derivatives at zero exist and

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$$\frac{\partial^k H}{\partial y_n^k}\Big|_{(y',0+)} = h_k(y').$$

Remark 5 The lemma is easily proved if we assume that $h_0, h_1, \dots, h_s \in C^\infty(Q'_d)$ since in this case it suffices to take

$$H(y', y_n) = \sum_{k=0}^{s} \frac{y_n^k}{k!} h_k(y').$$

The proof of the general case is an adaptation of this idea using approximation and a partition of unity.

Proof of Lemma 5 Let $\{\varphi_j\}$, $j = 2, 3, \dots$, $\varphi_j \in C^\infty(\mathbf{R})$ such that $\operatorname{supp}(\varphi_j) \subset (\frac{1}{j+1}, \frac{1}{j-1})$, $0 \le \varphi_j \le 1$, and $\sum_{j=2}^\infty \varphi_j \equiv 1$ on $(0, 1/2)$. Let $\|\varphi_j^{(k)}\|_0 =: \lambda_{kj}$ and $M := \max_{0 \le k \le s} \|h_k\|_{0,Q'_d}$. Let $\{\varepsilon_j\}_{j=2}^\infty \subset (0,1)$ be a sequence of decreasing numbers tending to zero. By the Weierstrass approximation theorem in several variables, for each k and j, $0 \le k < s$ and $j = 2, 3, \dots$, we can find $h_{kj} \in C^\infty(Q')$ (in fact polynomials) such that

$$\|h_{kj} - h_k\|_{0,Q'_d} < \varepsilon_j.$$

We claim that the function

$$H(y', y_n) = \sum_{k=0}^{s} \sum_{j=2}^{\infty} \frac{y_n^k}{k!} h_{kj}(y')\varphi_j(y_n), \quad \text{when } y_n > 0,$$
$$H(y', 0) = h_0(y')$$

has the desired properties whenever the sequence $\{\varepsilon_j\}$ is chosen to satisfy $\sum_{j \ge 2} \varepsilon_j \lambda_{kj} < \infty$, for each k, $0 \le k < s$. Indeed let us assume that $0 < y_n < \frac{1}{j_0+1} < 1/2$. Then

$$|H(y', y_n) - h_0(y')| = \Big|\sum_{j=2}^{\infty} (h_{0j}(y') - h_0(y')\varphi_j(y_n) + \sum_{k=1}^{s} \sum_{j=2}^{\infty} \frac{y_n^k}{k!} h_{kj}(y')\varphi_j(y_n)\Big|$$
$$\le 2\varepsilon_{j_0} + (M+1) \sum_{k=1}^{s} \frac{y_n^k}{k!},$$

and thus $|H(y', y_n) - h_0(y')| \to 0$ uniformly as $y_n \to 0$. Similarly, since

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$\sum_{j\geq 2}\varphi_j'(y_n) = 0$, $0 < y_n < 1/2$, we have

$$\begin{aligned}\Big|\frac{\partial H}{\partial y_n}(y', y_n) - h_1(y')\Big| &= \Big|\sum_{k=1}^{s}\sum_{j=2}^{\infty}\frac{y_n^{k-1}}{(k-1)!}h_{kj}(y')\varphi_j(y_n)\\ &\qquad + \sum_{k=0}^{s}\sum_{j=2}^{\infty}\frac{y_n^k}{k!}h_{kj}(y')\varphi_j'(y_n) - \sum_{j=2}^{\infty}h_1(y')\varphi_j(y_n)\Big|\\ &\leq \Big|\sum_{j=2}^{\infty}\big(h_{1j}(y') - h_1(y')\big)\,\varphi_j(y_n)\Big|\\ &\qquad + \Big|\sum_{k=2}^{s}\sum_{j=2}^{\infty}\frac{y_n^{k-1}}{(k-1)!}h_{kj}(y')\varphi_j(y_n)\Big|\\ &\qquad + \Big|\sum_{k=0}^{s}\sum_{j=2}^{\infty}\frac{y_n^k}{k!}\big(h_{kj}(y') - h_k(y')\big)\,\varphi_j'(y_n)\Big|\\ &\leq 2\varepsilon_{j_0} + (M+1)\sum_{k=2}^{s}\frac{y_n^{k-1}}{(k-1)!} + \sum_{k=0}^{s}\sum_{j\geq j_0}\frac{y_n^k}{(k)!}\varepsilon_j\lambda_{1j},\end{aligned}$$

assuming that $0 < y_n < \frac{1}{j_0+1}$. Thus $|\frac{\partial H}{\partial y_n}(y', y_n) - h_1(y')| \to 0$ uniformly as $y_n \to 0$. The proof of the other cases is very similar. ∎

