Can. J. Math., Vol. XXXII, No. 2, 1980, pp. 395–413

GENERALIZED RIEMANN INTEGRATION AND AN INTRINSIC TOPOLOGY

RALPH HENSTOCK

In generalized Riemann integration theory it is becoming increasingly clear that a particular collection of sets has some properties of a topology; it is a useful topology when general requirements hold, and the present paper examines the background. Thomson [**23**, **24**] altered my original theory of the variation and Riemann-type integration that has Lebesgue properties, defining the variation of a function of interval-point pairs over the whole of a space T by using partial divisions of T instead of divisions covering T entirely, and also defining a Lebesgue-type integral. His reason might have been that a decomposable division space seems impossible in a general compact or locally compact space. McGill mentioned this to me, and in [**15**] connected Thomson's setting with topological measure and Topsøe [**25**], giving an interesting theorem on the variation of the limit of a monotone increasing generalized sequence of open sets. The variation is different from Topsøe's content in the vital sense that the content is defined for special sets and has to be extended to other sets, whereas the variation is defined for all sets. However, Topsøe's theory and other extension theories such as Choksi [**1**], Kisyński [**13**], might be useful in finding properties of the variation, connections with other definitions, and proofs of uniqueness.

McGill did not wish to look for an analogue of his theory using an integration structure rather than an imposed topological structure. But, as this paper shows, a theory avoiding additivity can give deep results on generalized sequences of integrals and variations, and τ-smoothness, together with a useful intrinsic topology. This seems to be the best way of studying integration theory by way of divisions and generalized limits of sums.

1. The basic definitions for generalized Riemann integration. Following [**5**], [**6**] gave an axiomatic theory and [**7**] a special case, independently of the paper of Kurzweil [**14**], which gave the special case without looking into the theory. As [**8**] needed the general case, on J. J. McGrotty's suggestion [**8**] was given an arrangement like a Lebesgue measure space, namely, the division space. Most properties of such spaces are natural, while additivity is useful but not always necessary. We add to the definitions in [**11**] the definitions of co-divisional partial sets and $A|E$.

In the base space $\mathscr{T}$ of points we use a collection T of some non-empty subsets I, called (generalized) *intervals*. We associate points $x \in T$ with the $I \in \mathscr{T}$, using a

Received May 23, 1978 and in revised form December 18, 1978.

fixed collection U^1 of interval-point pairs (I, x). An *elementary set* E is an interval or a union of a finite number of mutually disjoint intervals. A subfamily $U \subseteq U^1$ *divides* E if, for a finite subfamily $\mathscr{E} \subseteq U$, called a *division* of E from U, the $(I, x) \in \mathscr{E}$ have mutually disjoint I that have union E. These I are called *partial intervals* of E. A non-empty subset of $\mathscr{E}$, including $\mathscr{E}$ itself, is called a *partial division* of E, and the union of the corresponding I is called a *partial set* P of E that comes from $\mathscr{E}$ and U, and P is *proper* if $P \neq E$. If there are mutually disjoint partial sets $P_1, \ldots, P_n$ formed from various I with $(I, x) \in \mathscr{E}$, then $P_1, \ldots, P_n$ are called *co-divisional*, and the union of these sets is also a partial set. A division $\mathscr{E}_0$ of E is a *refinement* of a division $\mathscr{E}$ of E if there is a division of each I with $(I, x) \in \mathscr{E}$, formed of those $(J, t) \in \mathscr{E}_0$ with $J \subseteq I$. We write $\mathscr{E}_0 \leqq \mathscr{E}$ to correspond to $J \subseteq I$.

We use Moore-Smith limits (usually stronger than refinement limits) along a direction in the particular family A of subsets $U \subseteq U^1$ employed; in lectures I have called this the direction "as U shrinks". Choices of A give many special integrals such as Riemann, Lebesgue, special and general Denjoy, and approximate Perron integrals. The referee has pointed out that, once an elementary set E has been chosen, the theory ignores any $(I, x) \in U^1$ for which I is not a partial interval of E. Thus we write

$$U.E = \{(I, x) \in U: I \text{ a partial interval of } E\},$$

$$A|E = \{U.E: U \in A,\ U \text{ divides } E\}.$$

The notation $A|E$ is borrowed from number theory where $b|c$ denotes that the integer b divides the integer c. A is directed (in the sense of divisions of elementary sets E) if, given $U_1, U_2 \in A|E$, there is $U_3 \in A|E$ with $U_3 \subseteq U_1 \cap U_2$. This is the direction as U shrinks. If $A|E$ is not empty then A divides E, and if the two properties hold for E, $(T, \mathscr{T}, A)$ is called a division system for E.

If U divides E, a restriction of U to a partial set P is a family $U_1 \subseteq U.P$. If, for each elementary set E, each partial set P, and each $U \in A|E$, there is in $A|P$ a restriction of U to P, we say that A has the *restriction* property. If this holds for A and if $(T, \mathscr{T}, A)$ is a division system for all elementary sets, we call $(T, \mathscr{T}, A)$ a *non-additive division space*.

If also A is additive (i.e. given disjoint elementary sets E_j, and $U_j \in A|E_j$ $(j = 1, 2)$, there is $U \in A|E_1 \cup E_2$ with $U \subseteq U_1 \cup U_2$) then $(T, \mathscr{T}, A)$ is called a *division space*. This is the additivity we avoid.

Two small but important results are inserted here. $(T, \mathscr{T}, A)$ is called *infinitely divisible* if each elementary set contains a proper partial set.

LEMMA 1. *If A has the restriction property with E an elementary set, P a partial set, and $U \in A|E$, then U divides P. If $(T, \mathscr{T}, A)$ is also infinitely divisible and N any integer, there is a division of E from U containing at least N interval-point pairs (I, x).*

Proof. As a restriction of U to P divides P, U itself divides P. For P proper in the second result, $E \backslash P$ is a partial set and U divides P and $E \backslash P$. Thus there is a division of E from U with two or more (I, x). For each such I there is similarly a division from U having two or more (J, t), and so on.

To obtain Lebesgue-type limit theorems, for $X \subseteq T$ and $U \subseteq U^1$ we define

$$U[X] = \{(I, x): (I, x) \in U, x \in X\}.$$

$(T, \mathscr{T}, A)$ is *fully decomposable* (respectively, *decomposable*) if to every elementary set E, every family (respectively, countable family) $\mathscr{X}$ of mutually disjoint subsets X of T, and every function $U(.):\mathscr{X} \to A|E$, there is $U \in A|E$ with $U[X] \subseteq U(X)$ $(X \in \mathscr{X})$.

The intrinsic topology is built up using star-sets. If E is an elementary set let $E^*(U)$ be the set of all x with $(I, x) \in U.E$, and define the star-set E^* as the intersection

$$E^* = \cap \{E^*(U): U \in A|E\}.$$

Then for P a proper partial set of E, the frontier star-set $F(E; P)$ is $P^* \cap (E \backslash P)^*$. If to each E there is a $U(E) \in A|E$ such that every $U_1 \in A|E$ with $U_1 \subseteq U(E)$, has $E^*(U_1) = E^*(U(E))$, we say that $(T, \mathscr{T}, A)$ is *stable*. Here the referee has extended my definition of E^*, originally restricted to stable $(T, \mathscr{T}, A)$.

In a topological T_3-space T there is a fully decomposable stable non-additive division space. Let $\bar{X}$, X^0 be the closure and interior of $X \subseteq T$, respectively, with H the family of non-empty compact sets. Let the generalized intervals be those $I = X \backslash Y$ $(X, Y \in H)$ with I^0 non-empty. Let A be the family of all U defined by an elementary set E and a function $J: T \to \mathscr{T}$ with $x \in J(x)^0$, such that U contains all (K, x) with $x \in \bar{E} \cap \bar{K}$ and $K \subseteq J(x) \cap E$. Then A divides each E: we show that A divides each interval $I = X \backslash Y$. As $\bar{I} \subseteq \bar{X} = X \in H$, $\bar{I}$ is compact and the union of a finite number $J(x_1)^0, \ldots, J(x_n)^0$ contains $\bar{I}$ with each $x_j \in \bar{I}$. As T is a T_3-space there are open sets G_j containing x_j, with disjoint $\bar{G}_j \subseteq J(x_j)^0$, and if

$$G_j^+ = J(x_j)^0 \backslash \cup_{k \neq j} \bar{G}_k \; (j = 1, \ldots, n),$$

the G_j^+ are open neighbourhoods of the x_j with union containing $\bar{I}$, such that $G_j^+ \cap \bar{G}_k$ is empty for $k \neq j$. The following mutually disjoint I_j form the division of $X \backslash Y$.

$$I_1 = X \cap G_1^+ \backslash Y, \; I_j = X \cap G_j^+ \backslash (G_1^+ \cup G_2^+ \cup \ldots \cup G_{j-1}^+ \cup Y) \quad (j = 2, \ldots, n),$$

each $I_j \in T$ and lies in $X \backslash Y$, which is their union. The other properties are now easily shown. This construction is better than that of [**8**, pp. 224–225, Ex. 43.14].

