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PERSISTENCE PROBABILITY FOR A CLASS OF GAUSSIAN PROCESSES RELATED TO RANDOM INTERFACE MODELS

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Abstract

We consider a class of Gaussian processes which are obtained as height processes of some $(d+1)$-dimensional dynamic random interface model on $\mathbb{Z}^d$. We give an estimate of persistence probability, namely, large T asymptotics of the probability that the process does not exceed a fixed level up to time T. The interaction of the model affects the persistence probability and its asymptotics changes depending on the dimension d.

Keywords: Persistence probability; random interface; Gaussian process; interacting diffusion process

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1. Introduction

Consider a real valued stochastic process $\{X_t\}_{t\geq 0}$ starting from zero in discrete or continuous time. To study the asymptotics of the probability $p_T := \mathbb{P}\{X_t \leq 1 \text{ for every } t \in [0, T]\}$ as $T \to \infty$ is called the problem of persistence probability. This is also referred to as survival probability or the one-sided exit problem and is one of the classical problems in probability theory. For example, it is well-known that $p_T \asymp T^{-1/2}$ for Brownian motion or symmetric random walk. Also, for fractional Brownian motion with Hurst parameter H, p_T behaves as $T^{-(1-H)+o(1)}$ (see [2], [21]). In many cases p_T exhibits power law decay and the main problem is to identify its exponent. Even for a Gaussian process with given covariances this is not an easy problem and the precise exponent is known only for a handful of cases (see [5], [19]). Recently, in connection with several applications, persistence probabilities for some stochastic processes such as the integrated process or weighted random walks etc. have been actively investigated (see, for example, [3], [4], and [10]). Many of these processes do not have the Markov property nor stationary increments and this makes the problem difficult. See a review [5] for recent developments.

As one of the motivations and related topics, there are extensive studies of persistence probability of fluctuating interface in the literature. A typical model is the Kardar–Parisi–Zhang (KPZ) equation given by the following:

$$\frac{\partial}{\partial t}h(x,t) = \frac{1}{2}\lambda(\nabla h)^2 + \nu\Delta h + \eta(x,t), \qquad x \in \mathbb{R}, t \geq 0, \tag{1.1}$$

where $\eta(x,t)$ is a space-time white noise. This describes the time evolution of a $(1+1)$-dimensional random interface and $h(x,t)$ represents the height at position $x \in \mathbb{R}$ and time $t \geq 0$.

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When $\lambda = 0$, (1.1) is called the Edwards–Wilkinson (EW) equation and the solution is the infinite dimensional Ornstein–Uhlenbeck process in this case. Studies about the KPZ/EW equation have been quite active in the last several decades and particularly, a lot of analytic, numerical, and experimental results have been obtained about the asymptotics of $p(0, T)$ or $\lim_{t_0 \to \infty} p(t_0, T)$ as $T \to \infty$, where $p(t_0, T) := \mathbb{P}\{h(x, s) \neq h(x, t_0)$ for every $s \in (t_0, t_0 + T)\}$ is the persistence probability focused on the height at a fixed position $x \in \mathbb{R}$ (see [17], [20], [24] and references therein).

However, to the author's knowledge, analytic results have been obtained only for the EW equation and almost all of those results are mathematically non-rigorous. Also, most of these studies focused on the one-dimensional model. In the viewpoint of the study of random interfaces, it seems natural to consider the higher dimensional model. The main purpose of this paper is to study persistence probability for a multi-dimensional dynamic random interface model in a mathematically rigorous way and to give an estimate which clarifies the dependence on the dimension. For this purpose we discretize the underlying space and consider a dynamic Gaussian interface model on $\mathbb{Z}^d$. In particular, if we focus on the height process at site $0 \in \mathbb{Z}^d$ then a class of non-Markov Gaussian processes appears.

1.1. Model and result

In this paper we consider the following lattice system of interacting diffusion processes as a model of dynamic $(d + 1)$-dimensional random interfaces

$$\mathrm{d}\phi_t(x) = \left\{ -\phi_t(x) + \sum_{y \neq x} q(y - x)\phi_t(y) \right\} \mathrm{d}t + \sqrt{2}\, \mathrm{d}B_t(x), \qquad x \in \mathbb{Z}^d, \tag{1.2a}$$

$$\phi_0(x) = 0, \qquad x \in \mathbb{Z}^d, \tag{1.2b}$$

where $\{B_t(x)\}_{x \in \mathbb{Z}^d}$ is a family of independent standard one-dimensional Brownian motions and we assume the following conditions for $\{q(x)\}_{x \in \mathbb{Z}^d}$:

(i) That $q(x) = q(-x) \geq 0$ for every $x \in \mathbb{Z}^d$.

(ii) There exists $R > 0$ such that $q(x) = 0$ for every $x \in \mathbb{Z}^d$ with $|x| \geq R$.

(iii) The summation $\sum_{x \neq 0} q(x) = 1$.

(iv) Additive group generated by $\{x \in \mathbb{Z}^d; q(x) > 0\}$ is $\mathbb{Z}^d$.

The physical meaning of (1.2) is as follows. For a configuration $\phi = \{\phi(x)\}_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, which describes a phase separating interface embedded in $(d + 1)$-dimensional space, consider a (formal) Hamiltonian

$$H(\phi) = \frac{1}{2} \sum_{<x,y>} q(y - x)(\phi(x) - \phi(y))^2, \tag{1.3}$$

where we take the summation for the pair $x, y \in \mathbb{Z}^d$. Then (1.2) corresponds to the Langevin equation associated with $H(\phi)$,

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t) + \sqrt{2} dB_t(x), \qquad x \in \mathbb{Z}^d, \tag{1.4}$$

and this describes the time evolution of a $(d+1)$-dimensional phase separating random interface starting from flat initial configuration at height 0. The term $\phi_t(x)$ corresponds to the height

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of the interface at position $x \in \mathbb{Z}^d$ and time $t \geq 0$. Note that though (1.3) is a formal sum, $\partial H/\partial\phi(x)$ makes sense by the assumption (ii) on q. Since the energy of the interface ϕ is determine by its height differences, this model is called the $\nabla\phi$ interface model and its studies have been active in both of static and dynamic aspects (see [14] and references therein).

Since the stochastic differential equation (SDE) (1.2) is determined from the constant diffusion coefficient and linear drift with the weight satisfying condition (iii), it is also called the critical Ornstein–Uhlenbeck process and has been investigated in connection with the study of infinite dimensional interacting diffusion processes (see, for example, [11], [15]). By standard approximation arguments (1.2) has a unique strong solution. In particular, we have a random walk representation of space-time correlations of this model (see [8], [12, Proposition 1.3]). By these facts, the following holds.

Lemma 1.1. *Set* $g_t := \phi_t(0), t \geq 0$. *Then,* $\{g_t\}_{t\geq 0}$ *is a continuous Gaussian process on* $[0, \infty)$ *with mean* 0 *and the covariance is given by*

$$\Gamma(s,t) := \mathbb{E}[g_s g_t] = \int_0^s \mathcal{P}(S_{t-s+2u} = 0)\,\mathrm{d}u, \qquad 0 \leq s \leq t, \tag{1.5}$$

where $\{S_u\}_{u\geq 0}$ *is a continuous time random walk on* $\mathbb{Z}^d$ *with the generator*

$$QF(x) = \sum_{y\neq x} q(y-x)(F(y) - F(x)), \qquad x \in \mathbb{Z}^d, \tag{1.6}$$

for $F\colon \mathbb{Z}^d \to \mathbb{R}$ *and* $\mathcal{P}$ *denotes its law starting at* $0 \in \mathbb{Z}^d$.

