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THERMIC MINORANTS AND REDUCTIONS OF SUPERTEMPERATURES

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Abstract

Let u be a supertemperature on an open set E, and let v be a related temperature on an open subset D of E. For example, v could be the greatest thermic minorant of u on D, if it exists. Putting $w = u$ on $E\backslash D$ and $w = v$ on D, we investigate whether w, or its lower semicontinuous smoothing, is a supertemperature on E. We also give a representation of the greatest thermic minorant on E, if it exists, in terms of PWB solutions on an expanding sequence of open subsets of E with union E. In addition, in the case of a nonnegative supertemperature, we prove inequalities that relate reductions to Dirichlet solutions. We also prove that the value of any reduction at a given time depends only on earlier times.

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1. Introduction, notation and terminology

It is an elementary fact that, if u is a superharmonic function on an open subset E of $\mathbb{R}^n$, and B is a ball whose closure is contained in E, then replacing u on B by the Poisson integral of its restriction to ∂B gives a superharmonic function which is majorized by u on E. See, for example, [1, Corollary 3.2.5]. Moreover, that Poisson integral is the greatest harmonic minorant of u on B, by [1, Theorem 3.6.5].

The situation in heat potential theory is more complicated. Let now u be a supertemperature on an open subset E of $\mathbb{R}^{n+1}$, and let $C = B \times]a, b[$ be an open circular cylinder whose closure is contained in E. We denote by $\partial_{\mathrm{n}}C$ the normal boundary $\partial C\backslash(B \times \{b\})$ of C. The Poisson integral of the restriction of u to $\partial_{\mathrm{n}}C$ exists and is a temperature on $\overline{C}\backslash\partial_{\mathrm{n}}C$. If we replace u on $\overline{C}\backslash\partial_{\mathrm{n}}C$ by that Poisson integral, then the resultant function is a supertemperature which is majorized by u on E; see [16, Theorem 10] or [17, Theorem 3.21]. Note that replacing u by its Poisson integral only on C does not in general produce a supertemperature. Moreover, that Poisson integral is not in general equal to the greatest thermic minorant of u on C, as is implied by [17, Remark 3.24]. A similar situation occurs if we use a rectangle instead of C, as was noted in [12].

In this paper, we give the corresponding result for a heat ball. The heat ball is of increasing importance, and can now be found in several books, including [5, 7, 8, 17]. It was first studied by Pini [11] in the case $n = 1$, and Fulks [10] for general n. Specifically, we show that if u is a supertemperature on an open subset E of $\mathbb{R}^{n+1}$, and $\Omega = \Omega(p; c)$ is a heat ball whose closure is contained in E, then replacing u on Ω by the PWB solution S_u^Ω of the Dirichlet problem with boundary function the restriction of u to $\partial\Omega$, gives a supertemperature on $E\backslash\{p\}$ whose lower semicontinuous smoothing is a supertemperature on E. Furthermore, S_u^Ω is the greatest thermic minorant of u on Ω. Thus, if we take a heat ball instead of a circular cylinder (or rectangle), we obtain a much closer analogy with the superharmonic case.

The proofs are not elementary. On the way, we prove general results about changing a supertemperature to a temperature on an open subset, and whether the resultant function, or its lower semicontinuous smoothing, is a supertemperature. We also give a representation of the greatest thermic minorant in terms of PWB solutions on an expanding sequence of open subsets of E with union E. In addition, in the case of a nonnegative supertemperature, we prove inequalities that relate reductions to Dirichlet solutions. Finally, we show that the value of any reduction at a given time depends only on earlier times.

Notation and terminology are the same as in [17], where full details can be found. We acknowledge that Bauer's theory of harmonic spaces [2] includes the heat equation. However, his approach is very different, and in particular his notion of the Dirichlet problem is different for a general open set. We illustrate this by the following simple example, where E consists of two circular cylinders one on top of the other. Let B be a ball in $\mathbb{R}^n$, and let $E = B \times (]a, b[\cup]b, c[)$. We put $E_1 = B \times]a, b[$, $E_2 = B \times]b, c[$ and, for any circular cylinder $D = B \times]\alpha, \beta[$, we put $\partial_{\mathrm{n}} D = (B \times \{\alpha\}) \cup (\partial B \times [\alpha, \beta])$. If $f \in C(\partial E)$, then the restriction of f to ∂E_1 is continuous and real-valued, and hence there is a function $u_f^{(1)} \in C(\overline{E}_1)$ which is a temperature on $\overline{E}_1\backslash\partial_{\mathrm{n}} E_1$ and equal to f on $\partial_{\mathrm{n}} E_1$. Thus, we cannot hope to prescribe the boundary values of $u_f^{(1)}$ on $\partial E_1\backslash\partial_{\mathrm{n}} E_1$. Similarly, there is a function $u_f^{(2)} \in C(\overline{E}_2)$ which is a temperature on $\overline{E}_2\backslash\partial_{\mathrm{n}} E_2$ and equal to f on $\partial_{\mathrm{n}} E_2$. The temperature u_f on E that corresponds to f is given by $u_f = u_f^{(i)}$ on E_i for each $i \in \{1, 2\}$. At all points of $\partial B \times (]a, b[\cup]b, c[)$, the boundary values are attained on any approach through E. For each point $x \in B$,

$$u_f(y, s) \to f(x, b) \quad \text{as } (y, s) \to (x, b+),$$

but in general

$$u_f(y, s) \not\to f(x, b) \quad \text{as } (y, s) \to (x, b-).$$

Thus, we can expect the boundary values to be attained on approach from above, but not on approach from below. The version of the Dirichlet problem in [12, 17] takes this into account, and considers all points of $B \times \{b\}$ to be regular. All points of $B \times \{c\}$ are considered to be irrelevant. By contrast, the version adopted by Bauer [2], Doob [6]

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and others, treats all boundary points in the same way, requiring the boundary values to be attained on any approach through E. They regard all points of $B \times \{b\}$ and $B \times \{c\}$ to be irregular. Thus a parabolic problem is treated as if it were an elliptic problem. This may be inevitable if one wants a theory which applies to both elliptic and parabolic equations.

We do not need any results from harmonic space theory that are not also given for the present context in [17]. Indeed, we make no reference to Bauer's book [2] in any of the proofs of our results. Moreover, the only essential references to Doob's book that we make in such proofs are to [6, page 287], an elementary lemma. For other references to [6], we give alternatives.

We briefly list the notation and terminology that we shall use here.

We denote by W the *fundamental temperature*, defined for all $(x, t) \in \mathbb{R}^{n+1}$ by

$$W(x,t) = \begin{cases} (4\pi t)^{-(n/2)} \exp\Big(-\dfrac{|x|^2}{4t}\Big) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Given any two points $p = (x, t)$ and $q = (y, s)$ in $\mathbb{R}^{n+1}$, we put $G(p; q) = W(x - y, t - s)$. For any point $p_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and any positive number c, the set

$$\Omega(p_0; c) = \{q \in \mathbb{R}^{n+1} : G(p_0; q) > (4\pi c)^{-(n/2)}\}$$

is called the *heat ball* with *centre* p_0 and *radius* c. The boundary of a heat ball is called a *heat sphere*.