But A need not be additive. For example, if T is the real line with the usual topology, and $a < b < c$, then $[a, b)$ and $[b, c)$ are disjoint intervals with b in their closures. A U for $[a, c)$ includes $([u, v), b)$ for various $u < b < v$, and these interval-point pairs do not lie in $\{U.[a, b)\} \cup \{U.[b, c)\}$. Similar situations occur in other examples, particularly those that are not Cartesian products of the real line.

If the generalized intervals are to separate all points we take T locally compact. For closed non-compact sets F we use an arrangement reminiscent of the Alexandroff one-point compactification as in [**7**, pp. 115–118, Section 47], [**8**, pp. 221–222, Ex. 43.4]. For, taking elementary sets $E \subset F$, we let the function value for $(F\backslash E, x)$ be 0, arranging the U so that in some sense E tends to F as U shrinks, while E is divided by U. i.e., the limit L is as follows. Given $\epsilon > 0$, there exist an elementary set $E \subset F$ and a $U \in A$ with a restriction to E_1 belonging to $A|E_1$, for all E_1 in $E \subseteq E_1 \subset F$, such that

$$|(\mathscr{E}) \sum h(I, x) - L| < \epsilon$$

for all $\mathscr{E}$ over such E_1 and from U. Compare the definition of integral given later.

Similarly we have a division system for Thomson's, see [**23**, **24**]. Let $\mathscr{T}_0$ consist of certain $I \subseteq T$, let $\mathscr{T}_1$ consist of all non-empty complements $\bigcup_{j=1}^n I_j$ for mutually disjoint $I_j \in \mathscr{T}_0$, and let $\mathscr{T} = \mathscr{T}_0 \cup \mathscr{T}_1$. If h is a function of interval-point pairs with $h(K, x) = 0$ for all $K \in \mathscr{T}_1 \backslash \mathscr{T}_0$ then Thomson's variation is ours with $F = T$ and E the union of a finite number of mutually disjoint $I \in \mathscr{T}_0$. McGill [**15**] puts a topology $\mathscr{G}$ in Thomson's system such that all $I \in \mathscr{T}_0$ are closed, and uses neighbourhood functions $N: T \rightarrow \mathscr{G}$ with $x \in N(x)$ for all $x \in T$, such that if $I \subseteq N(x)$ then $(I, x) \in U$. This is McShane's system [**16**, pp. 37–39, Example 4]. Given $I \in \mathscr{T}_0$, we can choose N so that $N(x) \subseteq \backslash I$ when $x \notin I$, so that if P is a finite union of disjoint sets of $\mathscr{T}_0$ then $P^* = \bar{P} = P$ and the system is stable. If P_1, P_2 are two such disjoint sets then $P_1^* \cap P_2^*$ is empty. This $(T, \mathscr{T}, A)$ is compatible with $\mathscr{G}$, i.e., if $G \in \mathscr{G}$, there is $U_G \in A$ such that if $(I, x) \in U_G$ and $x \in G$ then $I \subseteq G$. For we need only choose $N(x) \subseteq G$ when $x \in G$. McGill then follows Topsøe [**25**], with new proofs.

Here we first remove the imposed topology by defining a Q-set to be a set Q that has a $U_Q \in A$ associated with it, such that $(I, x) \in U_Q$ and $x \in Q$ imply $I \subseteq Q$. Thus T and the empty set are trivially Q-sets. Instead of assuming P closed we assume that if P is the finite union of disjoint intervals from $\mathscr{T}_0$ then $\backslash P^*$ is a Q-set. Next, we observe that unions of the $\backslash P^*$ (and, for one theorem here, the $\backslash(E\backslash P)^*$) are the only Q-sets really necessary in the theory. Thus the following assumptions contain McGill's as a special case, and they are used in Section 5 and subsequently.

If E is an elementary set and P a proper partial set of E from $\mathscr{T}_0$, we say that the non-additive division space $(T, \mathscr{T}, A)$ is *weakly compatible* with P when

(W) there is $U\{P\} \in A|E$ such that $(I, x) \in U\{P\}$ and $x \notin P^*$ imply $I \subseteq E\backslash P$.

The referee says that the space is then stable, for $P^*(U\{P\}) \subseteq P^*$; and also

https://doi.org/10.4153/CJM-1980-031-0 Published online by Cambridge University Press

that the existence of a $U^+ \in A|E$ such that, for each $x \in T$,

$$(\mathrm{W}') \quad U^+[\text{sing } x] \cap (U^+.P) \text{ is empty implies that } U^+[\text{sing } x] \subseteq U^+.(E \backslash P),$$

is equivalent to (W) with P^* replaced by $P^*(U^+)$. Thus (W) and (W′) are equivalent if also $U^+ \subseteq U(P)$ for the $U(P)$ of stability.

When $(T, \mathcal{T}, A)$ is a stable division space the existence of $U\{P\}$ follows from [**8**, p. 215, Theorem 43.2]. For if P is given by the D in [**8**], with $(I, x) \in D'$ and $x \notin P^*(S)$ for the S of [**8**], then $I \nsubseteq P$ and so $I \subseteq E \backslash P$. We need only take $P^*(S) = P^*$. Thus weak compatibility of a non-additive division space gives a system lying between a non-additive and an additive stable division space.

Further, we say that $(T, \mathcal{T}, A)$ is *strongly compatible* with P if it is weakly compatible and if the $(I, x) \in U\{P\}$ with $x \notin P^*$ also have $I^* \subseteq E^* \backslash P^*$.

LEMMA 2. *If the non-additive division space* $(T, \mathcal{T}, A)$ *is weakly compatible with every partial set from* $\mathcal{T}_0$, *and if the* $P_1, P_2, P_1 \cup P_2$ *are partial sets from* $\mathcal{T}_0$, *then* $(P_1 \cup P_2)^* = P_1^* \cup P_2^*$.

(Compare [**9**, p. 335], using an earlier definition of I^*, with $E^* \supseteq F$.)

Proof. $P \equiv P_1 \cup P_2 \supseteq P_1$, $U.P \supseteq U.P_1$ for $U \in A|E$, and $P^*(U) \supseteq P_1^*(U) \supseteq P_1^*$, $P^* \supseteq P_1^*$, $P^* \supseteq P_1^* \cup P_2^*$. On the other hand, weak compatibility implies stability, so that if $x \in P^* \backslash (P_1^* \cup P_2^*)$ and $U \in A|E$, $U \subseteq U\{P_1\} \cap U\{P_2\}$, there is $I \subseteq P$ with $(I, x) \in U$ and $I \subseteq E \backslash P_j$ $(j = 1, 2)$, $I \subseteq E \backslash P$. This contradiction gives the result.

A binary relation $\geqq$ directs a non-empty set B if it is transitive, reflexive, and if, for each $\alpha, \beta \in B$, there is $\gamma \in B$ with $\gamma \geqq \alpha$, $\gamma \geqq \beta$. A real or complex valued function x_β of the $\beta \in B$ is called a *generalized sequence*. It has a (Moore-Smith) limit x if, given $\epsilon > 0$, there is an $\alpha \in B$ such that

$$|x_\beta - x| < \epsilon \text{ (all } \beta \geqq \alpha).$$

See, for example, Moore and Smith [**18**] and Kelley [**12**, Chapter 2]. More generally, the values of x_β can be sets of real or complex values, in which case we use an open interval on the real line, or an open circle on the complex plane, with centre 0 and radius ϵ, denoting the neighbourhood by N. For a single number x we replace the inequality by

$$x_\beta - x \subseteq N \text{ (all } \beta \geqq \alpha),$$

$x_\beta - x$ being the set of $y - x$ for all $y \in x_\beta$. If X, Y are two sets, $X - Y$ is the set of $x - y$ for all $x \in X$, all $y \in Y$. Then the set-valued generalized sequence (x_β) is *fundamental* if, given $\epsilon > 0$, there is an $\alpha \in B$ such that for all $\beta \geqq \alpha$, all $\gamma \geqq \alpha$,

$$x_\beta - x_\gamma \subseteq N.$$

For monotone decreasing x_β i.e., $x_\beta \subseteq x_\alpha$ when $\beta \geqq \alpha$, we can put $\gamma = \beta$. It is easy to prove that a fundamental real or complex generalized sequence is convergent.