Theorem 6 *Let $L \in \mathcal{L}_r^n$ and let Ω be a domain of class C^{r+1} in $\mathbf{R}^n$. Let h_k, $k = 0, 1, \dots, r-1$, be σ-measurable functions which are finite σ-almost everywhere, where σ is the $n-1$ dimensional Lebesgue measure on $\partial\Omega$. Then there exists $h \in L(\Omega)$ such that, for $k = 0, \dots, r-1$, and for σ-almost all $x \in \partial\Omega$, the limit of $\frac{\partial^k h}{\partial \vec{n}_x^k}(y)$ is equal to $h_k(x)$, where the derivatives are taken in the direction of the outer normal at x, and $y \in \Omega$ tends to $x \in \partial\Omega$ along that normal direction.*

Proof We will begin the proof by constructing a special family of C^r-diffeomorphisms from n-dimensional closed cubes into $\bar{\Omega}$. We will use the notations introduced in Lemma 5. Fix a point b on the boundary of Ω and choose an (orthonormal) coordinate system $y = (y_1, \dots, y_n)$ such that $y(b) = 0$ and for some $\delta > 0$ there is $\psi \in C^{r+1}(Q_\delta')$ with $\psi(0') = 0$, $\frac{\partial\psi}{\partial y_k}\Big|_{0'} = 0$ $(k = 1, 2, \dots, n-1)$ such that

$$\{y \mid y = (y', y_n) \in \partial\Omega, y' \in Q_\delta', |y_n| < 2\delta\} = \{y \mid y_n = \psi(y'), y' \in Q_\delta'\}.$$

Moreover we suppose that

$$\{y \mid \psi(y') < y_n < 2\delta, y' \in Q_\delta'\} \subset \Omega.$$

Let us define $\Psi\colon Q_\delta' \times \mathbf{R} \to \mathbf{R}^n$ by:

$$\Psi(y', y_n) = \big(y', \psi(y')\big) - y_n\vec{n}_{\tilde{y}}.$$

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Here $\vec{n}_{\tilde{y}}$ denotes the outer normal (unit) vector to $\partial\Omega$ at the point $\tilde{y} = \big(y', \psi(y')\big)$. The Jacobian of Ψ at the origin is the identity. By the inverse mapping theorem, there exists d, $0 < d < \delta$, such that Ψ is a C^r-diffeomorphism of Q_d on $\Psi(Q_d)$ and such that $\Psi(Q_d) \subset \bar{\Omega}$.

Using the fact that $\partial\Omega$ is compact, we now choose a finite family of maps Ψ_ν and closed cubes $Q_{(\nu)} := Q_{d_\nu} = Q'_{d_\nu} \times [0, 2d_\nu] =: Q'_{(\nu)} \times [0, 2d_\nu]$ such that $\Psi_\nu|_{Q_{(\nu)}}$ is a C^r-diffeormophism, $\Psi_\nu(Q'_{(\nu)} \times \{0\}) \subset \partial\Omega$, $\Psi_\nu\big(Q_{(\nu)} \setminus (Q'_{(\nu)} \times \{0\})\big) \subset \Omega$ and $\partial\Omega \subset \cup_\nu \Psi_\nu(U_{(\nu)} \times \{0\})$, where $U_{(\nu)} := (-d_\nu, d_\nu)_{y_1} \times \cdots \times (-d_\nu, d_\nu)_{y_{n-1}}$.

Let $h_0, \dots, h_{r-1}$ be any r σ-measurable functions defined and σ-finite almost everywhere on $\partial\Omega$. We can construct a family $\{E_m\}_{m=1}^\infty$, $E_m \subset \partial\Omega$ with the following properties:

a) The sets E_m, $m = 1, 2, \dots$, are compact, pairwise disjoint, nowhere dense subsets of $\partial\Omega$ with $\sigma(E_m) \neq 0$.
b) For each $k \in \{0, 1, \dots, r-1\}$ and $m \in \{1, 2, \dots\}$, we have $h_k \in C(E_m)$.
c) $\sigma\big(\partial\Omega \setminus (\cup_m E_m)\big) = 0$.
d) For each $m \in \{1, 2, \dots\}$, there exists ν_m such that $\Psi_{\nu_m}^{-1}(E_m) \subset (U_{(\nu_m)} \times 0)$ where Ψ_{ν_m} belongs to the finite family of diffeomorphisms chosen above.
e) For some fixed $\mu \in (0, 1)$ and for each $m \in \{1, 2, \dots\}$ there is a $c > 0$ such that for any $x \in E_m$ and $\varepsilon < d_{\nu_m}$ one has

$$M^{n-2+\mu}\Big(\big\{ B(x, \varepsilon) \cap \Psi_{\nu_m}(Q'_{(\nu_m)} \times \{0\})\big\} \setminus E_m\Big) \geq c\varepsilon^{n-2+\mu}, \tag{15}$$

where M^λ denotes the λ-dimensional Hausdorff *content*.

For example, the first three properties are obtained using Lusin's theorem [12, Theorem 2.24], and the fourth follows easily. In order to have additionally property (e), we use the following lemma, taking products of the set E from this lemma with $n-2$-dimensional closed cubes which gives an $n-1$-dimensional analog of the lemma, that is (15).