The fundamental *mean value over the heat sphere* $\partial\Omega(p_0; c)$ is defined by

$$\mathcal{M}(u; p_0; c) = (4\pi c)^{-(n/2)} \int_{\partial\Omega(p_0;c)} Q(p_0 - p)u(p)\, d\sigma(p)$$

for any function u such that the integral exists. Here σ denotes surface area measure on $\partial\Omega(p_0; c)$, $p = (x, t) \in \partial\Omega(p_0; c)$ and

$$Q(p_0 - p) = \frac{|x_0 - x|^2}{(4|x_0 - x|^2(t_0 - t)^2 + (|x_0 - x|^2 - 2n(t_0 - t))^2)^{\frac{1}{2}}}.$$

Let u be a lower finite and lower semicontinuous function on an open subset E of $\mathbb{R}^{n+1}$. If, given any point $p \in E$ and a positive number ϵ, there is a positive number $c < \epsilon$ such that the closed heat ball $\overline{\Omega}(p; c)$ is a subset of E and the inequality $u(p) \geq \mathcal{M}(u; p; c)$ holds, then u is called a *hypertemperature* on E. If, in addition, $u < +\infty$ on a dense subset of E, then u is called a *supertemperature* on E. Bauer [3] proved that our hypertemperatures are the same as his hyperharmonic functions (associated with the heat equation). A more natural and elementary proof of this is given in both [16] and [17].

The negative $-u$ of a hypertemperature u is called a *hypotemperature*, and that of a supertemperature is called a *subtemperature*. A function which is both a supertemperature and a subtemperature is called a *temperature*, and is a solution of

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the heat equation. The use of the term 'temperature' for a solution of the heat equation goes back at least as far as Widder's paper [19], and our terminology above is a natural extension of this.

Given an open subset E of $\mathbb{R}^{n+1}$ and a point $p \in E$, we denote by $\Lambda(p;E)$ the set of points $q \in E$ that are lower than p relative to E, in the sense that there is a polygonal path $\gamma \subseteq E$ joining p to q along which the temporal variable t is strictly decreasing.

Given any point $p = (x,t) \in \mathbb{R}^{n+1}$ and a number $r > 0$, we denote by $H(p,r)$ the open lower half-ball $B(p,r) \cap (\mathbb{R}^n \times]-\infty, t[)$, and by $H^*(p,r)$ the open upper half-ball $B(p,r) \cap (\mathbb{R}^n \times]t, +\infty[)$.

Let E be an open set, and let $q \in \partial E$. We call q a *normal* boundary point if either q is the point at infinity, or $q \in \mathbb{R}^{n+1}$ and $H(q,r)\backslash E \neq \emptyset$ for each $r > 0$. Otherwise we call q an *abnormal* boundary point. The abnormal boundary points are of two kinds. If there is an $r > 0$ such that $H^*(q,r) \cap E = \emptyset$, then q is called a *singular* boundary point. On the other hand, if for every $r > 0$ we have $H^*(q,r) \cap E \neq \emptyset$, then q is called a *semisingular* boundary point.

The set of all normal boundary points of E is denoted by $\partial_{\mathrm{n}} E$, that of all abnormal ones by $\partial_{\mathrm{a}} E$, that of all singular ones by $\partial_{\mathrm{s}} E$ and that of all semisingular ones by $\partial_{\mathrm{ss}} E$. Thus, $\partial E = \partial_{\mathrm{n}} E \cup \partial_{\mathrm{a}} E$ and $\partial_{\mathrm{a}} E = \partial_{\mathrm{s}} E \cup \partial_{\mathrm{ss}} E$. The *essential boundary* $\partial_{\mathrm{e}} E$ is defined by $\partial_{\mathrm{e}} E = \partial_{\mathrm{n}} E \cup \partial_{\mathrm{ss}} E$.

Let f be a function on $\partial_{\mathrm{e}} E$. The *upper class* determined by f, denoted by $\mathfrak{U}_f^E$, consists of all lower bounded hypertemperatures on E that satisfy

$$\liminf_{(x,t)\to(y,s)} w(x,t) \geq f(y,s) \quad \text{for all } (y,s) \in \partial_{\mathrm{n}} E$$

and

$$\liminf_{(x,t)\to(y,s+)} w(x,t) \geq f(y,s) \quad \text{for all } (y,s) \in \partial_{\mathrm{ss}} E.$$

The *lower class* determined by f, denoted by $\mathfrak{L}_f^E$, consists of all upper bounded hypotemperatures on E that satisfy

$$\limsup_{(x,t)\to(y,s)} w(x,t) \leq f(y,s) \quad \text{for all } (y,s) \in \partial_{\mathrm{n}} E$$

and

$$\limsup_{(x,t)\to(y,s+)} w(x,t) \leq f(y,s) \quad \text{for all } (y,s) \in \partial_{\mathrm{ss}} E.$$

The function $U_f^E = \inf\{w : w \in \mathfrak{U}_f^E\}$ is called the *upper solution* for f on E, and $L_f^E = \sup\{w : w \in \mathfrak{L}_f^E\}$ is called the *lower solution* for f on E. We say that f is *resolutive* for E if $L_f^E = U_f^E$ and is a temperature on E. In particular, if $f \in C(\partial_{\mathrm{e}} E)$ then f is resolutive for E, by [17, Theorem 8.26]. For any resolutive function f, we define $S_f^E = L_f^E = U_f^E$ to be the *PWB solution* for f on E.

A point $(y,s) \in \partial_{\mathrm{e}} E$ is called *regular* if, for every function $f \in C(\partial_{\mathrm{e}} E)$,

$$\lim_{(x,t)\to(y,s)} S_f^E(x,t) = f(y,s)$$

if $(y,s) \in \partial_n E$, or

$$\lim_{(x,t)\to(y,s+)} S_f^E(x,t) = f(y,s)$$

if $(y,s) \in \partial_{ss} E$. The set E is called *regular* if every point $(y,s) \in \partial_e E$ is regular.

If u is an extended real-valued function on an open set E, then the *lower semicontinuous smoothing* $\widehat{u}$ of u is defined by

$$\widehat{u}(p) = u(p) \wedge \liminf_{q\to p} u(q)$$

for all $p \in E$. It is the greatest lower semicontinuous minorant of u on E.

If u is a supertemperature on E that is minorized by a subtemperature on E, then there is a greatest such minorant, which is in fact a temperature on E. It is called the *greatest thermic minorant* of u on E.

Let u be a nonnegative supertemperature on an open set E. If $L \subseteq E$, then the *reduction of u over L* (relative to E), denoted by R_u^L, is the infimum of the family of nonnegative supertemperatures on E that majorize u on L. The lower semicontinuous smoothing $\widehat{R_u^L}$ is called the *smoothed reduction of u over L* (relative to E).

The above form of the PWB solution gives more general results than the form mentioned by Doob [6]. Here is yet another illustration of that fact. Here, and below, we abbreviate $(x,t) \to (y,s\pm)$ to $p \to q\pm$, respectively.