We stress that though the whole system $\{\phi_t(x); x \in \mathbb{Z}^d\}_{t\geq 0}$ has the Markov property, if we focus on the height at $0 \in \mathbb{Z}^d$ then $\{g_t\}_{t\geq 0} := \{\phi_t(0)\}_{t\geq 0}$ is a non-Markov process. Also, this is a non-stationary process.

By the above lemma our main problem is to study the persistence probability for a class of continuous Gaussian processes whose covariances are given by (1.5). Now we are in the position to state the main result of this paper. We have the following estimate on the persistence probability for $\{g_t\}_{t\geq 0}$.

Theorem 1.1. *There exist* $C_-, C_+ > 0$ *such that the following holds for every* $T > 0$ *large enough.*

$$\left.\begin{matrix} \mathrm{e}^{-C_-(\log T)} \\ \mathrm{e}^{-C_-(\log T)^3} \\ \mathrm{e}^{-C_-\sqrt{T}\log T} \\ \mathrm{e}^{-C_-T} \\ \mathrm{e}^{-C_-T} \end{matrix}\right\} \leq \mathbb{P}\{g_t \leq 1 \textit{ for every } t \in [0,T]\} \leq \begin{cases} \mathrm{e}^{-C_+(\log T)} & \textit{if } d = 1, \\ \mathrm{e}^{-C_+(\log T)^2} & \textit{if } d = 2, \\ \mathrm{e}^{-C_+\sqrt{T}} & \textit{if } d = 3, \\ \mathrm{e}^{-C_+(T/\log T)} & \textit{if } d = 4, \\ \mathrm{e}^{-C_+T} & \textit{if } d \geq 5. \end{cases}$$

Roughly speaking, the dynamics (1.2) represent averaging of the height of the interface at each site with those of surrounding sites (plus random noises). The number of surrounding sites increases as the dimension increases and the influence of the height of the original site decreases in the averaging. Hence, the correlation of the process $\{g_t\}_{t\geq 0}$ decays faster as the dimension increases (see Lemma 1.2 below) and the persistence event becomes hard to occur. Though the order of upper and lower bounds does not match in $2 \leq d \leq 4$, the above result shows that the persistence probability decays faster as the dimension becomes large.

We give several remarks about the result.

Remark 1.1. If we assume $\eta := \sum_{x\neq 0} q(x) < 1$ then (1.2) corresponds to the Langevin equation associated with a massive Hamiltonian

$$H(\phi) = \frac{1}{2}\sum_{<x,y>} q(y-x)(\phi(x)-\phi(y))^2 + \frac{1}{2}(1-\eta)\sum_{x\in\mathbb{Z}^d}(\phi(x))^2.$$

In this case, the generator (1.6) can be considered as

$$QF(x) = \sum_{y\neq x} \eta \frac{q(y-x)}{\eta}(F(y)-F(x)), \qquad x \in \mathbb{Z}^d,$$

with $\sum_{y\neq x} q(y-x)/\eta = 1$. Namely, this is a generator of continuous time random walk with killing rate $1-\eta$. Hence, we have

$$\mathbb{E}[g_s g_t] = \int_0^s \mathcal{P}(S_{t-s+2u} = 0)\,\mathrm{d}u \leq \int_0^s \mathrm{e}^{-C(t-s+2u)}\,\mathrm{d}u \leq C'\mathrm{e}^{-C(t-s)},$$

for some $C, C' > 0$ and the covariance of $\{g_t\}_{t\geq 0}$ decays exponentially fast. In this case, the proof of Theorem 1.1 for the case $d \geq 5$ works well and we can prove that

$$\mathrm{e}^{-C_- T} \leq \mathbb{P}\{g_t \leq 1 \text{ for every } t \in [0,T]\} \leq \mathrm{e}^{-C_+ T},$$

for every $T > 0$ large enough for arbitrary dimensions.

Remark 1.2. We can also consider the discrete time dynamics of a $(d+1)$-dimensional Gaussian random interface model. Define a $\mathbb{R}^{\mathbb{Z}^d}$-valued discrete time process $\{h_n(x); x \in \mathbb{Z}^d, n \geq 0\}$ by

$$h_n(x) := \begin{cases} 0 & \text{if } n = 0, \\ \sum_{y\in\mathbb{Z}^d} q(y-x)h_{n-1}(y) + \eta_n(x) & \text{if } n \geq 1, \end{cases} \tag{1.7}$$

where $\{\eta_n(x); x \in \mathbb{Z}^d, n \geq 0\}$ is a family of independent, identically distributed (i.i.d.) random variables. This model is called serial harness and was introduced by Hammersley as a discrete time model of evolving random interfaces (see [13], [16]). By iterating (1.7), we have

$$h_n(x) = \sum_{k=0}^{n-1}\sum_{y\in\mathbb{Z}^d} p_k(x,y)\eta_{n-k}(y),$$

where $p_k(x,y)$ is k-step transition probability of a random walk on $\mathbb{Z}^d$ with transition probability $\{q(x)\}_{x\in\mathbb{Z}^d}$. Therefore, if the law of noise variables $\eta_n(x)$ is given by $\mathcal{N}(0,\sigma^2)$ then the height process at the origin $\{h_n(0)\}_{n\geq 1}$ is a family of centered Gaussian random variables whose covariances are given by

$$\mathbb{E}[h_m(0)h_n(0)] = \sigma^2 \sum_{k=n-m}^{n-1}\sum_{y\in\mathbb{Z}^d} p_k(0,y)p_{k-n+m}(0,y) = \sigma^2\sum_{k=0}^{m-1} p_{n-m+2k}(0,0)$$

for $1 \leq m \leq n$. Hence, the covariance structure is the same as (1.5) and Theorem 1.1 would hold also for this model. Actually, our proof of the upper bound also works well for this model. For the lower bound some extra work might be needed.

Remark 1.3. (i) One of the difficulties for the estimate of the persistence probability for our model in $d \geq 2$ is that the process has long range correlations (see Lemma 1.2). Namely, our model has logarithmic correlations in $d = 2$ and polynomially decaying correlations in $d \geq 3$. Though there are several general theorems for the persistence probability for Gaussian processes (e.g. [19]), most of them require fast decaying correlations of the process (summability or integrability of correlations). Recently, Dembo and Mukherjee [9] gave a general criterion for the persistence probability for Gaussian processes. But they also require the summability of correlations and their result does not work well for our model in dimensions 2, 3, 4.

To the author's knowledge, even for the stationary Gaussian process with similar (and simpler) covariance structure to our model, there is no result with the same order of the upper and lower bound. The best known result is a classical result in [22] which studied stationary Gaussian processes whose correlation at time s and $s+t$ behaves as $\asymp t^{-\alpha}$ $(\alpha > 0)$ as $t \to \infty$. Though the Gaussian process considered in this paper is not a stationary process, large time behavior of the correlation is similar to this when $d \geq 3$ and our result for this case is the same as [22].