2. Division systems. Here we take $(T, \mathscr{T}, A)$ a division system for an elementary set E, and functions $h: U^1 \to K$, K being the real line $\mathbf{R}$ or complex plane $\mathbf{C}$. Thus definitions of integrals and variation are simple. For other K see [**11**]. The generalized sequence x_U is the set $S(U)$ of sums

$$(\mathscr{E}) \sum h(I, x)$$

for all divisions $\mathscr{E}$ over E from U, $S(U)$ being monotone decreasing relative to the downward direction as U shrinks, and if the limit exists we say that $S(U)$ is convergent $(A; E)$. The limit, sometimes with $h(I, x) = f(x)k(I, x)$ or $f(x)k(I)$, is written as

$$H = H(E) = (A)\int_E dh = \int_E dh = (A)\int_E fdk = \int_E fdk,$$

omitting A when it is understood. If $S(U)$ is fundamental we say that it is fundamental $(A; E)$.

For real-valued h we have upper and lower generalized Riemann integrals, respectively

$$\text{(A)} \quad \overline{\int_E} dh \equiv \inf_U\{\sup_{\mathscr{E}}(\mathscr{E}) \sum h(I, x)\} \equiv \limsup (\mathscr{E}) \sum h(I, x)$$

as U shrinks, for the infimum over all $U \in A|E$, the supremum over all $\mathscr{E}$ of E from U, and

$$\text{(A)} \quad \underline{\int_E} dh \equiv \sup_U\{\inf_{\mathscr{E}}(\mathscr{E}) \sum h(I, x)\} \equiv \liminf (\mathscr{E}) \sum h(I, x)$$

as U shrinks, for the supremum over all $U \in A|E$, the infimum over all $\mathscr{E}$ of E from U. The integral clearly exists if and only if the upper and lower integrals are finite and equal. The (norm) *variation* $V(h; A; E)$ of h over E (relative to A) is the upper integral of $|h|$, or,

$$V(h; A; E) \equiv \inf_U V(h; U; E) \text{ where } V(h; U; E) \equiv \sup_{\mathscr{E}} (\mathscr{E}) \sum |h(I, x)|.$$

If $X \subseteq T$, the variation of h over X (relative to E, A) is

$$V(X) \equiv V(h; A; E; X) \equiv V(h.\chi(X;.); A; E) \text{ with}$$
$$V(h; U; E; X) \equiv V(h.\chi(X;.); U; E),$$

where $\chi(X; x)$ is the indicator function of X (1 when $x \in X$, 0 when $x \notin X$). If $V(h; A; E)$ is finite we say that, relative to A, h is of bounded variation in E, and if $V(h; A; E) = 0$, h is of variation zero in E. Similarly for $V(X)$. If a property holds except in a set X with $V(X) = 0$, we say that the property holds *h-almost everywhere*.

The integral is linear in h and (when h is real-valued) order properties carry through. See [**8**, p. 227, Theorems 44.2, 44.3, 44.4], [**10**, p. 230, Theorem 4]. If also $(T, \mathscr{T}, A)$ is decomposable and (X_j) is a sequence of subsets of T with union X, then

$$(1)\qquad V(X) \leqq \sum_{j=1}^{\infty} V(X_j).$$

See the proof of [**8**, p. 232, Theorem 44.10] replacing E_j by E. If only a finite number of the X_j are not empty, (1) only needs direction, not decomposability.

3. Non-additive division spaces. Here $(T, \mathscr{T}, A)$ is a non-additive division space. A union of two disjoint partial sets need not be a partial set, for let $T = R$ and $\mathscr{T}$ the family of $[u, v)$ with $v - u$ rational, or $u = 0$, or $v = 1$. The disjoint $[0, \frac{1}{2})$, $[2^{-1/2}, 1)$ are partial sets of $(0, 1)$ for divisions from T of $[0, 1)$ with mesh less than $\epsilon > 0$. But the union is not a partial set, for a division with division point $\frac{1}{2}$ has only rational division points and so cannot include $2^{-1/2}$. This is why we need co-divisional partial sets.

LEMMA 3. *If P is a proper partial set of E, if $\mathscr{E}$ is a division of P from a $U \in A|E$, and if $U_1 \in A|E\backslash P$, then there is a division $\mathscr{E}_1$ of $E\backslash P$ from U_1, depending only on P and not on the particular $\mathscr{E}$ dividing P, such that $\mathscr{E} \cup \mathscr{E}_1$ is a division of E from U.*

Proof. See [**8**, p. 215, Theorem 43.2 (43.4)] or [**10**, p. 228, Theorem 2].

THEOREM 1. *If $H(E) \equiv \int_E dh$ exists, then $H(P) \equiv \int_P dh$ exists and is finitely additive over co-divisional partial sets P of E.*

Proof. Not having a division space, we change the proofs in [**8**, pp. 228–229, Theorem 44.5], [**10**, pp. 230–231, Theorem 5]. If P is a proper partial set of E, $E\backslash P$ is a co-divisional partial set. By Lemma 1, given $\epsilon > 0$, if $U \in A|E$ and all divisions $\mathscr{E}$ of E from U satisfy

$$(2)\qquad |(\mathscr{E}) \textstyle\sum h(I, x) - H(E)| < \epsilon,$$

U divides P and $E\backslash P$, and if s_1, s_2 are two sums over divisions of P from U, and s_3 a sum over a division of $E\backslash P$ from U, $s_1 + s_3$ and $s_2 + s_3$ satisfy (2). Hence

$$(3)\qquad |s_1 - s_2| = |(s_1 + s_3 - H(E)) - (s_2 + s_3 - H(E))| < 2\epsilon.$$

As $\epsilon > 0$ is arbitrary, the set of sums over divisions of P from U, is fundamental $(A; P)$ and $H(P)$ exists. Letting s_2 in (3) tend to $H(P)$ we have

$$(4)\qquad |s_1 - H(P)| \leqq 2\epsilon,$$

uniform in the partial sets P of E and all $U \in A|E$ satisfying (2). If P_1, P_2 are co-divisional partial sets of E, $P_1 \cup P_2$ is a partial set and $H(P_1)$, $H(P_2)$, $H(P_1 \cup P_2)$ exist. For the same U, a sum for P_1 plus a sum for P_2 is a sum for

https://doi.org/10.4153/CJM-1980-031-0 Published online by Cambridge University Press

$P_1 \cup P_2$, with (4) for the three sets. Hence finite additivity follows from

$$|H(P_1 \cup P_2) - H(P_1) - H(P_2)| \leqq 6\epsilon.$$

McShane [**16**, pp. 12–14] calls (4) Henstock's lemma. It should be Saks' lemma, see [**20**, p. 214] for Burkill integration and [**3**, p. 208, (3.4)], [**4**, pp. 120–121, (1.2) and Corollary] for upper and lower Burkill and Moore-Pollard integration. My contribution is its use to prove the monotone convergence theorem, see [**7**, pp. 82–85, Theorem 36.1], [**8**, pp. 238–240, Theorem 46.1, particularly p. 240], [**10**, pp. 232–234, Theorem 7, particularly p. 233].

THEOREM 2. *If $H(E)$ exists, with $\mathscr{E}$ a division of E satisfying* (2), *from $U \in A|E$, then*

(5) $\quad (\mathscr{E}) \sum |h(I, x) - H(I)| \leqq 8\epsilon,$

(6) $\quad V(h; A; E; X) = V(H; A; E; X).$

(5) first appears in [**6**, p. 408, Theorem 3(19)].