Lemma 1.1. *Let u be a lower bounded supertemperature on an open set E, and let v be its greatest thermic minorant on E. If there is a continuous function f on $\partial_e E$ such that $\lim_{p\to q} u(p) = f(q)$ for all $q \in \partial_n E$ and $\lim_{p\to q+} u(p) = f(q)$ for all $q \in \partial_{ss} E$, and v is upper bounded on E, then f is resolutive for E with $S_f^E = v$ on E.*

Proof. Under these hypotheses, we have $u \in \mathfrak{U}_f^E$ and $v \in \mathfrak{L}_f^E$. Therefore $v \le L_f^E \le U_f^E \le u$ on E, so that U_f^E is lower finite on E and is upper finite on a dense subset of E. Hence, U_f^E is a temperature on E, by [18, Lemma 15] or [17, Lemma 8.15]. The definition of v now shows that $v = L_f^E = U_f^E$, as required. □

A similar result was claimed by Doob [6, Example (e), page 331] under the hypotheses that f is continuous on ∂E and $\lim_{p\to q} u(p) = f(q)$ for all $q \in \partial E$.

2. Resolutivity and reductions

We begin with an essential lemma, which is more general than [15, Lemma 2] and [17, Lemma 7.20].

Lemma 2.1. *Let u be a supertemperature on an open set E, and let v be a supertemperature on an open subset D of E. If*

$$\liminf_{p\to q,\, p\in D} v(p) \ge u(q) \qquad \textit{for all } q \in E \cap \partial_n D, \tag{2.1}$$

$$\liminf_{p\to q+,\, p\in D} v(p) \ge u(q) \qquad \textit{for all } q \in E \cap \partial_{ss} D \tag{2.2}$$

and

$$\liminf_{p\to q-} v(p) > -\infty \qquad \textit{for all } q \in E \cap \partial_{\mathrm{a}} D, \tag{2.3}$$

then the function w, defined by

$$w(q) = \begin{cases} (v \wedge u)(q) & \textit{if } q \in D, \\ u(q) & \textit{if } q \in E\backslash(D \cup \partial_{\mathrm{a}} D), \\ \left(\liminf_{p\to q-} v(p)\right) \wedge u(q) & \textit{if } q \in E \cap \partial_{\mathrm{a}} D, \end{cases}$$

is a supertemperature on E.

Proof. It is clear that w is a supertemperature on $E\backslash\partial D$, and that $w < +\infty$ on a dense subset of E. Condition (2.3) ensures that $w > -\infty$ on E.

If $q \in E \cap \partial_{\mathrm{n}} D$, then condition (2.1) implies that

$$w(q) = u(q) \le \left(\liminf_{p\to q, p\in D} v(p)\right) \wedge \left(\liminf_{p\to q} u(p)\right) = \liminf_{p\to q} w(p),$$

so that w is lower semicontinuous at q. On the other hand, if $q \in E \cap \partial_{\mathrm{s}} D$, we have

$$w(q) = \left(\liminf_{p\to q-} v(p)\right) \wedge u(q) \le \left(\liminf_{p\to q, p\in D} v(p)\right) \wedge \left(\liminf_{p\to q} u(p)\right) = \liminf_{p\to q} w(p),$$

so that w is lower semicontinuous at q. Moreover, if $q \in E \cap \partial_{\mathrm{ss}} D$, then condition (2.2) implies that

$$w(q) \le \left(\liminf_{p\to q-} v(p)\right) \wedge \left(\liminf_{p\to q} u(p)\right) \wedge \left(\liminf_{p\to q+, p\in D} v(p)\right) = \liminf_{p\to q} w(p),$$

so that w is lower semicontinuous at q. Hence w is lower semicontinuous on E.

It remains only to check that the inequality $w(q) \ge \mathcal{M}(w; q; c)$ holds whenever $q \in E \cap \partial D$ and c is sufficiently small. If $q \in E \cap \partial D$ and $w(q) = u(q)$, then

$$w(q) \ge \mathcal{M}(u; q; c) \ge \mathcal{M}(w; q; c)$$

whenever $\overline{\Omega}(q; c) \subseteq E$. Otherwise $q \in E \cap \partial D$ and $w(q) \neq u(q)$, so that $q \in \partial_{\mathrm{a}} D$ and $w(q) = \liminf_{p\to q-} v(p)$. Condition (2.3) shows that there is an open half-ball $H(q, \delta) \subseteq D$ such that v is lower bounded on $H(q, \delta)$. We can assume that $\overline{H}(q, \delta) \subseteq E$. We choose a positive number c_0 such that $\overline{\Omega}(p; c) \subseteq H(q, \delta)$ whenever $p \in H(q, \delta/2)$ and $0 < c \le c_0$. For all such p and c, we have $v(p) \ge \mathcal{M}(v; p; c) \ge \mathcal{M}(w; p; c)$, so that $w(q) \ge \liminf_{p\to q-} \mathcal{M}(w; p; c)$. Since $\overline{H}(q, \delta) \subseteq E$, the function u is lower bounded on $H(q, \delta)$, and so the same is true of w. We may therefore use Fatou's lemma to obtain $w(q) \ge \mathcal{M}(w; q; c)$. This completes the proof. □

Remark 2.2. If, in Lemma 2.1, v is defined on an open superset of $\overline{D} \cap E$, then $\liminf_{p\to q-} v(p) = v(q)$ for all $q \in \overline{D} \cap E$, by [17, Lemma 3.16] or [16, Lemma 2]. Therefore w takes the simpler form

$$w(q) = \begin{cases} (v \wedge u)(q) & \text{if } q \in E \cap (D \cup \partial_{\mathrm{a}} D), \\ u(q) & \text{if } q \in E\backslash(D \cup \partial_{\mathrm{a}} D). \end{cases}$$

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REMARK 2.3. If, in Lemma 2.1, $\partial_e D \subseteq E$ and u is lower bounded on $\partial_e D$, then conditions (2.1) and (2.2) combine with the minimum principle to show that v is lower bounded on D, so that condition (2.3) is automatically satisfied. In particular, this occurs whenever $\overline{D} \subseteq E$.

Lemma 2.1 is necessarily more complicated than its superharmonic counterpart. Doob [6, page 297] neglected this extra complication, and the following example shows that his argument is flawed.

EXAMPLE 2.4. We choose a positive real number a, put $H = \mathbb{R}^n \times]2a, +\infty[$ and denote by χ_H the characteristic function of H. Then the function $v = G(\cdot; 0) + \chi_H$ is a nonnegative supertemperature on $\mathbb{R}^{n+1}$. We choose $A = \mathbb{R}^n \times \{a, 3a\}$, $q_0 = (0, 3a)$, and put $\Lambda = \Lambda(q_0; \mathbb{R}^{n+1}) = \mathbb{R}^n \times]-\infty, 3a[$. According to Doob [6, page 297], if u' is a nonnegative supertemperature on Λ that majorizes v on $A \cap \Lambda$, and u is a nonnegative supertemperature on $\mathbb{R}^{n+1}$ that majorizes v on A, then the function

$$u'' = \begin{cases} u & \text{on } \mathbb{R}^{n+1} \backslash \Lambda, \\ u \wedge u' & \text{on } \Lambda \end{cases}$$

is a supertemperature on $\mathbb{R}^{n+1}$ that majorizes v on A. However, $G(\cdot; 0)$ is a nonnegative supertemperature on Λ that equals v on $A \cap \Lambda = \mathbb{R}^n \times \{a\}$, and so we can take $u' = G(\cdot; 0)$. This gives

$$\liminf_{p \to q_0-} u''(p) \leq \lim_{p \to q_0-} G(p; 0) = G(q_0; 0) < v(q_0) \leq u(q_0) = u''(q_0),$$

so that u'' is not lower semicontinuous. Of course, u'' can be redefined on $\mathbb{R}^n \times \{3a\}$ to make it lower semicontinuous, by putting $u''(q) = \liminf_{p \to q-} (u \wedge u')(p)$ for all $q \in \mathbb{R}^n \times \{3a\}$, but then u'' would not majorize v on A.