(ii) From the viewpoint of the study of dynamic interface models, it might be natural to consider Langevin's equation (1.4), which is associated with the more general Hamiltonian $H(\phi) = \frac{1}{2}\sum_{<x,y>} V(\phi(x) - \phi(y))$ instead of (1.3) where $V : \mathbb{R} \to \mathbb{R}$ denotes interaction potential (see [14]). However, since several parts of our proof rely on the Gaussian property of our model, at this moment we do not have an estimate about persistence probability for this general model.

1.2. Strategy of the proof

For the proof of Theorem 1.1 we first investigate the covariance structure of the process $\{g_t\}_{t\geq 0}$. We recall the local central limit theorem

$$\left| \mathcal{P}(S_t = x) - \frac{1}{(2\pi t)^{d/2}\sqrt{\det A}} \mathrm{e}^{-(x\cdot A^{-1}x/2t)} \right| \leq Ct^{-(d/2)-1} \tag{1.8}$$

for every $x \in \mathbb{Z}^d$ and $t > 0$ where A is the covariance matrix of random walk on $\mathbb{Z}^d$ with transition probability $\{q(x)\}_{x\in\mathbb{Z}^d}$ (see [18, Theorem 2.1.3]). By combining this with the random walk representation (1.5), we have the following lemma.

Lemma 1.2. *There exists $C > 0$ such that the following holds for every $0 \leq s \leq t$:*

- *For $d = 1$,* $\quad |\Gamma(s,t) - \kappa_1(\sqrt{(t+s)\vee 1} - \sqrt{(t-s)\vee 1})| \leq C.$
- *For $d = 2$,* $$\left|\Gamma(s,t) - \frac{\kappa_2}{2}(\log((t+s)\vee 1) - \log((t-s)\vee 1))\right| \leq C. \tag{1.9}$$
- *For $d \geq 3$,* $$0 \leq \Gamma(s,t) \leq C((t-s)\vee 1)^{-(d/2)+1}, \tag{1.10}$$

where $\kappa_d = (1/(2\pi)^{d/2}\sqrt{\det A})$.

In particular we remark that our model exhibits aging phenomena when $d \leq 2$. Namely, asymptotics of the correlation $(\Gamma(s, s+t)/\sqrt{\Gamma(s,s)}\sqrt{\Gamma(s+t,s+t)})$ as $s, t \to \infty$ depends on the choice of s, t (see [8]).

Now we explain the strategy of the proof of Theorem 1.1. The main part of the proof of upper bound is the case $d = 2$. The important property of the process $\{g_t\}_{t\geq 0}$ in this case is that the process has logarithmic correlations. This is similar to the two-dimensional discrete Gaussian free field (DGFF). In the recent study of DGFF, [7] introduced modified branching random walk (MBRW) and proved tightness of the (centered) maximum of DGFF by comparing with MBRW. We also introduce MBRW and consider comparison with $\{g_t\}_{t\geq 0}$. This reduces the estimate on the persistence probability for $\{g_t\}_{t\geq 0}$ to the problem about the partial maximum of MBRW. This problem can be handled by a multi-scale argument which is based on the hierarchical structure of MBRW. For the case $d \geq 3$, we follow the argument of [22] which studied the persistence probability for stationary Gaussian processes whose correlation behaves as $\asymp t^{-\alpha}$, $\alpha > 0$. The case $d = 1$ follows from comparison with Brownian motion.

For the proof of the lower bound in the case of $d \leq 3$, we use a measure change argument. We first add a suitable drift to the process $\{g_t\}_{0\leq t\leq T}$ so that the persistence event in time interval $[\delta T, T]$ $(0 < \delta < 1)$ occurs with large probability. We can estimate its cost by using the Cameron–Martin formula and we obtain the lower bound of the probability of the persistence event in time interval $[\delta T, T]$. Then, by using Slepian's lemma we obtain the lower bound of the probability of the event in time interval $[0, T]$. The lower bound in the case of $d \geq 4$ simply follows from Slepian's lemma.

In Sections 2 and 3 we give the proofs of the upper and the lower bounds of Theorem 1.1, respectively. At the end of this section, we remark that throughout this paper C represents a positive constant which does not depend on T but may depend on other parameters. Also, this C in estimates may change from place to place in the paper.

2. Proof of the upper bound

2.1. The case $d = 2$

For the proof of the upper bound in $d = 2$, we introduce MBRW which was originally considered by [7] in the study of the maximum of two-dimensional discrete Gaussian free fields. For $k = 0, 1, 2, \ldots$, let $\mathcal{I}_k = \{([1, 2^k] \cap \mathbb{Z}) + z;\ z \in \mathbb{Z}\}$ be the set of all intervals on $\mathbb{Z}$ with size 2^k and define $\mathcal{I}_k(x) = \{I \in \mathcal{I}_k;\ I \ni x\}$ for $x \in \mathbb{Z}$. For $L \in \mathbb{N}$, $\mathcal{I}_k^L$ denotes all intervals in $\mathcal{I}_k$ whose left end point belongs to the interval $[1, 2^L]$. The set $\{a_{k,I};\ I \in \mathcal{I}_k^L,\ k \geq 0\}$ is a family of centered Gaussian random variables which are independent with respect to $k \geq 0$ and i.i.d. with respect to $I \in \mathcal{I}_k^L$ with variance 2^{-k}. We extend it periodically over $\mathbb{Z}$, namely we define $\{a_{k,I}^L;\ I \in \mathcal{I}_k,\ k \geq 0\}$ as

$$a_{k,I}^L := \begin{cases} a_{k,I} & \text{if } I \in \mathcal{I}_k^L, \\ a_{k,I'} & \text{if } I = I' + 2^L y \quad \text{for } I' \in \mathcal{I}_k^L \quad \text{and } y \in \mathbb{Z}. \end{cases}$$

Then MBRW $\{M_x^L\}_{x=1}^{2^L}$ on $[1, 2^L] \cap \mathbb{Z}$ is defined by $M_x^L := \sum_{k=0}^{L} \sum_{I\in\mathcal{I}_k(x)} a_{k,I}^L$. By definition $\mathbb{E}[(M_x^L)^2] = L + 1$ for every $x \in [1, 2^L] \cap \mathbb{Z}$. We also define

$$\widetilde{M}_x^L := \frac{M_x^L + vg}{\sqrt{\mathbb{E}[(M_x^L + vg)^2]}} = \frac{M_x^L + vg}{\sqrt{L + 1 + v^2}}$$

where $v > 0$, and g is a standard normal random variable independent of $\{M_x^L\}_{x=1}^{2^L}$.

Now, we recall Slepian's lemma which is used frequently throughout this paper.

Proposition 2.1. ([23].) *Let $\xi = \{\xi_i\}_{1\leq i\leq n}$, $\zeta = \{\zeta_i\}_{1\leq i\leq n}$ be centered $\mathbb{R}^n$-valued Gaussian random variables such that $\mathbb{E}[\xi_i^2] = \mathbb{E}[\zeta_i^2]$ for every $1 \leq i \leq n$ and $\mathbb{E}[\xi_i\xi_j] \leq \mathbb{E}[\zeta_i\zeta_j]$ for every $1 \leq i, j \leq n$. Then, we have*

$$\mathbb{P}\{\xi_i \leq \lambda_i \text{ for every } 1 \leq i \leq n\} \leq \mathbb{P}\{\zeta_i \leq \lambda_i \text{ for every } 1 \leq i \leq n\},$$

for every $\{\lambda_i\}_{1\leq i\leq n} \in \mathbb{R}^n$.