Proof. If $\mathscr{E}_1$ is the part of $\mathscr{E}$ with real $(h - H) \geqq 0$, and P the union of the corresponding I,

$$\begin{aligned}(\mathscr{E}) \sum |\text{real } (h - H)| &= (\mathscr{E}_1) \sum \text{real } (h - H) - (\mathscr{E}\backslash\mathscr{E}_1) \sum \text{real } (h - H) \\ &= \text{real } \{(\mathscr{E}_1) \sum h - H(P)\} - \text{real } \{(\mathscr{E}\backslash\mathscr{E}_1) \sum h \\ &\qquad - H(E\backslash P)\} \leqq 4\epsilon,\end{aligned}$$

by (4). Then (5), (6) follow and $h - H$ is of variation zero in E, since in a similar way,

$$(\mathscr{E}) \sum |\text{ imag } (h - H)| \leqq 4\epsilon, \text{ and } |h| \leqq |h - H| + |H|, |H| \leqq |H - h| + |h|.$$

We can generalize [**3**], [**4**] (upper and lower Burkill integral) and Scanlon [**22**] (McShane's P-integral) to upper and lower generalized Riemann integrals on a non-additive division space; and, using the methods of [**3**], [**4**], coupling sequences of special divisions with the arrangements for A, we can have some results of division space type.

4. Decomposable non-additive division spaces. Adding decomposability to the previous section's hypotheses, we prove various monotone convergence theorems. First let $k(I, x) \geqq 0$ and $f_j(x)$ monotone increasing in the integer j, where the integral $H_j(E)$ of $f_j k$ over E exists for each j and is bounded in j. Using Theorems 1, 2, the proof of [**10**, pp. 232–234, Theorem 7] gives the weak case in which $f_j(x)$ tends to a finite limit as $j \to \infty$ for each fixed x. Dropping the last assumption on $f_j(x)$, the strong case is proved in [**10**, pp. 236–237, Theorem 11] using [**10**, pp. 235–236, Theorem 10]. Here this needs more assumptions, see Theorem 12. Instead we use (1), as follows.

As the monotone increasing $H_j(E)$ is bounded above, it tends to a limit as $j \to \infty$, so that by taking a subsequence if necessary we can assume that

(7) $\quad 0 \leqq H_{j+1}(E) - H_j(E) \leqq 4^{-j} \; (j = 1, 2, \ldots,).$

As $H_{j+1} - H_j \geqq 0$ is finitely additive over divisions of E, if X_j is the set of x where $f_{j+1}(x) - f_j(x) \geqq 2^{-j}$, with $V_1(X) = V(k; A; E; X)$, Theorem 2(6) and (7), (1) give

$$2^{-j}.V_1(X_j) \leqq V((f_{j+1} - f_j)k; A; E) = V(H_{j+1} - H_j; A; E)$$
$$= H_{j+1}(E) - H_j(E) \leqq 4^{-j}, \; V_1(X_j) \leqq 2^{-j}, X^N \equiv \bigcup_{j=N}^{\infty} X_j,$$
$$X \equiv \bigcap_{N=1}^{\infty} X^N, \; V_1(X) \leqq V_1(X^N) \leqq \sum_{j=N}^{\infty} V_1(X_j) \leqq 2^{1-N}, \; V_1(X) = 0.$$

If $x \notin X$ then for some N, $x \notin X^N$, $x \notin X_j$ $(j \geqq N)$, and $f_j(x)$ tends to a finite limit since

$$0 \leqq f_{j+1}(x) - f_j(x) < 2^{-j} \; (j \geqq N).$$

As $f_j(x)$ is monotone increasing in j for each fixed x, the original sequence tends to the same limit as the subsequence for $x \notin X$ and so k-almost everywhere.

Changing to a monotone increasing generalized sequence $f_\beta(\beta \in B)$, if $H(E) = \sup_\beta H_\beta\,(E)$, finite, there is a sequence $(\alpha(n))$ such that

(8) $\quad H_{\alpha(n)}(E) > H(E) - 4^{-n}.$

As f_β is monotone increasing in β and $k \geqq 0$, by direction in B we can take $\alpha(n)$monotone increasing in n. By the preceding proof, $\lim_{n\to\infty} f_{\alpha(n)} = f$, finite, k-almost everywhere. The weak monotone convergence theorem gives fk integrable to $H(E)$ over E. If for a monotone increasing sequence $(\alpha'(n))$ satisfying (8) we have a limit f_0, a third monotone increasing sequence $(\alpha''(n))$ exists with (8) and a limit f_*, such that

$$\alpha''(n) \geqq \alpha(n), \quad \alpha''(n) \geqq \alpha'(n), \quad f_* \geqq f, \quad f_* \geqq f_0,$$
$$\int_E f_* dk = \int_E f dk = H(E), \; V((f_* - f)k; A; E) = \int_E (f_* - f)dk = 0.$$

Hence $f_* < f + 1/n$ k-almost everywhere. By (1), $f_* = f$ k-almost everywhere. Similarly $f_* = f_0$ k-almost everywhere, so that $f_0 = f$ k-almost everywhere, and the limit is independent of the particular sequence used, modulo values in sets of k-variation zero. If $\gamma \geqq \alpha(n)$ (all n) then $f_\gamma \geqq f$, $H_\gamma \geqq H$, $H_\gamma = H$, $f_\gamma = f$ k-almost everywhere.

But we can have $f \neq \lim_\beta f_\beta$ everywhere. For let β be a finite set of real numbers, $\alpha \leqq \beta$ meaning $\alpha \subseteq \beta$, and $f_\beta(x)$ the supremum of $f(x, y)$ for all $y \in \beta$, where $f(x, y) = 0(x \neq y)$, $f(x, x) = 1$. Then $f_\beta = 0 = f$ almost everywhere but $\lim_\beta f_\beta = 1$ everywhere. We avoid this, writing the limit function

f as $\lim^*_\beta f_\beta$, and have a majorized convergence theorem using

(9) $\inf^*_{\alpha \leqq \beta} f_\beta(x) \equiv \lim^*_\gamma \{\inf_{\alpha \leqq \beta < \gamma} f_\beta(x)\}$,

(10) $\liminf^*_\beta f_\beta(x) \equiv \lim^*_\alpha \{\inf^*_{\alpha \leqq \beta} f_\beta(x)\}$,

(11) $\sup^*_{\alpha \leqq \beta} f_\beta(x) \equiv \lim^*_\gamma \{\sup_{\alpha \leqq \beta < \gamma} f_\beta(x)\}$,

(12) $\limsup^*_\beta f_\beta(x) \equiv \lim^*_\alpha \{\sup^*_{\alpha \leqq \beta} f_\beta(x)\}$.

THEOREM 3. *Let* $k(I, x) \geqq 0$ *with* $f_+(x)k$, $f^+(x)k$, $f_\beta(x)k$ *integrable in* E *and*

(13) $f_\beta(x) \geqq f_+(x) \quad (\beta \in B)$,

(14) $\inf_{\alpha \leqq \beta < \gamma} f_\beta(x).k$ *integrable in* E *for each fixed* $\alpha < \gamma$ *in* B. *Then*

(15) $$\int_E \liminf^*_\beta f_\beta(x)dk \leqq \liminf_\beta \int_E f_\beta(x)dk.$$

If (13), (14) *are replaced by*

(16) $f_\beta(x) \leqq f^+(x) \quad (\beta \in B)$,

(17) $\sup_{\alpha \leqq \beta < \gamma} f_\beta(x).k$ *integrable in* E *for each fixed* $\alpha < \gamma$ *in* B, *then*

(18) $$\int_E \limsup^*_\beta f_\beta(x)dk \geqq \limsup_\beta \int_E f_\beta(x)dk.$$

If (13), (14), (16), (17) *are true with*

$$\liminf^*_\beta f_\beta(x) = \limsup^*_\beta f_\beta(x) = \lim^*_\beta f_\beta(x)$$

k-almost everywhere, then there exists

(19) $$\int_E \lim^*_\beta f_\beta(x)dk = \lim_\beta \int_E f_\beta(x)dk.$$

Proof. By replacing f_β by $f_\beta - f_+$, we can take $f_\beta(x) \geqq 0$. The function in (14) is monotone decreasing in γ and bounded below by 0, so that

$$f_\alpha(x) - \inf_{\alpha \leqq \beta < \gamma} f_\beta(x) \geqq 0$$

is monotone increasing in γ and bounded above by $f_\alpha(x)$. By the monotone convergence theorem, (9) is integrable with respect to k in E with (20) below,

(20) $$\int_E \inf^*_{\alpha \leqq \beta} f_\beta(x)dk = \lim_\gamma \int_E \inf_{\alpha \leqq \beta < \gamma} f_\beta(x)dk \leqq \inf_{\alpha \leqq \beta} \int_E f_\beta(x)dk.$$

Again (9) is monotone increasing as α increases, and we can clearly assume that the right-hand side of (15) is finite. Hence by (20) the integral of (9) with respect to k is bounded as α increases, and the monotone convergence theorem shows that (10) is finite k-almost everywhere, and with (20) we have (15), and (18) on replacing f_β by $f^+ - f_\beta$. Then (15), (18) give (19).