We use the following theorem in two situations, namely when $u \geq 0$ and $f = 0$, and when $\overline{D} \subseteq E$. The analogous situations for superharmonic functions are treated separately in both [1, page 191] and [6, page 122], but it seems desirable to have a general result that covers both cases.

THEOREM 2.5. *Let u be a supertemperature on an open set E, let D be an open subset of E such that u is lower bounded on $E \cap \partial_e D$, and suppose that there is a lower bounded Borel measurable function f on $\partial E \cap \partial_e D$ such that*

$$f(q) \leq \liminf_{p \to q, p \in D} u(p) \quad \text{for all } q \in \partial E \cap \partial_n D$$

and

$$f(q) \leq \liminf_{p \to q+, p \in D} u(p) \quad \text{for all } q \in \partial E \cap \partial_{ss} D.$$

If $\bar{u}$ is defined on $\partial_e D$ by

$$\bar{u} = \begin{cases} u & \text{on } E \cap \partial_e D, \\ f & \text{on } \partial E \cap \partial_e D, \end{cases}$$

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then $\bar{u}$ is resolutive for D, and the function h, defined by

$$h = \begin{cases} u & \text{on } E \backslash \overline{D}, \\ S_{\bar{u}}^{D} & \text{on } D, \end{cases}$$

can be extended to a supertemperature majorized by u on E.

PROOF. The function $\bar{u}$ is Borel measurable, and the conditions on f ensure that the restriction of u to D belongs to the class $\mathfrak{U}_{\bar{u}}^{D}$. Therefore, $U_{\bar{u}}^{D} \leq u < +\infty$ on a dense subset of D. Since $\bar{u}$ is also lower bounded, it follows from [17, Lemma 8.15] or [18, Lemma 15] that $U_{\bar{u}}^{D}$ is a temperature on D. Now [17, Corollary 8.33] or [18, Corollary 26] shows that $\bar{u}$ is resolutive for D.

Let v be any supertemperature in the class $\mathfrak{U}_{\bar{u}}^{D}$, and put $m = \inf_{\partial_e D} \bar{u}$. Then $\liminf_{p \to q} v(p) \geq m$ for all points $q \in \partial_n D$, and $\liminf_{p \to q+} v(p) \geq m$ for all points $q \in \partial_{ss} D$, so that $v \geq m$ on D by the minimum principle. Therefore v satisfies all the conditions in Lemma 2.1, so that the function $w = w_v$ of that lemma is a supertemperature on E, and $w_v \geq m$ on D. We now put

$$g = \inf\{w_v : v \text{ is a supertemperature in } \mathfrak{U}_{\bar{u}}^{D}\} \leq w_u = u$$

on E. Then $g \geq m$ on D and, if K is any compact subset of E then $g \geq m \wedge (\inf_K u)$ on K, so that g is locally lower bounded on E. Now [17, Theorem 7.13] or [6, page 295] shows that the lower semicontinuous smoothing $\widehat{g}$ is a supertemperature on E, and is equal to g at every point q where $g(q) = \liminf_{p \to q} g(p)$. Clearly $g = u$ on $E \backslash \overline{D}$, so that $\widehat{g} = u$ there too, in view of [17, Lemma 3.16] or [16, Lemma 2]. Moreover, on D we have $g = \inf\{v \wedge u : v \in \mathfrak{U}_{\bar{u}}^{D}\}$. Since $u \in \mathfrak{U}_{\bar{u}}^{D}$ we have $v \wedge u \in \mathfrak{U}_{\bar{u}}^{D}$ whenever $v \in \mathfrak{U}_{\bar{u}}^{D}$, and it follows that $S_{\bar{u}}^{D} = \inf\{v \wedge u : v \in \mathfrak{U}_{\bar{u}}^{D}\} = g = \widehat{g}$ on D. Hence, $\widehat{g} = h$ wherever the latter is defined. □

COROLLARY 2.6. *Let u be a supertemperature on an open set E, and let D be a bounded, regular open set such that $\overline{D} \subseteq E$ and $\partial_n D = \partial D$. Then the restriction of u to ∂D is resolutive for D, and if*

$$h = \begin{cases} u & \text{on } E \backslash D, \\ S_{u}^{D} & \text{on } D, \end{cases}$$

then h is a supertemperature majorized by u on E.

PROOF. By Theorem 2.5, the restriction of u to ∂D is resolutive for D, and $S_u^D \leq u$ on D. Since u is lower semicontinuous, D is regular, and $\partial_n D = \partial D$, we have

$$\liminf_{p \to q} S_u^D(p) \geq \liminf_{p \to q, p \in \partial D} u(p) \geq u(q)$$

for all $q \in \partial D$, by [17, Theorems 8.46 and 8.44]. Therefore, by [17, Lemma 7.20] with $V = D$ and $v = S_u^D$, the function h is a supertemperature on E. □

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Corollary 2.7. *Let u be a supertemperature on an open set E, and suppose that there is a lower bounded Borel measurable function f on $\partial_{\mathrm{e}}E$ such that*

$$f(q) \le \liminf_{p \to q} u(p) \quad \text{for all } q \in \partial_{\mathrm{n}}E$$

and

$$f(q) \le \liminf_{p \to q+} u(p) \quad \text{for all } q \in \partial_{\mathrm{ss}}E.$$

Then f is resolutive and $S_f^E \le u$ on E.

Proof. Take $D = E$ in Theorem 2.5. □

In the next theorem, we obtain inequalities between two particular reductions of a nonnegative supertemperature u on E, and the PWB solution on an open subset D of E with boundary function as given in Theorem 2.5 with $f = 0$. The result is analogous to [1, Theorem 6.9.1] and a result in [6, page 122], but is less satisfactory insofar as in the superharmonic case there is an equality rather than two inequalities.

Theorem 2.8. *Let u be a nonnegative supertemperature on an open set E, let D be an open subset of E, and let u_0 be defined on $\partial_{\mathrm{e}}D$ by*

$$u_0 = \begin{cases} u & \text{on } E \cap \partial_{\mathrm{e}}D, \\ 0 & \text{on } \partial E \cap \partial_{\mathrm{e}}D. \end{cases}$$

Then u_0 is resolutive for D, and

$$R_u^{E\backslash(D\cup\partial_{\mathrm{a}}D)} \le S_{u_0}^D \le R_u^{E\backslash(D\cup\partial_{\mathrm{s}}D)} \tag{2.4}$$

on D.

Moreover, if $E \cap \partial_{\mathrm{ss}}D$ is polar, then $R_u^{E\backslash(D\cup\partial_{\mathrm{a}}D)} = R_u^{E\backslash(D\cup\partial_{\mathrm{s}}D)}$ on $E\backslash\partial_{\mathrm{ss}}D$ and equality holds in (2.4).

Proof. The fact that u_0 is resolutive for D follows from Theorem 2.5 by taking $f = 0$.