We remark that this inequality also holds for the continuous parameter setting (see [1, Theorem 2.2.1]). Define the normalized process of $\{g_t\}_{t\geq 0}$ as $\widetilde{g}_t := g_t/\sqrt{\mathbb{E}[g_t^2]}$, for $t > 0$. Then we have the following comparison estimate.

Lemma 2.1. *Let* $0 < \delta < 1$, $0 < \beta < 1$ *be fixed and set* $\Delta = 2^{[\beta L]}$ *and* $N = [(1-\delta)2^{L-[\beta L]}]$. *There exists* $v_0 = v_0(\delta) > 0$ *such that if* $v \geq v_0$, *then we have*

$$\begin{aligned}&\mathbb{P}\{\widetilde{g}_x \leq \lambda_x \text{ for every } x \in \{[\delta 2^L] + i\Delta; i = 0, 1, \dots, N\}\}\\ &\qquad \leq \mathbb{P}\{\widetilde{M}_x^L \leq \lambda_x \text{ for every } x \in \{[\delta 2^L] + i\Delta; i = 0, 1, \dots, N\}\},\end{aligned}$$

for every $\{\lambda_x\} \in \mathbb{R}^{N+1}$ *and* L *large enough.*

Proof. By Slepian's lemma, we have only to show that $\mathbb{E}[\widetilde{g}_x\widetilde{g}_y] \leq \mathbb{E}[\widetilde{M}_x^L\widetilde{M}_y^L]$ for every $x, y \in \{[\delta 2^L] + i\Delta; i = 0, 1, \dots, N\}$, $x \neq y$. By (1.9),

$$\mathbb{E}[g_x^2] \geq \frac{\kappa_2}{2}\log(2x) - C \geq \frac{\kappa_2 \log 2}{2} L - C_1, \tag{2.1}$$

for some $C_1 = C_1(\delta) > 0$ and

$$\begin{aligned}\mathbb{E}[g_x g_y] &\leq \frac{\kappa_2}{2}(\log((x+y)\vee 1) - \log(|x-y|\vee 1)) + C\\ &\leq \frac{\kappa_2 \log 2}{2}(L - \log_2(|x-y|+1)) + C_2,\end{aligned}$$

for some $C_2 > 0$. Therefore,

$$\mathbb{E}[\widetilde{g}_x\widetilde{g}_y] \leq \frac{L - \log_2(|x-y|+1) + C_2'}{L - C_1'}, \tag{2.2}$$

for some $C_1' = C_1'(\delta) >$ and $C_2' > 0$. Next, by definition of M_x^L,

$$\begin{aligned}\mathbb{E}[M_x^L M_y^L] &= \sum_{k=0}^{L}\sum_{I\in\mathcal{I}_k(x)}\sum_{I'\in\mathcal{I}_k(y)}\mathbb{E}[a_{k,I}^L a_{k,I'}^L]\\ &= \sum_{k=[\log_2(d^L(x,y)+1)]}^{L}(2^k - d^L(x,y))2^{-k}\\ &\geq L - \log_2(|x-y|+1) - C_3\end{aligned}$$

for some $C_3 > 0$ where $d^L(x, y) := \min\{|x - z|; z \in y + 2^L\mathbb{Z}\}$. Therefore,

$$\mathbb{E}[\widetilde{M}_x^L\widetilde{M}_y^L] \geq \frac{L - \log_2(|x-y|+1) - C_3 + v^2}{L + 1 + v^2}. \tag{2.3}$$

Now, by using the elementary fact that $(a + x)/(b + x) \geq a/b$ for every $x \geq 0$ if $0 < a \leq b$ and $\log_2|x - y| \geq [\beta L]$ for $x, y \in \{[\delta 2^L] + i\Delta; i = 0, 1, \dots, N\}$ with $x \neq y$, (2.2) and (2.3) yield that $\mathbb{E}[\widetilde{g}_x\widetilde{g}_y] \leq \mathbb{E}[\widetilde{M}_x^L\widetilde{M}_y^L]$ if $v > 0$, $L > 0$ large enough.

Proof of Theorem 1.1 upper bound; the case $d = 2$. It is sufficient to show that there exists $C > 0$ such that

$$\mathbb{P}\{g_t \leq 1 \text{ for every } t \in [0, 2^L]\} \leq \mathrm{e}^{-CL^2},$$

for every $L \in \mathbb{N}$ large enough. By Lemma 2.1,

$$\begin{aligned}
&\mathbb{P}\{g_t \le 1 \text{ for every } t \in [0, 2^L]\} \\
&\quad \le \mathbb{P}\bigg\{\widetilde{g}_x \le \frac{1}{\sqrt{\mathbb{E}[g_x^2]}} \text{ for every } x \in \{[\delta 2^L] + i\Delta;\, i = 0, 1, \dots, N\}\bigg\} \\
&\quad \le \mathbb{P}\bigg\{\widetilde{M}_x^L \le \frac{1}{\sqrt{\mathbb{E}[g_x^2]}} \text{ for every } x \in \{[\delta 2^L] + i\Delta;\, i = 0, 1, \dots, N\}\bigg\} \\
&\quad \le \mathbb{P}\bigg\{g \le -\frac{\gamma}{v} L\bigg\} \\
&\qquad + \mathbb{P}\bigg\{M_x^L \le \gamma L + \frac{\sqrt{L+1+v^2}}{\sqrt{\mathbb{E}[g_x^2]}} \text{ for every } x \in \{[\delta 2^L] + i\Delta;\, i = 0, 1, \dots, N\}\bigg\},
\end{aligned}$$

for every $\gamma > 0$. The first term in the right-hand side is less than $\exp(-CL^2)$ by Gaussian tail estimate. By (2.1) there exists $C > 0$ such that $\sqrt{L+1+v^2}/\sqrt{\mathbb{E}[g_x^2]} \le C$ for every $x \in \{[\delta 2^L] + i\Delta;\, i = 0, 1, \dots, N\}$ and $L > 0$ large enough. Hence, Proposition 2.2 yields that the second term is also less than $\exp(-CL^2)$ for small $\gamma > 0$ and we obtain the upper bound.

Proposition 2.2. *Let $0 < \delta < 1$, $0 < \beta < 1$ be fixed and set $\Delta = 2^{[\beta L]}$, $N = [(1-\delta)2^{L-[\beta L]}]$. There exist $\gamma_0 > 0$ and $C > 0$ such that for every $\gamma \le \gamma_0$, it holds that*

$$\mathbb{P}\{M_x^L \le \gamma L \textit{ for every } x \in \{[\delta 2^L] + i\Delta;\, i = 0, 1, \dots, N\}\} \le \mathrm{e}^{-CL^2},$$

for every L large enough.