With more conditions we can improve this theorem slightly. (See Theorem 7.)

5. Non-additive division spaces that are weakly compatible with every partial set. The following is true in division spaces.

(21) Given a division $\mathscr{E}$ of E from a $U \in A|E$, we can find $U_1 \in A|E$ such that all divisions $\mathscr{E}_1$ of E from U_1 satisfy $\mathscr{E}_1 \leqq \mathscr{E}$.

If $\mathscr{E}$ has two members (I, x), (J, y) only, with (21), and $(K, z) \in U_1$ with $K \cap I$ and $K \cap J$ not empty, (K, z) cannot be used in any division of E from U_1. Thus, for the purposes of dividing E, all such (K, z) could be dropped from U_1, to give additivity in this case, and (21) is very near to additivity in general. Also (21) is not satisfied by the example in a topological T_3-space. Needing something near to (21), we turn to other hypotheses. First we assume that the non-additive division space $(T, \mathscr{T}, A)$ is weakly compatible with every proper partial set P of an elementary set E. Thus by definition there is $U\{P\} \in A|E$ such that $(I, x) \in U\{P\}$ and $x \notin P^*$ imply $I \subseteq E\backslash P$. (21) follows partially, and the only trouble occurs when for an $(I, x) \in \mathscr{E}$ there is $(K, z) \in U_1$ with $z \in F(E; I)$. Then K could overlap with several J with $(J, y) \in \mathscr{E}$, without contravening the existence of the $U\{P\}$.

One result of [**11**] is that if $(T, \mathscr{T}, A)$ is compatible with a topology $\mathscr{G}$ and if $\bar{X} \subseteq G$, $\bar{Y} \subseteq G_1$, where $G, G_1 \in \mathscr{G}$ are disjoint and the bar denotes closure, then

$$(22) \quad V(X) + V(Y) = V(X \cup Y).$$

The referee points out that a better way to view these results is to define: $A|E$ separates X and Y if there is $U \in A|E$ with $I \cap J$ empty whenever $(I, x) \in U[X]$ and $(J, y) \in U[Y]$. Then (22) is true, so that my original Theorem 4 can be given as follows:

THEOREM 4. *If* $P_2 = E\backslash P_1$ *where* P_1, P_2 *are partial sets of* E, *and if* $X \subseteq \backslash P_1^*$, $Y \subseteq \backslash P_2,^*$ *then* $A|E$ *separates* X *and* Y *and* (22) *is true.*

Proof. Let $U \in A|E$ satisfy $U \subseteq U\{P_1\} \cap U\{P_2\}$. If $(I, x) \in U$, $x \in X$, then $(I, x) \in U\{P_1\}$, $x \in \backslash P_1^*$, and $I \subseteq E\backslash P_1 = P_2$. Similarly, if $(J, y) \in U$, $y \in Y$, then $J \subseteq P_1$ and I, J are disjoint.

Note that if the open G, G_1 are disjoint in a Euclidean space with the Pythagorean metric topology, we can usually find P_1 and $P_2 = E\backslash P_1$ such that $G \subseteq P_1$, $G_1 \subseteq P_2$.

A function $h(I)$ of the $I \in \mathscr{T}$ is finitely sub-additive if, for each $J \in \mathscr{T}$ and each division $\mathscr{E}$ of J from some $U \in A|J$, we have

$$(\mathscr{E}) \sum h(I) \geqq h(J).$$

https://doi.org/10.4153/CJM-1980-031-0 Published online by Cambridge University Press

THEOREM 5. *Let $h(I)$ be a finitely sub-additive function satisfying*

(23) $V(h; A; E; F(E; I)) = 0$

for every partial interval I of E. Given a division $\mathscr{E}$ of E from a $U \in A|E$, suppose that to each $U_1 \in A|E$ with $U_1 \subseteq U$, there is a $U_2 \in A|E$ and depending on $\mathscr{E}$, with $U_2 \subseteq U_1$, such that if $t \in F(E; I)$ for some $(I, x) \in \mathscr{E}$, and if $(J, t) \in U_2$, there are disjoint intervals $J_j (1 \leqq j \leqq k)$ with union J, with $(J_j, t) \in U_1$, and with every $J_j \subseteq K_j$ for some $(K_j, y) \in \mathscr{E}$. If the set of sums s of h over divisions of E from U is bounded above with supremum H, then h integrates over E to the value H. If the set of sums is unbounded, for each integer n there is a $U_n \in A|E$ such that all sums s of h over divisions of E from U_n satisfy $s \geqq n$.

Proof. Given $\epsilon > 0$, if $\mathscr{E}$ is a division of E from a $U \in A|E$, then from (23), and (1) for a finite number of X_j, there is a $U_3 \in A|E$ such that

(24) $V(h; U_3; E; \bigcup_{(I,x)\in\mathscr{E}} F(E; I)) < \epsilon.$

For each $(I, x) \in \mathscr{E}$, $E\backslash I$ is a partial set and there are $U\{I\}$ and $U\{E\backslash I\}$ in $A|E$ such that if $(J, t) \in U\{I\}$ and $t \notin I^*$ then $J \subseteq E\backslash I$, and if $(J, t) \in U\{E\backslash I\}$ and $t \notin (E\backslash I)^*$ then $J \subseteq I$. By direction there is a $U_1 \in A|E$ that lies in U, U_3, and $U\{I\}$, $U\{E\backslash I\}$ for all $(I, x) \in \mathscr{E}$, a finite number of pairs. If U_2 corresponds to this U_1 and if $(J, t) \in U_2$ with $J \not\subseteq I$ for all $(I, x) \in \mathscr{E}$, then $t \in (E\backslash I)^*$ for all $(I, x) \in \mathscr{E}$. Also, if $J \cap I$ is not empty, for some $(I, x) \in \mathscr{E}$, then $J \not\subseteq E\backslash I$ and so $t \in I^*$. Hence if $(J, t) \in U_2$ and $J \not\subseteq I$ for all $(I, x) \in \mathscr{E}$, then $t \in F(E; I)$ for all $(I, x) \in \mathscr{E}$ with $J \cap I$ not empty. Thus we can replace (J, t) by $(J_j, t) \in U_1 (1 \leqq j \leqq k)$, and this for all such (J, t), which by (24) changes the sum s by at most 2ϵ.

If the set of sums s has supremum H we can choose $\mathscr{E}$ so that for the corresponding s', $H - \epsilon < s' \leqq H$, and then $H - 3\epsilon < s \leqq H$ for every sum s over a division of E from U_2, by the finite sub-additivity of h. If the set of s is unbounded we can choose $\mathscr{E}$ so that the corresponding s' satisfies $s' \geqq n + 2\epsilon$, and then $s \geqq n$. Hence the results.

THEOREM 6. *In Theorem* 5 *let k satisfy* (23), *with f, g, g_1 integrable in E relative to $k \geqq 0$, and $f \leqq g_1$, $g \leqq g_1$. Then* max $(f, g).k$ *is integrable over E.*

Proof. Theorem 5 gives the integrability of max (F, G) where F, G are the integrals of fk, gk, respectively. To finish we use [**8**, pp. 236–237, Theorem 45.2 and Ex. 45.2].

We can now add to Theorem 3.

THEOREM 7. *In Theorems* 3, 5 *let B be the set of positive integers and let k satisfy* (23). *If* (13) *is true then so is* (14). *If* (16) *is true then so is* (17).

Proof. Use Theorem 6 repeatedly.