If v is a nonnegative supertemperature on E such that $v \ge u$ on $E\backslash(D \cup \partial_{\mathrm{s}}D)$, then

$$\liminf_{p \to q, p \in D} v(p) \ge v(q) \ge u(q)$$

for all points $q \in E \cap \partial_{\mathrm{e}}D$, and

$$\liminf_{p \to q, p \in D} v(p) \ge 0$$

for all $q \in \partial E \cap \partial_{\mathrm{e}}D$, so that the restriction of v to D belongs to the class $\mathfrak{U}_{u_0}^D$. Therefore on D we have $v \ge S_{u_0}^D$, and hence $R_u^{E\backslash(D\cup\partial_{\mathrm{s}}D)} \ge S_{u_0}^D$.

On the other hand, if v is now any supertemperature in the class $\mathfrak{U}_{u_0}^D$, then v satisfies all the conditions in Lemma 2.1. Therefore the function w, defined as in Lemma 2.1, is a supertemperature on E. Since $w = u$ on $E\backslash(D \cup \partial_{\mathrm{a}}D)$, we have $w \ge R_u^{E\backslash(D\cup\partial_{\mathrm{a}}D)}$ on E, and hence $v \ge R_u^{E\backslash(D\cup\partial_{\mathrm{a}}D)}$ on D. It follows that $S_{u_0}^D \ge R_u^{E\backslash(D\cup\partial_{\mathrm{a}}D)}$ on D.

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If $E \cap \partial_{ss} D$ is polar, we put $L = E \backslash (D \cup \partial_a D)$ and $Z = E \cap \partial_{ss} D$, so that $L \cup Z = E \backslash (D \cup \partial_s D)$. Given a point $p_0 \in E \backslash Z$, we can find a nonnegative supertemperature w on E such that $w = +\infty$ on Z and $w(p_0) < +\infty$, by [12, Theorem 27] or [17, Theorem 7.3]. If v is a nonnegative supertemperature on E such that $v \geq u$ on L, then for each $\epsilon > 0$ we have $v + \epsilon w \geq u$ on $L \cup Z$, and so $v + \epsilon w \geq R_u^{L \cup Z}$ on E. In particular, $v(p_0) + \epsilon w(p_0) \geq R_u^{L \cup Z}(p_0)$ for all $\epsilon > 0$, so that $v(p_0) \geq R_u^{L \cup Z}(p_0)$, and hence $R_u^L(p_0) \geq R_u^{L \cup Z}(p_0)$. Therefore $R_u^L(p_0) = R_u^{L \cup Z}(p_0)$, because $R_u^L \leq R_u^{L \cup Z}$ on E. Thus, $R_u^{E \backslash (D \cup \partial_a D)} = R_u^{E \backslash (D \cup \partial_s D)}$ on $E \backslash \partial_{ss} D$, which gives the result. □

REMARK 2.9. Theorem 2.8 leaves open the question of whether the two reductions in (2.4) are equal if $E \cap \partial_{ss} D$ is not polar. For two arbitrary disjoint subsets L and Z of E, the hypothesis that Z is not polar is insufficient to guarantee that $R_u^L \neq R_u^{L \cup Z}$ on E. For example, if $E \cap (\mathbb{R}^n \times \{a\}) \neq \emptyset$, and we take $L = E \cap (\mathbb{R}^n \times]-\infty, a[)$ and $Z = E \cap (\mathbb{R}^n \times \{a\})$, then whenever v is a nonnegative supertemperature on E such that $v \geq u$ on L, for every point $q \in Z$ we have

$$v(q) = \liminf_{p \to q-} v(p) \geq \liminf_{p \to q-} u(p) = u(q),$$

by [17, Lemma 3.16] or [16, Lemma 2]. Thus, $v \geq u$ on $L \cup Z$, so that $v \geq R_u^{L \cup Z}$, and hence $R_u^L \geq R_u^{L \cup Z}$ on E. The reverse inequality is always true, so that equality holds even though Z is not polar.

The following example shows that the two reductions in (2.4) may not be equal. In it, we are able to evaluate the reductions explicitly, and to show that $S_{u_0}^D$ is equal to the larger one.

EXAMPLE 2.10. Let F be a closed subset of $\mathbb{R}^n$ with Lebesgue measure $m_n(F) > 0$, let $E = \mathbb{R}^n \times]-\infty, 1[$ and let $D = E \backslash (F \times \{0\})$. Then $\partial_n D$ contains only the point at infinity, $\partial_s D = \mathbb{R}^n \times \{1\}$ and $\partial_{ss} D = F \times \{0\}$. Therefore $E \backslash (D \cup \partial_a D) = \emptyset$, so that $R_1^{E \backslash (D \cup \partial_a D)} = 0$ on E. Moreover, $E \backslash (D \cup \partial_s D) = F \times \{0\}$ and, if v is a nonnegative supertemperature on E such that $v \geq 1$ on $F \times \{0\}$, then

$$\liminf_{(x,t) \to (y, 0+)} v(x,t) \geq v(y,0) \geq \chi_{F \times \{0\}}(y,0)$$

for all $y \in \mathbb{R}^n$, where χ_A denotes the characteristic function of a set A. It follows that, if W_F is defined on E by

$$W_F(x,t) = \begin{cases} \displaystyle\int_F W(x-y,t)\,dy & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

then $v \geq W_F$ on E by [6, page 287]. Since $m_n(F) > 0$, we have $W_F(x,t) > 0$ if $t > 0$, and it follows that $R_1^{F \times \{0\}} \geq W_F > 0 = R_1^{\emptyset}$ on $\mathbb{R}^n \times]0, 1[$. Thus, $R_1^{E \backslash (D \cup \partial_s D)} > R_1^{E \backslash (D \cup \partial_a D)}$ on $\mathbb{R}^n \times]0, 1[$. Furthermore, given any function v as above, and any positive number c, the function v_c, defined by

$$v_c(x,t) = \begin{cases} v(x,t) & \text{if } t > -c, \\ 0 & \text{if } t \leq -c, \end{cases}$$

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satisfies the same conditions as v, so that $v_c \geq R_1^{F\times\{0\}}$ on E. Therefore, $R_1^{F\times\{0\}}(x,t) = 0$ whenever $t < 0$. Because $R_1^{F\times\{0\}}$ is a temperature on D, it follows that $R_1^{F\times\{0\}}(x,t) = 0$ whenever $(x,t) \in D$ and $t \leq 0$. Furthermore, since the restriction of $R_1^{F\times\{0\}}$ to $\mathbb{R}^n\times\,]0,1[$ is a temperature that takes values only in the interval $[0,1]$, it is, in view of [13, Theorem 5.5], the Gauss–Weierstrass integral of the function

$$f(x) = \liminf_{t\to 0+} R_1^{F\times\{0\}}(x,t) \leq \chi_F(x).$$

Therefore, $R_1^{F\times\{0\}} \leq W_F$ on $\mathbb{R}^n\times\,]0,1[$, and so equality holds there. Thus,

$$R_1^{F\times\{0\}}(x,t) = \begin{cases} W_F(x,t) & \text{if } t \neq 0, \\ \chi_F(x) & \text{if } t = 0. \end{cases}$$

We now put $u_0 = 1$ on $F \times \{0\}$ and $u_0(\infty) = 0$. Theorem 2.8 shows that u_0 is resolutive for D, and that $S_{u_0}^D \leq R_1^{F\times\{0\}}$ on D. If w is any supertemperature in the class $\mathfrak{U}_{u_0}^D$, then $w \geq 0$ on D by the minimum principle, and $w \geq W_F$ on $\mathbb{R}^n\times\,]0,1[$ by [6, page 287]. Therefore, $w \geq W_F$ on D, so that $S_{u_0}^D \geq W_F = R_1^{F\times\{0\}}$ on D, and hence equality holds.