For the proof of this proposition, we introduce some notation. For $0 < \delta < 1$ and $0 < \alpha < 1$, let $\Lambda_{L,\delta}^{(\alpha)} := \{[\delta 2^L] + i 2^{[\alpha L]};\, i = 0, 1, \dots, [(1-\delta)2^{L-[\alpha L]}]\}$. Note that if $0 < \beta < \alpha < 1$ then $\Lambda_{L,\delta}^{(\alpha)} \subset \Lambda_{L,\delta}^{(\beta)}$. We also define $\mathcal{F}^{(\alpha)} := \sigma(a_{k,I}^L;\, I \in \mathcal{I}_k, [\alpha L] \le k \le L)$, and $M_x^{L,(\alpha)} := \mathbb{E}[M_x^L | \mathcal{F}^{(\alpha)}] = \sum_{k=[\alpha L]}^{L} \sum_{I \in \mathcal{I}_k(x)} a_{k,I}^L$. The main idea of the proof is a multi-scale argument which is based on the hierarchical structure of MBRW. This is often used in the study of branching random walks or two-dimensional discrete Gaussian free fields (see [6]).

Proof of Proposition 2.2. Let $0 < \delta < 1$, $0 < \beta < 1$ be fixed and consider events

$$\begin{aligned}
\mathcal{E} &= \{M_x^L \le \gamma_0 L \text{ for every } x \in \Lambda_{L,\delta}^{(\beta)}\}, \\
\mathcal{A} &= \{M_x^{L,(\alpha)} \le L^2 \text{ for every } x \in \Lambda_{L,\delta}^{(\alpha)}\}, \\
\mathcal{B} &= \{\sharp\{x \in \Lambda_{L,\delta}^{(\alpha)};\, M_x^{L,(\alpha)} \ge -\gamma_1 L\} \ge 2^{\varepsilon_1 L}\}, \\
\mathcal{C} &= \{\sharp\{x \in \Lambda_{L,\delta}^{(\alpha)};\, M_x^{L,(\beta)} \ge \gamma_2 L\} \ge 2^{\varepsilon_2 L}\},
\end{aligned}$$

where $\alpha \in (\beta, 1)$, $\gamma_0, \gamma_1, \gamma_2, \varepsilon_1, \varepsilon_2 > 0$ are to be specified later on. Then we have

$$\mathbb{P}\{\mathcal{E}\} \le \mathbb{P}\{\mathcal{A}^c\} + \mathbb{P}\{\mathcal{A} \cap \mathcal{B}^c\} + \mathbb{P}\{\mathcal{B} \cap \mathcal{C}^c\} + \mathbb{P}\{\mathcal{C} \cap \mathcal{E}\}.$$

We estimate each term in the right-hand side.

For $\mathbb{P}\{\mathcal{A}^c\}$, we have

$$\mathbb{P}\{\mathcal{A}^c\} \le |\Lambda_{L,\delta}^{(\alpha)}| \max_{x \in \Lambda_{L,\delta}^{(\alpha)}} \mathbb{P}\{M_x^{L,(\alpha)} \ge L^2\} \le C 2^{L-[\alpha L]} \max_{x \in \Lambda_{L,\delta}^{(\alpha)}} \exp\bigg\{-\frac{L^4}{2\,\mathrm{var}(M_x^{L,(\alpha)})}\bigg\}.$$

Since $\mathrm{var}(M_x^{L,(\alpha)}) = L - [\alpha L] + 1$ for every $x \in \Lambda_{L,\delta}^{(\alpha)}$, we obtain $\mathbb{P}\{\mathcal{A}^c\} \le \exp(-CL^3)$ for every L large enough and this term is negligible. Next, on $\mathcal{B}^c$ the number of $x \in \Lambda_{L,\delta}^{(\alpha)}$ which satisfies $M_x^{L,(\alpha)} < -\gamma_1 L$ is at least $|\Lambda_{L,\delta}^{(\alpha)}| - 2^{\varepsilon_1 L}$. Therefore, on $\mathcal{A} \cap \mathcal{B}^c$ we have

$$\frac{1}{|\Lambda_{L,\delta}^{(\alpha)}|} \sum_{x \in \Lambda_{L,\delta}^{(\alpha)}} M_x^{L,(\alpha)} \le \frac{1}{|\Lambda_{L,\delta}^{(\alpha)}|} \{-\gamma_1 L(|\Lambda_{L,\delta}^{(\alpha)}| - 2^{\varepsilon_1 L}) + L^2 2^{\varepsilon_1 L}\} \le -\tfrac{1}{2}\gamma_1 L,$$

for every L large enough if $\varepsilon_1 + \alpha < 1$. In this case, Gaussian tail estimate yields that

$$\mathbb{P}\{\mathcal{A} \cap \mathcal{B}^c\} \le \exp\left\{-C \frac{L^2}{\mathrm{var}((1/|\Lambda_{L,\delta}^{(\alpha)}|) \sum_{x \in \Lambda_{L,\delta}^{(\alpha)}} M_x^{L,(\alpha)})}\right\}.$$

By definition of $M_x^{L,(\alpha)}$,

$$\begin{aligned}\mathrm{var}\left(\frac{1}{|\Lambda_{L,\delta}^{(\alpha)}|} \sum_{x \in \Lambda_{L,\delta}^{(\alpha)}} M_x^{L,(\alpha)}\right) &= \frac{1}{|\Lambda_{L,\delta}^{(\alpha)}|^2} \sum_{x \in \Lambda_{L,\delta}^{(\alpha)}} \sum_{y \in \Lambda_{L,\delta}^{(\alpha)}} \sum_{k=[\alpha L]}^{L} \sum_{I \in \mathcal{I}_k(x)} \sum_{I' \in \mathcal{I}_k(y)} \mathbb{E}[a_{k,I}^L a_{k,I'}^L] \\ &= \frac{1}{|\Lambda_{L,\delta}^{(\alpha)}|^2} \sum_{x \in \Lambda_{L,\delta}^{(\alpha)}} \sum_{y \in \Lambda_{L,\delta}^{(\alpha)}} \sum_{k=[\alpha L]}^{L} ((2^k - d^L(x,y)) \vee 0) 2^{-k}.\end{aligned}$$

For given $x \in \Lambda_{L,\delta}^{(\alpha)}$ and $[\alpha L] \le k \le L$, we have

$$\sum_{y \in \Lambda_{L,\delta}^{(\alpha)}} ((2^k - d^L(x,y)) \vee 0) \le 2 \sum_{i=0}^{2^{k-[\alpha L]}} (2^k - i 2^{[\alpha L]}) = 2^k (2^{k-[\alpha L]} + 1).$$

By these computations,

$$\mathrm{var}\left(\frac{1}{|\Lambda_{L,\delta}^{(\alpha)}|} \sum_{x \in \Lambda_{L,\delta}^{(\alpha)}} M_x^{L,(\alpha)}\right) \le \frac{1}{|\Lambda_{L,\delta}^{(\alpha)}|} \sum_{k=[\alpha L]}^{L} (2^{k-[\alpha L]} + 1) = O(1),$$

and we obtain $\mathbb{P}\{\mathcal{A} \cap \mathcal{B}^c\} \le \exp(-CL^2)$ for every L large enough if $\varepsilon_1 + \alpha < 1$.