THEOREM 8. *In Theorems* 3, 5 *let k satisfy* (23). *If in* (15) *or* (18), $\lim^*_\beta f_\beta(x)$ *exists, we do not need the integrability of* (14) *or* (17) *as an extra assumption.*

https://doi.org/10.4153/CJM-1980-031-0 Published online by Cambridge University Press

Proof. For some sequence $(\alpha(n))$, the integral of $f_{\alpha(n)}.k$ tends to the right-hand side of (15) or (18), respectively. As

$$\lim{}^*{}_\beta f_\beta(x) = \lim_{n\to\infty} f_{\alpha(n)}(x) \text{ } k\text{-almost everywhere,}$$

Theorem 7 gives the result.

6. Non-additive division spaces that are strongly compatible with partial sets from $\mathcal{T}_0$. We suppose that there is a subset $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $h(I, x) = 0$ when $I \in \mathcal{T}\backslash\mathcal{T}_0$, and that the non-additive division space $(T, \mathcal{T}, A)$ is either strongly compatible with every partial set of E from $\mathcal{T}_0$, or, if P_1, P_2 are disjoint partial sets of E from $\mathcal{T}_0$, satisfies

(25) $\quad V(P_1{}^* \cap P_2{}^*) = 0,$

together with weak compatibility with every partial set of E from $\mathcal{T}_0$.

THEOREM 9. *Given $\epsilon > 0$, $V(X)$ finite, there is a proper partial set P_1 of E from $\mathcal{T}_0$ with*

$$V(X\backslash P_1{}^*) < \epsilon, \; V(X) < V(X \cap P_1{}^*) + \epsilon,$$

if $(T, \mathcal{T}, A)$ is infinitely divisible or if $X = Y\backslash P^*$ for a partial set P of E from $\mathcal{T}_0$, when $P \cap P_1$ is empty, or if $E = T$ in Thomson's setting.

COROLLARY. *If $V(\backslash P^*)$ is finite, it is the supremum of $V(P_1{}^*)$ for all $P_1{}^* \subseteq \backslash P^*$, when strong compatibility is used, or with decomposability and* (25) *there is a fixed set X of h-variation zero with $P_1{}^* \subseteq X \cup \backslash P^*$.*

The idea of the proof, due to McGill [**15**, p. 33, Lemma 2] in a similar result, is that if a sum is within ϵ of a supremum, then an extra sum cannot be greater than ϵ.

Proof. As $V(X)$ is finite let $U \in A|E$ have $V(h; U; E; X)$ finite. There is a partial division $\mathcal{E}_1$ from a division $\mathcal{E}$ of E from U, with $x \in X$ for $(I, x) \in \mathcal{E}_1$, $x \notin X$ or $I \in \mathcal{T}\backslash\mathcal{T}_0$ for $(I, x) \in \mathcal{E}\backslash\mathcal{E}_1$, and

(26) $\quad V(h; U; E; X) - \epsilon < (\mathcal{E}_1) \sum |h(I, x)| \leqq V(h; U; E; X).$

For the $(I, x) \in \mathcal{E}_1$, with P_1 the union of I, by infinite divisibility and Lemma 1 the least $|h(I, x)|$ is as small as we please and can eventually be omitted, giving (26) with P_1 proper. If $X = Y\backslash P^*$ let $U \subseteq U\{P\}$. Then $(I, x) \in U\{P\}$, $x \in \backslash P^*$, so that $I \in E\backslash P$, $P_1 \subseteq E\backslash P$, and P_1 is proper with $P \cap P_1$ empty. In Thomson's setting, $E = T$, no elementary set from $\mathcal{T}_0$ equals T, and P_1 is necessarily proper. Let $U_1 \in A|E$ have $U_1 \subseteq U \cap U\{P_1\}$. Then by (26),

(27) $\quad (\mathcal{E}_1) \sum |h(I, x)| + V(h; U_1; E; X\backslash P_1{}^*) \leqq V(h; U; E; X)$

$$< (\mathcal{E}_1) \sum |h(I, x)| + \epsilon.$$

For in the definition of the first V of (27) we omit the (I, x) in a division of E

https://doi.org/10.4153/CJM-1980-031-0 Published online by Cambridge University Press

with $\chi(X\backslash P_1^*; x) = 0$, and the rest form a partial division $\mathscr{E}_2$ with $x \in X\backslash P_1^*$, so that $I \subseteq E\backslash P_1$, disjoint from P_1, while $x \in X$. Thus $\mathscr{E}_2$ can be used with E_1, (27) follows, and we have

$$V(X\backslash P_1^*) \leqq V(h; U_1; E; X\backslash P_1^*) < \epsilon.$$

As $V(X)$ is sub-additive in X the second result follows. For the Corollary, as $P \cap P_1$ is empty, strong compatibility and Lemma 2 give $P^* \cap P_1^*$ empty, while if (25) is used we take $\epsilon = 1/n$, P_n for P_1, and $X = \bigcup_{n=2}^{\infty} P_n^* \cap P^*$. Decomposability ensures that $V(X) = 0$.

THEOREM 10. *If* $Y \subseteq T$ *and* P *is a partial set of* E *from* $\mathscr{T}_0$, P^* *is Carathéodory* V*-measurable,*

$$(28) \quad V(Y) = V(Y \cap P^*) + V(Y\backslash P^*),$$

for strong compatibility or for every h *satisfying* (25).

COROLLARY. *With decomposability,* V *is countably additive over the* σ*-ring from the* P^*.

Compare McGill [**15**, p. 35, Lemma 4].

Proof. If $P = E$ or $V(Y) = +\infty$, (28) is trivial. If $P \neq E$, $V(Y) < \infty, \epsilon > 0$, let the U of the proof of Theorem 9 with $X = Y\backslash P^*$ also satisfy

$$V(h; U; E; Y) < V(Y) + \epsilon.$$

We find $P_1 \in T_0$, $P_1 \subseteq E\backslash P$, and U_1, but instead of (27) we use (26) and have

$$V(Y\backslash P^*) - \epsilon + V(Y\backslash P_1^*) \leqq (E_1) \sum |h(I, x)| + V(h; U_1; E; Y\backslash P_1^*)$$
$$\leqq V(h; U; E; Y) < V(Y) + \epsilon,$$

since $x \in Y\backslash P_1^*$, $(I, x) \in U\{P_1\}$, imply $I \subseteq E\backslash P_1$ and (I, x) can be used with E_1. Strong compatibility and Lemma 2 give $P^* \cap P_1^*$ empty, or (25) gives $V(P^* \cap P_1^*) = 0$, and

$$Y \cap P^* = (Y \cap P^* \cap P_1^*) \cup (Y \cap P^*\backslash P_1^*) \subseteq (P^* \cap P_1)^* \cup (Y\backslash P_1^*).$$

As $\epsilon > 0$ is arbitrary we prove (28) from

$$V(Y\backslash P^*) + V(Y \cap P^*) < V(Y) + 2\epsilon.$$

The corollary follows from Saks [**21**, pp. 44–45, Theorems (4.1), (4.4), (4.5)].

$$V(h; A; P) \leqq V(h; A; E; P^*) \equiv V(P^*)$$

if P is a partial set of E, for $V(P^*)$ uses all (I, x) used by $V(h; A: P)$, and possibly more. Even with (25), the further property that

$$(29) \quad V(h; A; P) = V(P^*)$$

is not always true. For $V(P^*)$ sometimes needs (I, x) with $x \in F(E; P)$ and

$I \cap P$, $I \backslash P$ non-empty. Such (I, x) are unaffected by $U\{P\}$, $U\{E\backslash P\}$, but cannot be used for $V(h; A; P)$ since $I \nsubseteq P$. With $h = 1$ for all such (I, x), $h = 0$ for the rest, we falsify (29). (This example cannot occur in a division space since eventually every I lies in P entirely or in $E\backslash P$ entirely.) Similarly, here $V(h; A; P)$ need not be finitely additive in P.

THEOREM 11. *Given $\epsilon > 0$ and P a partial set of E from $\mathscr{T}_0$, if $V(h; U; E) < V(E^*) + \epsilon$ then*

$$V(h; U; P) < V(P^*) + \epsilon.$$

Proof. If $x \notin P^*$, $(I, x) \in U\{P\}$, then $I \subseteq E\backslash P$ and I is disjoint from any $J \subseteq P$. Hence if $U_1 \in A|E$ satisfies $U_1 \subseteq U \cap U\{P\}$, the result follows from (28) and

$$\begin{aligned} V(h; U; P) + V(\backslash P^*) &\leqq V(h; U; P) + V(h; U_1; E;\backslash P^*) \\ &\leqq V(h; U; E) < V(E^*) + \epsilon. \end{aligned}$$

This proof is modelled on [**8**, p. 231, Theorem 44.8] for division spaces, and the use of $V(P^*)$ is due to Thomson [**24**, pp. 504–505, Lemma 1] in his system.