3. Greatest thermic minorants

In this section, we first give a characterization of the greatest thermic minorant of a given supertemperature on an open set. We then use the characterization to show that, if u is a nonnegative supertemperature on E and D is an open subset of E, then replacing u on D by its greatest thermic minorant on D gives a function whose lower semicontinuous smoothing is a supertemperature on E and equal to u on $E\backslash\overline{D}$. Specializing to the case where D is a heat ball Ω such that $\overline{\Omega} \subseteq E$, we show that the greatest thermic minorant of u on Ω is equal to S_u^Ω. Furthermore, if we replace u on Ω by S_u^Ω, then the resultant function, whose lower semicontinuous smoothing is a supertemperature on E, is itself a supertemperature except at the centre of the heat ball.

Theorem 3.1 is analogous to [1, Theorem 6.4.10], which is also mentioned in [6, page 123].

Theorem 3.1. *Let u be a supertemperature on an open set E, and let $\{E_k\}$ be an expanding sequence of bounded open sets such that $E_k \cup \partial_e E_k \subseteq E$ for all k and $\bigcup_{k=1}^\infty E_k = E$.*

(a) *For each positive integer m, the sequence $\{S_u^{E_k}\}_{k\geq m}$ is decreasing on E_m.*

(b) *If there is a point $p_0 \in E$ such that*

$$\lim_{k\to\infty} S_u^{E_k}(p_0) > -\infty,$$

then u has a thermic minorant on $\Lambda(p_0; E)$.

(c) *If u has a thermic minorant on E, then the greatest one is $\lim_{k\to\infty} S_u^{E_k}$.*

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Proof. For each positive integer k, $\partial_e E_k$ is a compact subset of E, so that u is lower bounded on $\partial_e E_k$. It therefore follows from Theorem 2.5 that the restriction of u to $\partial_e E_k$ is resolutive for E_k, and that $S_u^{E_k} \leq u$ on E_k. If $w \in \mathfrak{L}_u^{E_{k+1}}$, then for all $q \in \partial_n E_{k+1}$ we have

$$\limsup_{p\to q}(w-u)(p) \leq \limsup_{p\to q} w(p) - u(q) \leq 0,$$

and for all $q \in \partial_{ss} E_{k+1}$ we similarly have $\limsup_{p\to q+}(w-u)(p) \leq 0$. Therefore $w \leq u$ on E_{k+1} by the maximum principle [17, Theorem 8.2]. It follows that, for all $q \in \partial_n E_k$, we have

$$\limsup_{p\to q, p\in E_k} w(p) \leq \limsup_{p\to q} u(p) \leq u(q),$$

and for all $q \in \partial_{ss} E_k$ we similarly have $\limsup_{p\to q+, p\in E_k} w(p) \leq u(q)$. Therefore, since w is upper bounded, we have $w \in \mathfrak{L}_u^{E_k}$. It follows that $S_u^{E_{k+1}} \leq S_u^{E_k}$ on E_k. This proves (a).

To prove (b), we take any point $q_0 \in \Lambda(p_0; E)$ and choose a polygonal path γ in E which joins p_0 to q_0 along which the temporal variable is strictly decreasing. Since γ is compact, we can find a positive integer m such that $\gamma \subseteq E_m$. If $h = \lim_{k\to\infty} S_u^{E_k}$ on E, then $h(p_0) > -\infty$, and so the Harnack monotone convergence theorem shows that h is a temperature on $\Lambda(p_0; E_m)$. This holds for all sufficiently large values of m, and so h is a temperature on $\Lambda(p_0; E)$. Therefore, because $S_u^{E_k} \leq u$ on E_k for all k, h is a thermic minorant of u on $\Lambda(p_0; E)$.

To prove (c), we let w denote a thermic minorant of u on E. For each k, $\partial_e E_k$ is a compact subset of E, so that w is upper bounded on $\partial_e E_k$. Therefore the maximum principle implies that w is upper bounded on E_k. Moreover, for any $q \in \partial_e E_k$ we have $\lim_{p\to q, p\in E_k} w(p) = w(q) \leq u(q)$, and so it follows that $w \in \mathfrak{L}_u^{E_k}$. Therefore $w \leq S_u^{E_k}$ on E_k, and so $w \leq \lim_{k\to\infty} S_u^{E_k}$ on E. □

Theorem 3.2. *Let u be a supertemperature which has a thermic minorant on an open set E, and let D be an open subset of E. If h is the greatest thermic minorant of u on D, and*

$$w = \begin{cases} h & \text{on } D, \\ u & \text{on } E\backslash D, \end{cases}$$

then the lower semicontinuous smoothing $\widehat{w}$ is a supertemperature on E such that $\widehat{w} = w$ on $E\backslash\partial D$.

Proof. We first consider the case where $u \geq 0$ on E. By [17, Theorem 8.50], we can write D as the union of a sequence $\{D_k\}$ of bounded open sets such that, for each k, $\overline{D}_k \subseteq D_{k+1}$, $\partial_s D_k = \emptyset$ and $\partial_{ss} D_k$ has only finitely many points. By Theorem 2.8, for each k, the restriction of u to $\partial_e D_k$ is resolutive for D_k and, because $\partial_{ss} D_k$ is polar and $\partial_s D_k = \emptyset$, we have $S_u^{D_k} = R_u^{E\backslash D_k}$ on D_k. Therefore, $\widehat{R}_u^{E\backslash D_k} = S_u^{D_k}$ on D_k, by [17, Theorem 7.27(d)] or [6, page 297].

Since $\{E\backslash D_k\}$ is a contracting sequence of subsets of E, the sequence of smoothed reductions $\{\widehat{R}_u^{E\backslash D_k}\}$ is decreasing on E, and therefore tends to a limit v on E. Moreover, for each k the function $\widehat{R}_u^{E\backslash D_k}$ is a supertemperature on E, which is equal to $R_u^{E\backslash D_k} = u$

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on $E\backslash\overline{D}_k$ by [17, Theorem 7.13] or [6, Theorem 1.XVII.2], and hence on $E\backslash D$. It follows that $\widehat{R}_u^{E\backslash D_k} = u$ on $E\backslash D$, and hence $v = u$ on $E\backslash D$. Furthermore, by Theorem 3.1,

$$h = \lim_{k\to\infty} S_u^{D_k} = \lim_{k\to\infty} \widehat{R}_u^{E\backslash D_k} = v$$

on D. Since the sequence $\{\widehat{R}_u^{E\backslash D_k}\}$ is decreasing on E, and its limit $v = w$ on E, it follows from [17, Theorem 7.13] or [6, page 295] that $\widehat{w}$ is a supertemperature on E and equal to w on $E\backslash\partial D$.