For the third term, we have $\mathbb{P}\{\mathcal{B} \cap \mathcal{C}^c\} = \mathbb{E}[\mathbb{P}\{\mathcal{C}^c | \mathcal{F}^{(\alpha)}\}; \mathcal{B}]$. For given $\mathcal{F}^{(\alpha)}$ and on $\mathcal{B}$, the number of $x \in \Lambda_{L,\delta}^{(\alpha)}$ which satisfies $M_x^{L,(\alpha)} \ge -\gamma_1 L$ is at least $2^{\varepsilon_1 L}$. We denote this set as $\mathcal{S}_1$. Now, if $\mathcal{C}^c$ occurs then the number of $x \in \mathcal{S}_1$ which satisfy $M_x^{L,(\beta)} - M_x^{L,(\alpha)} \ge (\gamma_1 + \gamma_2)L$ is less than $2^{\varepsilon_2 L}$. For $x \in \Lambda_{L,\delta}^{(\alpha)}$,

$$M_x^{L,(\beta)} - M_x^{L,(\alpha)} = \sum_{k=[\beta L]}^{[\alpha L]-1} \sum_{I \in \mathcal{I}_k(x)} a_{k,I}^L$$

are independent centered Gaussian random variables with variance $[\alpha L] - [\beta L]$. Therefore, on $\mathcal{B}$ we have

$$\mathbb{P}\{\mathcal{C}^c \mid \mathcal{F}^{(\alpha)}\} \le \mathbb{P}\left\{\sum_{i=1}^{2^{\varepsilon_1 L}} I(\xi_i \ge (\gamma_1 + \gamma_2)L) \le 2^{\varepsilon_2 L}\right\},$$

where $\{\xi_i\}$ are i.i.d. centered Gaussian random variables with variance $[\alpha L] - [\beta L]$. Set $\theta_i := I(\xi_i \geq (\gamma_1 + \gamma_2)L)$. Then $\{\theta_i\}$ are i.i.d. and by Gaussian tail estimate, we have

$$\mathbb{E}[\theta_i] \geq \frac{C}{L^{3/2}} \exp\left\{-\frac{(\gamma_1+\gamma_2)^2 L^2}{2([\alpha L] - [\beta L])}\right\} \geq 2^{-\lambda L},$$

for every large enough L, where we set $\lambda := (\gamma_1 + \gamma_2)^2/(\alpha - \beta)\log 2$. Now, if $\varepsilon_1 - \lambda > \varepsilon_2$ then

$$\begin{aligned}\mathbb{P}\{\mathcal{C}^c \mid \mathcal{F}^{(\alpha)}\} &\leq \mathbb{P}\left\{\sum_{i=1}^{2^{\varepsilon_1 L}} \theta_i \leq 2^{\varepsilon_2 L}\right\} \\ &\leq \mathbb{P}\left\{\sum_{i=1}^{2^{\varepsilon_1 L}} (\theta_i - \mathbb{E}[\theta_i]) \leq -\frac{1}{2} 2^{(\varepsilon_1 - \lambda)L}\right\} \\ &\leq 2\exp\{-C 2^{(\varepsilon_1-\lambda)L}\},\end{aligned}$$

for every L large enough, where the last inequality follows from [6, Lemma 11]. Therefore, the third term is negligible under the condition $\varepsilon_1 - \lambda > \varepsilon_2$, $\lambda := (\gamma_1+\gamma_2)^2/(\alpha-\beta)\log 2$.

For the last term, we have $\mathbb{P}\{\mathcal{C} \cap \mathcal{E}\} = \mathbb{E}[\mathbb{P}\{\mathcal{E} \mid \mathcal{F}^{(\beta)}\}; \mathcal{C}]$. For given $\mathcal{F}^{(\beta)}$ and on $\mathcal{C}$, the number of $x \in \Lambda_{L,\delta}^{(\alpha)}$ which satisfy $M_x^{L,(\beta)} \geq \gamma_2 L$ is at least $2^{\varepsilon_2 L}$. We denote this set as $\mathcal{S}_2$. Now, if $\mathcal{E}$ occurs then $M_x^L - M_x^{L,(\beta)} \leq (\gamma_0 - \gamma_2)L$ for every $x \in \mathcal{S}_2$. For $x \in \Lambda_{L,\delta}^{(\alpha)}$, $M_x^L - M_x^{L,(\beta)} = \sum_{k=0}^{[\beta L]-1} \sum_{I \in \mathcal{I}_k(x)} a_{k,I}^L$ are independent centered Gaussian random variables with variance $[\beta L]$. Hence, we have

$$\mathbb{P}\{\mathcal{E} \mid \mathcal{F}^{(\beta)}\} \leq \mathbb{P}\{\xi \leq (\gamma_0 - \gamma_2)L\}^{2^{\varepsilon_2 L}} \leq \exp\left\{-\frac{(\gamma_0-\gamma_2)^2 L^2}{2[\beta L]}\right\}^{2^{\varepsilon_2 L}},$$

if $\gamma_0 - \gamma_2 < 0$ and this term is also negligible, where ξ is a centered Gaussian random variable with variance $[\beta L]$.

Finally, by choosing $\alpha, \gamma_0, \gamma_1, \gamma_2, \varepsilon_1, \varepsilon_2$ to satisfy all the conditions, we obtain the desired estimate. For example, for given $0 < \beta < 1$ it is sufficient to take $\alpha = (1+\beta)/2$, $\varepsilon_1 = (1-\beta)/3$, $\varepsilon_2 = (1-\beta)/6$, $\gamma_1 = \gamma_2 = \sqrt{\log 2}(1-\beta)/7 > \gamma_0$.

2.2. The cases $d = 1$ and $d \geq 3$

Next, we consider the cases $d = 1$ and $d \geq 3$. We prepare some notation which will be also used in the proof of the lower bound. By the inversion formula,

$$\mathcal{P}(S_u = x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta\cdot S_u}]\mathrm{e}^{-\mathrm{i}\theta\cdot x}\,\mathrm{d}\theta,$$

for $x \in \mathbb{Z}^d$, and $u \geq 0$. Since $\{S_u\}_{u\geq 0}$ is a continuous time random walk with generator (1.6), we have $\mathbb{E}[\exp(\mathrm{i}\theta\cdot S_u)] = \exp\{u(\hat{q}(\theta)-1)\}$ for every $u \geq 0$, $\theta \in \mathbb{R}^d$ where $\hat{q}(\theta) := \sum_{x\in\mathbb{Z}^d} \exp(\mathrm{i}\theta\cdot x)q(x)$ (cf. [18, Lemma 2.3.1]). By the assumption on q, $\phi(\theta) := 1 - \hat{q}(\theta) = 0$ if and only if all the values of θ are integer multiples of 2π and there exists $\varepsilon > 0$ such that $\phi(\theta) \geq \varepsilon|\theta|^2$ for every $\theta \in [-\pi,\pi]^d$ (see [18]). Also, by Taylor's theorem $\phi(\theta) = \theta\cdot A\theta/2 + O(|\theta|^4)$ as $|\theta| \to 0$. Then, by Fubini's theorem,

$$\int_s^t \mathcal{P}(S_u = 0)\,\mathrm{d}u = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{\mathrm{e}^{-s\phi(\theta)} - \mathrm{e}^{-t\phi(\theta)}}{\phi(\theta)}\,\mathrm{d}\theta, \tag{2.4}$$

for every $0 \leq s \leq t$.