In the notation of [**11**], Muldowney's example after [**11**, Lemma 4], gives

$$\begin{aligned} [0, 1)^* = [0, 1], \; VS(h; U; [0, 2); [0, 1]) &= (1 - \delta, 1 + \delta) \cup \{0\}, \\ \overline{VS}(h; A_1; [0, 2); [0, 1]) &= \{0, 1\}. \end{aligned}$$

Thus the analogue of Theorem 11 here, replacing $\overline{VS}(h; A; P)$ by $\overline{VS}(h; A; E; P^*)$ in [**11**, Lemma 4], is still false.

7. Decomposable and fully decomposable non-additive division spaces that are strongly compatible with partial sets from $\mathscr{T}_0$.

We now add full decomposability, or decomposability on occasion.

THEOREM 12. *Let (X_n) be a sequence of sets in T. If V satisfies (25) then*

$$V(\liminf_{n\to\infty} X_n) \leqq \liminf_{n\to\infty} V(X_n),$$

decomposability being sufficient. If also (X_n) is monotone increasing,

$$V(X) = \lim_{n\to\infty} V(X_n) \quad (X \equiv \lim_{n\to\infty} X_n).$$

(Compare [**8**, pp. 231–32, Theorem 44.9], [**11**, Theorem 3].)

Proof. As in [**11**] it is sufficient to prove the second part using $V(X_n) \leqq V(X)$, the result being trivial when $V(X_n) = +\infty$. Thus with $V(X_n) < \infty$, given $\epsilon > 0$ let U_n, $U \in A|E$ have

$$V(h; U_n; E; X_n) < V(X_n) + \epsilon . 2^{-n}, \; U[X_n \backslash X_{n-1}] \subseteq U_n \quad (X_0 \text{ empty}, n = 1, 2, \ldots).$$

For $\mathscr{E}$ a division of E from U let $\mathscr{E}_n$ be the $(I, x) \in \mathscr{E}$ with $x \in X_n \backslash X_{n-1}$. As $\mathscr{E}$ has only a finite number of (I, x), an N exists with $\mathscr{E}_n$ empty if $n > N$. For

https://doi.org/10.4153/CJM-1980-031-0 Published online by Cambridge University Press

P_n the union of I for $(I, x) \in \mathscr{E}_n$, or the empty set if $\mathscr{E}_n$ is empty, when $V(h; U; P_n; X_n)$ is replaced by 0,

$$(\mathscr{E}) \sum |h(I,x)|\chi(X;x) = \sum_{n=1}^{N} (\mathscr{E}_n) \sum |h(I,x)| \leqq \sum_{n=1}^{N} V(h;U;P_n;X_n).$$

Putting $h.\chi(X_n;.)$ in Theorem 11, with Theorem 10 Corollary and (25), we have

$$\sum_{n=1}^{N} V(h;\ U;\ P_n;\ X_n) < \sum_{n=1}^{N} V(P_n^* \cap X_n) + \epsilon$$
$$\leqq \sum_{n=1}^{N} V(P_n^* \cap X_n) + \epsilon \leqq V(X_N) + \epsilon.$$

Hence the second result follows by monotonicity.

In the notation of [**17**], a *paving* in E^* is a collection S of subsets of E^* that contains the empty set. If S is closed under finite unions and intersections a *Choquet S-capacity* on E^* is an extended-real-valued set function $\mathscr{I}$ defined for all subsets of E^* such that $\mathscr{I}$ is increasing, that if $X_n \subseteq E^*$, (X_n) monotone increasing, then $\mathscr{I}(\lim_{n=\infty} X_n) = \sup \mathscr{I}(X_n)$, and that for every monotone decreasing $(X_n) \subseteq S$, $\mathscr{I}(\lim_{n\to\infty} X_n) = \inf \mathscr{I}(X_n)$. If S is the collection of P^* then Theorems 12 and 10 Corollary show that for finite $V(E^*)$, V is a Choquet S-capacity. If true when $V(E^*) = +\infty$, then for (P_n^*) monotone decreasing, $V(P_n^*) = +\infty$ (all n), we need $V(\lim_{n\to\infty} P_n^*) = +\infty$. But in infinitely divisible spaces having such a (P_n^*) with a finite limit set X, and $h = 1$ (all (I, x)), then $V(P_n^*) = +\infty$, $V(X) < \infty$.

THEOREM 13. *Let $h(I, x) \geqq 0, f_n(x) \geqq 0$, where (f_n) is a monotone increasing sequence converging to f everywhere in E^*. If V satisfies* (25) *then*

$$V(fh;A;E) = \lim_{n\to\infty} V(f_nh;A;E).$$

Proof. Theorem 12 and the proof of the analogue, McGill [**15**, p. 41, Lemma 7], will suffice.

THEOREM 14. *Let (P_α^*), with intersection X, be the star sets of a monotone decreasing generalized sequence of partial sets of E from T_0. For $V(Y\backslash X)$ finite, $V(Y\backslash P_\alpha^*)$ tends to it. If $V(Y \cap P_\beta^*)$ is finite for some β, V is τ-smooth in the sense that $V(Y \cap P_\alpha^*)$ tends to $V(Y \cap X)$. Also X and $\backslash X$ are V-measurable.*

(Compare the analogues, McGill [**15**, pp. 34, 35, Lemma 3, Lemma 4, Corollary 1].)

Proof. Replacing h by $h.\chi(Y;.)$, we take $Y = E^*$ throughout. If $x \notin X$ then $x \notin P_{\alpha(x)}^*$ for some function $\alpha(x)$. Let $U \in A|E$ be such that

$$U[\text{sing } (x)] \subseteq U\{P_{\alpha(x)}\}.$$

In Theorem 9 let $\mathscr{E}_1$ be the partial division of E from U, and for all $(I, x) \in \mathscr{E}_1$ let $\alpha(x) \leqq \gamma$. As $(I, x) \in U\{P_{\alpha(x)}\}$, $I \subseteq E\backslash P_{\alpha(x)} \subseteq E\backslash P_\gamma$, $P \equiv \bigcup I \subseteq E\backslash P_\gamma$, and

P, P_γ are disjoint partial sets. By strong compatibility and Lemma 2,

$$I^* \subseteq E^*\backslash P_\gamma^*,\ P^* \subseteq E^*\backslash P_\gamma^* \subseteq \backslash X,\ P^*\backslash\backslash P_\gamma^* = P^* \cap P_\gamma^*,$$
$$V(\backslash X) \leqq V(P^*\backslash X) + V(\backslash(P^* \cup X)) \leqq V(P^*) + \epsilon \leqq V(\backslash P_\gamma^*) + \epsilon.$$

The X and P_1 of Theorem 9 are $\backslash X$ and P here. Replacing strong compatibility by (25) we have a similar proof, but without $P^* \subseteq \backslash X$. If $\alpha \geqq \beta$, $P_\alpha \subseteq P_\beta$, $X \subseteq P_\alpha^* \subseteq P_\beta^*$, $X = P_\beta^* \cap X$,

$$V(P_\beta^*) = V(P_\alpha^*) + V(P_\beta^*\backslash P_\alpha^*),$$

by Theorem 10. Taking $Y = P^*$ in the first result,

$$\lim_\alpha V(P_\beta^*\backslash P_\alpha^*) = V(P_\beta^*\backslash X),\ \lim_\alpha V(P_\alpha^*)$$
$$= V(P_\beta^*) - V(P_\beta^*\backslash X) = V(X),$$

giving the second result. Thus we have

$$V(Y\backslash X) + V(Y \cap X) = \lim_\alpha V(Y\backslash P_\alpha^*) + \lim_\alpha V(Y \cap P_\alpha^*)$$
$$= \lim_\alpha \{V(Y\backslash P_\alpha^*) + V(Y \cap P_\alpha^*)\} = V(Y),$$

and so X and $\backslash X$ are V-measurable.