We now consider the general case. If g is the greatest thermic minorant of u on E, then the case just proved can be applied to $u - g$. Thus, if

$$f = \begin{cases} h - g & \text{on } D, \\ u - g & \text{on } E\backslash D, \end{cases}$$

then $\widehat{f}$ is a supertemperature on E such that $\widehat{f} = f$ on $E\backslash\partial D$. Since g is continuous on E, we have $\widehat{f} + g = \widehat{w}$ on E. Hence, $\widehat{w}$ is a supertemperature on E and equal to $f + g = w$ on $E\backslash\partial D$. □

Corollary 3.3. *Let u be a supertemperature on an open set E, let D be a bounded open set such that $\overline{D} \subseteq E$, and let h be the greatest thermic minorant of u on D. If*

$$w = \begin{cases} h & \text{on } D, \\ u & \text{on } E\backslash D, \end{cases}$$

then $\widehat{w}$ is a supertemperature on E and equal to w on $E\backslash\partial D$.

Proof. Since $\overline{D}$ is a compact subset of E, we can find a bounded open superset C of $\overline{D}$ such that $\overline{C} \subseteq E$. Since $\overline{C}$ is compact, u is lower bounded on C. Applying Theorem 3.2 to u on C, we deduce that $\widehat{w}$ is a supertemperature on C and equal to w on $C\backslash\partial D$. The result follows easily. □

Theorem 3.4 is the analogue for heat balls of [1, Theorem 3.6.5], but is far harder to prove.

Theorem 3.4. *Let u be a supertemperature on an open set E, and let $\Omega = \Omega(p_0; c_0)$ be a heat ball such that $\overline{\Omega} \subseteq E$. Then the greatest thermic minorant of u on Ω is S_u^Ω.*

Proof. By Theorem 2.5, the restriction of u to $\partial\Omega$ $(= \partial_e\Omega)$ is resolutive for Ω, and the function h, defined by

$$h = \begin{cases} u & \text{on } E\backslash\overline{\Omega}, \\ S_u^\Omega & \text{on } \Omega, \end{cases}$$

can be extended to a supertemperature $v \leq u$ on E. By Corollary 3.3, if $w = u$ on $E\backslash\Omega$, and w is equal on Ω to the greatest thermic minorant of u on Ω, then $\widehat{w}$ is a supertemperature on E and equal to w on $E\backslash\partial\Omega$. By [14, Theorem 2] or [17, Theorem 6.43], the functions $\mathcal{M}(\widehat{w}; p_0; \cdot)$ and $\mathcal{M}(v; p_0; \cdot)$ are continuous at c_0, so that

$$\mathcal{M}(\widehat{w}; p_0; c_0) = \lim_{c\to c_0+} \mathcal{M}(u; p_0; c) = \mathcal{M}(v; p_0; c_0).$$

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Since $\widehat{w} \geq S_u^\Omega = v$ on Ω, and $\widehat{w} = u = v$ on $E\backslash\overline{\Omega}$, we have $\widehat{w} \geq v$ almost everywhere on E, so that [17, Theorem 3.59] implies that $\widehat{w} \geq v$ everywhere on E. Furthermore, [14, Theorem 4] or [17, Theorem 6.45] shows that, whenever $0 < c \leq c_0$,

$$\mathcal{M}(\widehat{w}; p_0; c) = \widehat{w}(p_0) = \mathcal{M}(\widehat{w}; p_0; c_0) = \mathcal{M}(v; p_0; c_0) = v(p_0) = \mathcal{M}(v; p_0; c).$$

Since $\widehat{w} - v$ is nonnegative and continuous on Ω, it follows that $\widehat{w} = v$ on Ω, as asserted. □

Theorem 3.4 shows that the greatest thermic minorant of u on Ω is equal to S_u^Ω regardless of whether the set of irregular points of $\partial\Omega$, namely $\{p_0\}$, is a null set for the Riesz measure associated with u. This is in contrast to an observation made by Brelot [4, page 116] concerning a formula of Frostman [9], for the superharmonic case.

We can now prove an analogue for heat balls of the elementary result [1, Corollary 3.2.5]. It is not, of course, covered by [2, Satz 4.1.4], because that result says nothing about the function values on $\partial\Omega$.

Theorem 3.5. *Let u be a supertemperature on an open set E, and let $\Omega = \Omega(p_0; c)$ be a heat ball such that $\overline{\Omega} \subseteq E$. Then the function w, defined by*

$$w = \begin{cases} S_u^\Omega & \text{on } \Omega, \\ u & \text{on } E\backslash\Omega, \end{cases}$$

is a supertemperature on $E\backslash\{p_0\}$, and its lower semicontinuous smoothing $\widehat{w}$ is a supertemperature on E.

Proof. Theorem 2.5 shows that the restriction of u to $\partial_e\Omega$ is resolutive for Ω. Theorem 3.4 shows that S_u^Ω is the greatest thermic minorant of u on Ω. Therefore, Corollary 3.3 shows that $\widehat{w}$ is a supertemperature on E and equal to w on $E\backslash\partial\Omega$. By [17, Corollary 3.41], every point $q \in \partial\Omega\backslash\{p_0\}$ is a regular point for Ω. It therefore follows from [17, Theorems 8.46 and 8.44], or [12, Theorem 34 and Lemma 32], that

$$\liminf_{p\to q} S_u^\Omega(p) \geq \liminf_{p\to q, p\in\partial\Omega} u(p) \geq u(q)$$

for every such point q. The lower semicontinuity of u on $E\backslash\Omega$ now implies that w is lower semicontinuous at every point $q \in \partial\Omega\backslash\{p_0\}$, and hence on $E\backslash\{p_0\}$. Thus $w = \widehat{w}$ on $E\backslash\{p_0\}$, which proves the result. □

Remark 3.6. In the context of Theorem 3.5, we cannot generally conclude that w is a supertemperature on E. For example, if $u(p) = -|p - p_0|^2$, then $\Theta u < 0$ on an open neighbourhood E of p_0. If $\Omega = \Omega(p_0; c_0)$ is chosen such that $\overline{\Omega} \subseteq E$, and we put $v = u - S_u^\Omega$ on Ω, then v is a positive supertemperature on Ω because $\Theta v < 0$. If w was a supertemperature on E, then it would be lower semicontinuous at p_0, and we would have

$$0 \leq \limsup_{p\to p_0} v(p) = u(p_0) - \liminf_{p\to p_0} S_u^\Omega(p) \leq 0,$$

so that v would be a barrier at p_0. The point p_0 is irregular for Ω by [17, Example 8.36], and so [12, Theorem 34] or [17, Theorem 8.46] shows that there is no barrier at p_0.

4. Reductions and the temporal variable

If the temporal variable truly represents time, then we would expect the values of the nonnegative supertemperature $u(y, s)$ for $s \geq a$ to have no effect on the values of the reduction $R_u^L(x, t)$ for $t < a$. The next theorem implies that this is indeed the case.

Theorem 4.1. *Let u be a nonnegative supertemperature on an open set E, and let L be any subset of E.*

(a) *If D is an open subset of E such that $E \cap \partial_e D = \emptyset$, then $R_u^L = R_u^{L\cap\overline{D}}$ on $E \cap \overline{D}$.*
(b) *More generally, if there is an expanding sequence $\{D_k\}$ of open subsets of E such that $E \cap \partial_e D_k = \emptyset$ for all k, and $M = \bigcup_{k=1}^{\infty} \overline{D}_k$, then $R_u^L = R_u^{L\cap M}$ on $E \cap M$.*

Proof. (a) Since $L \cap \overline{D} \subseteq L$, we have $R_u^{L\cap\overline{D}} \leq R_u^L$ on E.