Lemma 2.2. *Let $d = 1$ and $\{B_t\}_{t\geq 0}$ be a one-dimensional standard Brownian motion. Define $\widetilde{B}_t := B_t/\sqrt{\mathbb{E}[B_t^2]} = B_t/\sqrt{t}$, $t > 0$. Then, we have $\mathbb{E}[\widetilde{g}_s\widetilde{g}_t] \leq \mathbb{E}[\widetilde{B}_s\widetilde{B}_t]$ for every $s, t > 0$.*

Proof. By the random walk representation (1.5), all we need to show is that

$$\frac{\int_{t-s}^{t+s} \mathcal{P}(S_u = 0)\,\mathrm{d}u}{(\int_0^{2s} \mathcal{P}(S_u = 0)\,\mathrm{d}u)^{1/2}(\int_0^{2t} \mathcal{P}(S_u = 0)\,\mathrm{d}u)^{1/2}} \leq \frac{\sqrt{s}}{\sqrt{t}},$$

for every $0 < s < t$. By (2.4), this is equivalent to

$$\begin{aligned}&\left(\int_{-\pi}^{\pi} \frac{\mathrm{e}^{-(t-s)\phi(\theta)} - \mathrm{e}^{-(t+s)\phi(\theta)}}{2s\phi(\theta)}\,\mathrm{d}\theta\right)\\ &\qquad\leq \left(\int_{-\pi}^{\pi} \frac{1-\mathrm{e}^{-2s\phi(\theta)}}{2s\phi(\theta)}\,\mathrm{d}\theta\right)^{1/2}\left(\int_{-\pi}^{\pi} \frac{1-\mathrm{e}^{-2t\phi(\theta)}}{2t\phi(\theta)}\,\mathrm{d}\theta\right)^{1/2}.\end{aligned} \tag{2.5}$$

Since $(\mathrm{e}^x - 1)/x$ is increasing in $x > 0$, we have

$$\frac{\mathrm{e}^{-(t-s)a} - \mathrm{e}^{-(t+s)a}}{2sa} \leq \sqrt{\frac{1-\mathrm{e}^{-2sa}}{2sa}}\sqrt{\frac{1-\mathrm{e}^{-2ta}}{2ta}},$$

for every $0 < s < t$, $a > 0$ and (2.5) follows from Schwarz's inequality.

Proof of Theorem 1.1 upper bound; the case $d = 1$. By Lemma 2.2, we can use Slepian's lemma and we have

$$\mathbb{P}\{g_t \leq 1 \text{ for every } t \in [0, T]\} \leq \mathbb{P}\left\{B_t \leq \frac{\sqrt{t}}{\sqrt{\mathbb{E}[g_t^2]}} \text{ for every } t \in (0, T]\right\}.$$

By the local central limit theorem (1.8),

$$\mathbb{E}[g_t^2] = \frac{1}{2}\int_0^{2t} \mathcal{P}(S_u = 0)\,\mathrm{d}u \geq \frac{1}{2}\int_0^{t}\left(\frac{\kappa_1}{\sqrt{u}} - \frac{C}{u^{3/2}}\right)\mathrm{d}u \geq \sqrt{2}\kappa_1\sqrt{t} - C,$$

for some $C > 0$ and every $t > 0$. Also, by using the estimate

$$\mathcal{P}(S_u = 0) \geq \mathcal{P}(\text{there is no jump in } [0, u]) = \mathrm{e}^{-u},$$

we have $\mathbb{E}[g_t^2] \geq \frac{1}{2}\int_0^{2t} \exp(-u)\,\mathrm{d}u = \frac{1}{2}(1 - \exp(-2t))$. Hence, $\mathbb{E}[g_t^2] \geq \max\{\sqrt{2}\kappa_1\sqrt{t} - C, \frac{1}{2}(1 - \exp(-2t))\}$ for every $t > 0$ and this yields that there exist $C, C' > 0$ such that $\sqrt{t}/\sqrt{\mathbb{E}[g_t^2]} \leq Ct^{1/4} + C'$ for every $t > 0$. Therefore,

$$\mathbb{P}\{g_t \leq 1 \text{ for every } t \in [0, T]\} \leq \mathbb{P}\{B_t \leq Ct^{1/4} + C' \text{ for every } t \in (0, T]\},$$

and the right-hand side is bounded above by $CT^{-1/2}$ by the result about persistence probability for Brownian motion with drift (see [25]).

Proof of Theorem 1.1 upper bound; the case $d \geq 3$. In this case the upper bound follows from a similar argument to the proof of [22, Theorem 2] which studied a class of stationary Gaussian processes. For completeness we give the proof. Let $\Delta = \Delta(T) > 0$ which will be specified

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later on. Consider $n \times n$ matrix $\rho = (\rho_{jk})$ where $\rho_{jk} = \mathbb{E}[\widetilde{g}_{j\Delta}\widetilde{g}_{k\Delta}]$, $1 \leq j, k \leq n := [T/\Delta]$. We also denote $\rho^{-1} = (\rho^{-1}_{jk})$ as the matrix inverse of ρ. By (1.8), $\int_0^\infty \mathcal{P}(S_u = 0)\, \mathrm{d}u < \infty$ when $d \geq 3$. Therefore, there exists $c_0 > 0$ such that $\mathbb{E}[g_x^2] = \frac{1}{2}\int_0^{2x} \mathcal{P}(S_u = 0)\, \mathrm{d}u \geq 1/c_0^2$ for every $x \in \{j\Delta;\ j = 1, 2, \ldots, n\}$ if $T > 0$ is large enough and we have

$$\begin{aligned}
&\mathbb{P}\{g_t \leq 1 \text{ for every } t \in [0, T]\} \\
&\quad \leq \mathbb{P}\bigg\{\widetilde{g}_x \leq \frac{1}{\sqrt{\mathbb{E}[g_x^2]}} \text{ for every } x \in \{j\Delta;\ j = 1, 2, \ldots, n\}\bigg\} \\
&\quad \leq \mathbb{P}\{\widetilde{g}_x \leq c_0 \text{ for every } x \in \{j\Delta;\ j = 1, 2, \ldots, n\}\} \\
&\quad = \frac{1}{(2\pi)^{n/2}\sqrt{\det \rho}} \int_{-\infty}^{c_0} \mathrm{d}x_1 \cdots \int_{-\infty}^{c_0} \mathrm{d}x_n \exp\bigg\{-\frac{1}{2}\sum_{j,k=1}^{n} \rho^{-1}_{jk} x_j x_k\bigg\}.
\end{aligned}$$