8. The intrinsic topology. The sets P^* were originally defined as a location requirement, so that for the (I, x) in divisions, the x lay in well-defined sets. The definition of P^* has gradually been refined, see [**5**, p. 118, axiom (X1)], [**6**, p. 415, axiom (T1)], [**8**, p. 218], [**9**, p. 320], [**11**]. The first clues that it might be useful to regard the $\backslash P^*$ as open sets came since P^* is often the closure of P in some reasonable topology. For example, see McGill [**15**]. More especially note [**9**, p. 335, elementary *-sets and the Tychonoff analogue, Theorem 8] and earlier sections here. Now we show that the topology constructed from the $\backslash P^*$ has useful properties for integration on assuming full decomposability, strong compatibility relative to each partial set of E from $\mathscr{T}_0$, and another simple condition, as follows:

If P_1, P_2 are partial sets of E from $\mathscr{T}_0$, so is $P_1 \cup P_2$.

This holds in a division space. Thus from Lemma 2 the P^* are finitely additive, i.e.,

$$(P_1 \cup P_2)^* = P_1^* \cup P_2^*.$$

Let the intrinsic topology $\mathscr{G}$ in E^* be the empty set, E^*, and arbitrary unions of $E^*\backslash P^*$ and so the complements of arbitrary intersections of P^*, for all $P \in \mathscr{T}_0$.

THEOREM 15. *If $X \in \mathscr{G}$ with $V(X)$ finite, then X, $\backslash X$ are V-measurable and, given $\epsilon > 0$, there is a partial set P of E such that*

$$P^* \subseteq X,\ V(P^*) > V(X) - \epsilon, \backslash P^* \supseteq \backslash X,\ V(\backslash P^*) < V(\backslash X) + \epsilon.$$

Proof. Each $x \in X$ lies in some $\backslash P(x)^*$ making up X, and we can choose

$U \in A|E$ so that

(30) $U[\text{sing}\,(x)] \subseteq U\{P(x)\}\ (x \in X)$.

By (30), in the proof of Theorem 9 the $(I, x) \in \mathscr{E}_1$ have

$$x \in X,\ x \notin P(x)^*,\ I^* \subseteq E^*\backslash P(x)^* \subseteq X.$$

Hence by Lemma 2 and Theorems 9, 10,

$$P^* \subseteq X, \backslash P^* \supseteq \backslash X, V(\backslash X) + V(X) < V(\backslash X) + V(P^*) + \epsilon$$
$$= V(\backslash X) + V(E^*) - V(E^*\backslash P^*) + \epsilon \leqq V(E^*) + \epsilon.$$

Replacing h by $h.\chi(Y;.)$ and letting $\epsilon \to 0$ we have the V-measurability of X and can easily obtain the other result.

THEOREM 16. *The set E^* is compact in the intrinsic topology.*

Proof. If E^* is covered by a collection of unions of $\backslash P^*$ then each $x \in E^*$ lies in at least one of these sets and so in one of the $\backslash P^*$. Let $P(x)$ be one of the P with $x \notin P^*$. As in Theorem 15 there is a $U \in A|E$ with the corresponding (30). From U we have a division $\mathscr{E}$ of E formed of

$$(I, x) \in U\{P(x)\},\ I^* \subseteq E^*\backslash P(x)^*, \text{ with } E^* = \cup_{\mathscr{E}} I^* \subseteq \cup_{\mathscr{E}} E^*\backslash P(x)^*,$$

by Lemma 2, and so a finite number of the open sets cover E^*, and E^* is compact.

This theorem explains the remarks in [**11**, below Theorem 2] about the links with Rudin [**19**, pp. 50–51, Theorem 2.18]. Also, in the notation of [**9**, p. 335], the extra assumptions here ensure that E has the *-intersection property, as this is the property of families of compact sets in E^*, that if they have the finite intersection property there is a common point. Thus here, [**9**, Theorem 8] reduces to Tychonoff's theorem.

Inner and outer measure constructed from closed sets contained in the given set and open sets surrounding it, originated in de la Vallée Poussin [**2**, pp. 22–23]. Let

$$V^*(Y) \equiv \inf\{V(G): G \in \mathscr{G}, G \supseteq Y\},\ V^*(Y) \geqq V(Y),$$
$$V_*(Y) \equiv \sup\{V(F): \backslash F \in \mathscr{G}, F \subseteq Y\},\ V_*(Y) \leqq V(Y).$$

As E^* is compact, so are the F. The σ-algebra $\mathscr{F}$ of all Y for which

$$V_*(Y) = V(Y) = V^*(Y)$$

contains all open sets with finite V (Theorem 15), and so all Borel sets with finite V, and all $Y \in \mathscr{F}$ are Carathéodory V-measurable. But there might be sets not in $\mathscr{F}$ that are Carathéodory V-measurable. These details are proved in the usual way.

Here we have a compact containing space on using the intrinsic topology, so that in Rudin [**19**, p. 43], his notation, we can take $K = X$ in the definition of M to have $M \subseteq M_F$, and every set in M is regular. Thus [**19**, p. 61, example 18] cannot apply here. Also Kisyński [**13**] seems irrelevant here, since the conclusions of [**13**, pp. 142, 145, Theorems 1.2, 2.1] seem true directly, except

https://doi.org/10.4153/CJM-1980-031-0 Published online by Cambridge University Press

for the uniqueness. It is an open question whether we have regularity of V for all sets, even non-measurable ones. McGill [**15**, p. 37, Lemma 6] proves this for special h and A, but his proof cannot be generalized. A proof of another case is given in [**11**, Theorem 5].

My thanks are due to the referee for making a careful study of and improving the notation and results, as is remarked in various places of this paper.

References

1. J. R. Choksi, *On compact contents*, J. London Math. Soc. *33* (1958), 387–398.

2. C. de la Vallée Poussin, *Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire* (Paris, 1916).

3. R. Henstock, *On interval functions and their integrals*, J. London Math. Soc. *21* (1946), 204–209.

4. ——— *On interval functions and their integrals (II)*, J. London Math. Soc. *23* (1948), 118–128.

5. ——— *N-variation and N-variational integrals of set functions*, Proc. London Math. Soc. (3) *11* (1961), 109–133.

6. ——— *Definitions of Riemann type of the variational integrals*, Proc. London Math. Soc. (3) *11* (1961), 402–418.

7. ——— *Theory of integration* (Butterworths, London, 1963).

8. ——— *Linear analysis* (Butterworths, London, 1968).

9. ——— *Integration in product spaces, including Wiener and Feynman integration*, Proc. London Math. Soc. (3) *27* (1973), 317–344.

10. ——— *Additivity and the Lebesgue limit theorems*, The Greek Math. Soc. C. Carathéodory Symposium, Athens, 1973 (published 1974), 223–241.

11. ———*Integration, variation and differentiation in division spaces*, Proc. Royal Irish Academy *78A* (10) (1978), 69–85.

12. J. L. Kelley, *General topology* (Van Nostrand, Princeton, 1955).

13. J. Kisyński, *On the generation of tight measures*, Studia Math. *30* (1968), 141–151.

14. J. Kurzweil, *Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. Journal *7* (1957), 418–446.

15. P. McGill, *Constructing smooth measures on certain classes of paved sets*, Proc. Royal Irish Academy *77A* (1977), 31–43.

16. E. J. McShane, *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Memoirs Amer. Math. Soc. *88* (1969).

17. P. A. Meyer, *Probability and potentials* (Blaisdell, Waltham, Mass., 1966).

18. E. H. Moore and H. L. Smith, *A general theory of limits*, Amer. J. Math. *44* (1922), 102–121.

19. W. Rudin, *Real and complex analysis* (McGraw-Hill, New York, 1974).

20. S. Saks, *Sur les fonctions d'intervalle*, Fundamenta Math. *10* (1927), 211–216.

21. ——— *Theory of the integral* (Warsaw, 1937).

22. C. H. Scanlon, *Additivity and indefinite integration for McShane's P-integral*, Proc. Amer. Math. Soc. *39* (1973), 129–134.

23. B. S. Thomson, *Construction of measures and integrals*, Transactions Amer. Math. Soc. *160* (1971), 281–296.

24. ——— *A theory of integration*, Duke Math. Journal *39* (1972), 503–510.

25. F. Topsøe, *Topology and measure*, Lecture Notes in Math. *133* (Springer-Verlag, Berlin, 1970).

The New University of Ulster,
Northern Ireland