Let v be a nonnegative supertemperature on E such that $v \geq u$ on $L \cap \overline{D}$. The condition $E \cap \partial_e D = \emptyset$ implies that $E \cap \overline{D} = E \cap (D \cup \partial_a D)$ and $E\backslash\overline{D} = E\backslash(D \cup \partial_a D)$. Therefore, if w is defined by

$$w = \begin{cases} v \wedge u & \text{on } E \cap \overline{D}, \\ u & \text{on } E\backslash\overline{D}, \end{cases}$$

then w is a nonnegative supertemperature on E, by Lemma 2.1. Since $v \geq u$ on $L \cap \overline{D}$, we have $w \geq u$ on $L \cap \overline{D}$, and clearly $w = u$ on $L\backslash\overline{D}$. Therefore $w \geq R_u^L$ on E, and in particular $v \geq w \geq R_u^L$ on $E \cap \overline{D}$. It follows that $R_u^{L\cap\overline{D}} \geq R_u^L$ on $E \cap \overline{D}$, and so equality holds there.

(b) By part (a), we have $R_u^L = R_u^{L\cap\overline{D}_k}$ on $E \cap \overline{D}_k$ for all k. The sequence $\{L \cap \overline{D}_k\}$ is expanding and its union is $L \cap M$, so that [6, page 318, (e)] or [17, Theorem 9.33] shows that $\lim_{k\to\infty} R_u^{L\cap\overline{D}_k} = R_u^{L\cap M}$ on E. Given any point $p \in E \cap M$, there is a positive integer k_p such that $p \in E \cap \overline{D}_k$ for all $k \geq k_p$. Since $R_u^L(p) = R_u^{L\cap\overline{D}_k}(p)$ for all such k,

$$R_u^L(p) = \lim_{k\to\infty} R_u^{L\cap\overline{D}_k}(p) = R_u^{L\cap M}(p),$$

as required. □

Example 4.2. In the context of Theorem 4.1, if $b \in \mathbb{R}$ and $D = \{(x, t) \in E : t < b\}$, then $E \cap \partial_e D = \emptyset$, so that Theorem 4.1(a) shows that $R_u^L = R_u^{\{(x,t)\in L: t\leq b\}}$ on $\{(x, t) \in E : t \leq b\}$. Moreover, if $D_k = \{(x, t) \in E : t < b - (1/k)\}$ for all k, then the sequence $\{D_k\}$ is expanding and $E \cap \partial_e D_k = \emptyset$ for all k. Therefore, since $\bigcup_{k=1}^{\infty} \overline{D}_k = \{(x, t) \in \overline{E} : t < b\}$, Theorem 4.1(b) implies that $R_u^L = R_u^{\{(x,t)\in L: t<b\}}$ on D.

Example 4.3. In the context of Theorem 4.1, if $p_0 \in E$ and $\Lambda = \Lambda(p_0; E)$, then $E \cap \partial_e \Lambda = \emptyset$ by [17, Lemma 8.4] or [12, Lemma 1], so that Theorem 4.1(a) shows that $R_u^L = R_u^{L\cap\overline{\Lambda}}$ on $E \cap \overline{\Lambda}$. More generally, let $D_k = \bigcup_{j=1}^{k} \Lambda(q_j; E)$ for some points $q_1, \ldots, q_k \in E$. If $q \in \partial_n D_k$, then for every $r > 0$ we have $H(q, r)\backslash D_k \neq \emptyset$, so that $H(q, r)\backslash\Lambda(q_j; E) \neq \emptyset$ for any j, which implies that $q \in \partial_n \Lambda(q_j; E)$ for some j, and hence $q \in \partial_e E$ by [17, Lemma 8.4]. On the other hand, if $q \in \partial_{ss} D_k$, then for every

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$r > 0$ we have $H^*(q, r) \cap D_k \neq \emptyset$. Therefore there is an integer j_0, and a sequence $\{p_l\}$ in $H^*(q, 1) \cap \Lambda(q_{j_0}; E)$ such that $p_l \to q$ as $l \to \infty$. This implies that $q \in \partial_e \Lambda(q_{j_0}; E)$, and so $q \in \partial_e E$ by [17, Lemma 8.4]. Thus, $E \cap \partial_e D_k = \emptyset$ for all k, and Theorem 4.1(a) shows that $R_u^L = R_u^{L \cap \overline{D}_k}$ on $E \cap \overline{D}_k$.

Since $\Lambda(p_0; E) = \bigcup_{p \in \Lambda(p_0;E)} \Lambda(p; E)$, the Lindelöf property of $\mathbb{R}^{n+1}$ shows that there is a sequence of points $\{q_j\}$ in $\Lambda(p_0; E)$ such that $\Lambda(p_0; E) = \bigcup_{j=1}^{\infty} \Lambda(q_j; E)$. Taking D_k as above, the sequence $\{D_k\}$ satisfies the hypotheses of Theorem 4.1(b), and so if $M = \bigcup_{k=1}^{\infty} \overline{D}_k$ then $R_u^L = R_u^{L \cap M}$ on the proper subset $E \cap M$ of $\overline{\Lambda}(p_0; E)$.

For the case considered in Example 4.2 we can go further, as follows.

Theorem 4.4. *Let u be a nonnegative supertemperature on an open set E, let $L \subseteq E$, let $b \in \mathbb{R}$, and let $D = \{(x, t) \in E : t < b\}$. Then the reduction of u over L relative to E, is equal on D to the reduction of u over $L \cap D$ relative to D.*

Proof. For any open subset C of E, we denote the reduction of u over $L \cap C$ relative to C by ${}^C R_u^{L \cap C}$.

If v is a nonnegative supertemperature on E such that $v \geq u$ on L, then its restriction to D is a nonnegative supertemperature on D such that $v \geq u$ on $L \cap D$. Therefore $v \geq {}^D R_u^{L \cap D}$ on D, and it follows that ${}^E R_u^L \geq {}^D R_u^{L \cap D}$ on D.

To prove the reverse inequality, we now suppose that w is a nonnegative supertemperature on D such that $w \geq u$ on $L \cap D$. For each positive integer k, we put $E_k = \{(x, t) \in E : t \leq b - (1/k)\}$ and $D_k = \{(x, t) \in E : t < b - (1/k)\}$, and note that $E \cap \partial_e D_k = \emptyset$ for all k. Therefore, if w_k is defined on E by

$$w_k(q) = \begin{cases} (w \wedge u)(q) & \text{if } q \in D_k, \\ u(q) & \text{if } q \in E \backslash (D_k \cup \partial_a D_k), \\ \left(\liminf_{p \to q-} w(p) \right) \wedge u(q) & \text{if } q \in E \cap \partial_a D_k, \end{cases}$$

then w_k is a supertemperature on E, by Lemma 2.1. Noting that $\liminf_{p \to q-} w(p) = w(q)$ for all $q \in E \cap \partial_a D_k$, we see that w_k can be written as

$$w_k = \begin{cases} w \wedge u & \text{on } E_k, \\ u & \text{on } E \backslash E_k. \end{cases}$$

Since $w \geq u$ on $L \cap D \supseteq L \cap E_k$, it is now clear that $w_k \geq u$ on $L \cap E_k$, and hence on L. Therefore, $w_k \geq {}^E R_u^L$ on E, so that $w \geq {}^E R_u^L$ on E_k for every k, and hence on D. It follows that ${}^D R_u^{L \cap D} \geq {}^E R_u^L$ on D, and so equality holds. □

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