Lemma 6 *For each $\mu \in (0, 1)$ and $\eta > 0$, there exist a compact set $E \subset [0, 1]$ and a constant $c > 0$ (independent of η) such that $M^1(E) > 1 - \eta$ and for each $t \in \mathbf{R}$ and each $\varepsilon > 0$, one has*

$$M^\mu\Big(\big\{\tau \;\big|\; |\tau - t| < \varepsilon\big\} \setminus E\Big) \geq c\varepsilon^\mu.$$

Proof Fix μ and η. It is well known (see [8, Section 4.10] and use the fact that a Hausdorff *measure* and the *corresponding* Hausdorff *content* have the same zero sets) that there exists a Cantor-type set $K \subset [0, 1]$ with $M^1(K) = 0$ and $M^\mu(K) > 0$. For $m \in \mathbf{Z}_+$ and $j \in \{0, \dots, 2^m - 1\}$, define $K_m^j = \{(\tau + j)2^{-m} \mid \tau \in K\}$. Since $M^1(K_m^j) = 0$, there are open sets U_m^j containing K_m^j with $M^1(U_m^j) < \eta 2^{-2m-1}$. It suffices to take (as can be easily checked)

$$E = [0, 1] \setminus \bigcup_{m=0}^{\infty} \bigcup_{j=0}^{2^m-1} U_m^j. \qquad \blacksquare$$

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We now return to the proof of Theorem 6. Given $\{E_m\}$ as above, define

$$F_1 = \Psi_{\nu_1}\left(\Psi_{\nu_1}^{-1}(E_1) \times (0, \delta_1]\right),$$

where $0 < \delta_1 \leq d_{\nu_1}$, and for $m \geq 2$,

$$F_m = \Psi_{\nu_m}(\Psi_{\nu_m}^{-1}(E_m) \times (0, \delta_m]),$$

where $0 < \delta_m \leq \min\{d_{\nu_m}, \delta_{m-1}/2\}$ is so small that F_m is disjoint from $F_1 \cup \cdots \cup F_{m-1}$ and $\{F_m\}$ is a locally finite family in Ω.

Let $F = \bigcup_{m=1}^{\infty} F_m$. We note that F is a (relatively) closed Ω-RKL set with no interior. Let $G_m = \Psi_{\nu_m}^{-1}(E_m)$ and $h_{k,m}^{\star}(y') = h_k\left(\Psi_{\nu_m}(y', 0)\right)$ and note that $h_{k,m}^{\star}$ is (defined and) continuous on G_m. We extend $h_{k,m}^{\star}$ continuously to all of $Q'_{(\nu_m)}$ and still denote this extension by $h_{k,m}^{\star}$. Using Lemma 5 with $s = r-1$, for each $m \geq 1$, there exist functions $H_m^{\star} \in C^{\infty}\left(Q_{(\nu_m)} \setminus (Q'_{(\nu_m)} \times \{0\})\right) \cap C(Q_{(\nu_m)})$ such that for each k, $0 \leq k \leq r-1$,

$$\frac{\partial^k H_m^{\star}}{\partial y_n^k}(y', y_n) \to h_{k,m}^{\star}(y')$$

uniformly on $Q'_{(\nu_m)}$ as $y_n \to 0^+$. Define H_m in $C^r\left(\Psi_{\nu_m}(Q^{\circ}_{(\nu_m)})\right)$ by $H_m(x) = H_m^{\star}\left(\Psi_{\nu_m}^{-1}(x)\right)$.

From our construction, it follows that one can choose (open) neighbourhoods Ω_m of F_m such that the sets Ω_m are still pairwise disjoint and $\Omega_m \subset \Psi_{\nu_m}(Q^{\circ}_{(\nu_m)})$. Define

$$f|_{\Omega_m} = H_m|_{\Omega_m}.$$

If V is the space $BC^{r-1+\mu}(\Omega)$ then $f \in V_{\mathrm{loc}}(F)$ (note that f can be extended from (possibly smaller) neighbourhoods Ω'_m of F_m to a function in $V_{\mathrm{loc}}(\Omega)$).

It follows also from our construction of F (recalling (15)) that there exists an exhaustion of Ω by compact sets K_j such that

1) each $Y_j = K_j \cup (K_{j+2} \cap F)$ is an Ω-RKL set;
2) for each Y_j, there exists a constant $c_j = c(Y_j) > 1$ such that for all balls $B(x, \varepsilon) \subset \Omega$ we have

$$c_j M^{n-1+\mu}(B(x,\varepsilon) \setminus Y_j) \geq \varepsilon^{n-1+\mu} \geq M_*^{n-1+\mu}(B(x,\varepsilon) \setminus Y_j^{\circ}).$$

It then follows from Theorem 4((i) and (iv)) that $V_L(Y_j) = M_{LV}(Y_j) = E_{LV}(Y_j)$. Thus by Proposition 5, one can find $h \in L(\Omega)$ such that

$$\|f - h\|_{F \setminus K_j^{\circ}} < \frac{1}{j}.$$

The function h has the desired properties.

It can be proved that Theorem 6 remains true if we require only C^r-smoothness of $\partial\Omega$.

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