Since ρ is a real symmetric positive definite matrix, its eigenvalues are given by $0 < \lambda_1 \leq \cdots \leq \lambda_n$ and we have $\det \rho \geq \lambda_1^n$. Also, eigenvalues of ρ^{-1} are given by $0 < 1/\lambda_n \leq \cdots \leq 1/\lambda_1$ and we have $\sum_{j,k=1}^n \rho^{-1}_{jk} x_j x_k \geq (1/\lambda_n)\sum_{j=1}^n x_j^2$ for every $(x_j) \in \mathbb{R}^n$. Therefore, we obtain

$$\mathbb{P}\{g_t \leq 1 \text{ for every } t \in [0, T]\} \leq \left(b\frac{\lambda_n}{\lambda_1}\right)^{n/2}, \tag{2.6}$$

for some $b \in (0, 1)$. Now, we use the fact that for every $n \times n$ matrix $A = (a_{ij})$ and an eigenvalue λ, it holds that $|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|$ for some $1 \leq i \leq n$. By (1.10) and $\rho_{ii} = 1$ for every $1 \leq i \leq n$, there exists $C_2 > 0$ such that $\lambda_n \leq 1 + C_2\Delta^{-(d/2)+1}\sum_{j=1}^n |j|^{-(d/2)+1}$ and $\lambda_1 \geq 1 - C_2\Delta^{-(d/2)+1}\sum_{j=1}^n |j|^{-(d/2)+1}$. If we choose Δ as

$$\Delta = \Delta(T) = \begin{cases} K\sqrt{T} & \text{if } d = 3, \\ K \log T & \text{if } d = 4, \\ K & \text{if } d \geq 5, \end{cases}$$

with $K > 0$ large enough (independent of T), then $\Delta^{-(d/2)+1}\sum_{j=1}^n |j|^{-(d/2)+1} \leq C/K$ for every $T > 0$ large enough and every $d \geq 3$. Hence, we can choose $K > 0$ so that $b(\lambda_n/\lambda_1) \leq C_3 < 1$ for every $T > 0$ large enough and (2.6) yields the desired upper bound.

3. Proof of the lower bound

In this section we prove the lower bound of Theorem 1.1. We first consider the case $d \leq 3$. Let $T > 0$ and consider a continuous Gaussian process $\{g_t\}_{t \in [0,T]}$ with mean 0 and covariance (1.5). We first give an estimate on its maximum.

Lemma 3.1. *There exists $C > 0$ such that*

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} g_t\Big] \leq \begin{cases} CT^{1/4} & \textit{if } d = 1, \\ C\log T & \textit{if } d = 2, \\ C\sqrt{\log T} & \textit{if } d \geq 3. \end{cases} \tag{3.1}$$

Proof. Let $\rho(s, t) := \{\mathbb{E}[(g_s - g_t)^2]\}^{1/2}$ be a canonical metric induced by g. For $0 \leq s \leq t$,

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by (1.5) we have

$$\begin{aligned}\rho(s,t)^2 &= \frac{1}{2}\biggl(\int_0^{2s} + \int_0^{2t} -2\int_{t-s}^{t+s}\biggr)\mathcal{P}(S_u=0)\,\mathrm{d}u\\ &= \frac{1}{2}\biggl(\int_{t+s}^{(t+s)+(t-s)} - \int_{2s}^{2s+(t-s)} +2\int_0^{t-s}\biggr)\mathcal{P}(S_u=0)\,\mathrm{d}u\\ &= \frac{1}{2}\int_0^{t-s}\{\mathcal{P}(S_{u+t+s}=0)-\mathcal{P}(S_{u+2s}=0)+2\mathcal{P}(S_u=0)\}\,\mathrm{d}u.\end{aligned}$$

Then, we have a trivial bound $\rho(s,t)^2 \le 2|t-s|$ for every $d \ge 1$. Also, by the local central limit theorem (1.8) we have $\mathcal{P}(S_u=0) \le Cu^{-(d/2)}$ for every $u > 0$ and this yields

$$\rho(s,t)^2 \le \begin{cases} C|t-s|^{1/2} & \text{if } d=1,\\ C\log(|t-s|\vee 1)+C' & \text{if } d=2,\\ C & \text{if } d\ge 3,\end{cases}$$

for every $s, t \ge 0$.

Now, we use Dudley's bound:

$$\mathbb{E}\Bigl[\sup_{0\le t\le T} g_t\Bigr] \le C\int_0^{\frac{1}{2}\mathrm{diam}([0,T])}\sqrt{\log N(\varepsilon)}\,\mathrm{d}\varepsilon, \tag{3.2}$$

where $N(\varepsilon)$ is a metric entropy for $[0, T]$ induced by ρ, namely, the smallest number of ρ-balls with radius ε which cover $[0, T]$ (cf. [1, Theorem 1.3.3]). When $d = 2$, by the above estimates there exist $C_1, L_1 > 0$ such that

$$\rho(s,t) \le \begin{cases}\sqrt{2|t-s|} & \text{for every } s,t\ge 0,\\ \sqrt{C_1\log(|t-s|)} & \text{if } |t-s|\ge L_1.\end{cases}$$

If $\varepsilon \ge \sqrt{C_1 \log L_1}$ then $s, t \ge 0$ with $L_1 \le |t-s| \le \exp(\varepsilon^2/C_1)$ satisfy $\rho(s,t) \le \varepsilon$ and we have $N(\varepsilon) \le T\exp(-\varepsilon^2/C_1)$. If $\varepsilon < \sqrt{C_1\log L_1}$ then $s, t \ge 0$ with $|t-s| \le \frac{1}{2}\varepsilon^2$ satisfy $\rho(s,t) \le \varepsilon$ and we have $N(\varepsilon) \le 2T/\varepsilon^2$ in this case. Also, diam $([0, T]) = \sup_{s,t\in[0,T]}\rho(s,t) \le C\sqrt{\log T}$. By combining these estimates with (3.2), we obtain $\mathbb{E}[\sup_{0\le t\le T} g_t] \le C\log T$. When $d \ge 3$, by the trivial bound of ρ we have $N(\varepsilon) \le 2T/\varepsilon^2$ for every $\varepsilon > 0$. Also, since ρ is bounded, diam $([0, T]) \le C$. Hence, we obtain $\mathbb{E}[\sup_{0\le t\le T} g_t] \le C\sqrt{\log T}$ in this case.

Finally, for the case $d = 1$ let $\{B_t^\gamma\}_{t\ge 0}$ be a fractional Brownian motion with Hurst parameter $\gamma = \frac{1}{4}$, namely $\{B_t^\gamma\}_{t\ge 0}$ is a continuous centered Gaussian process with $\mathbb{E}[(B_t^\gamma - B_s^\gamma)^2] = |t-s|^{1/2}$. It is well known that $\mathbb{E}[\sup_{0\le t\le T} B_t^\gamma] \le CT^{1/4}$ and we have $\mathbb{E}[(g_t-g_s)^2] \le \mathbb{E}[(\sqrt{C}B_t^\gamma - \sqrt{C}B_s^\gamma)^2]$. Then, the Sudakov–Fernique inequality (see [1, Theorem 2.2.3]) yields that $\mathbb{E}[\sup_{0\le t\le T} g_t] \le \sqrt{C}\mathbb{E}[\sup_{0\le t\le T} B_t^\gamma] \le CT^{1/4}$.

Next, let $(H(\Gamma), \|\cdot\|_H)$ denote a reproducing kernel Hilbert space associated with $\Gamma = \{\Gamma(s,t);\, s,\, t \in [0, T]\}$. We define $\eta = \eta^T := \int_0^T g_s\,\mathrm{d}s$ and $h = \{h_t^T\}_{t\in[0,T]}$, $h_t^T := \mathbb{E}[\eta g_t] = \int_0^T E[g_s g_t]\,\mathrm{d}s$. Then, by definition, $h \in H(\Gamma)$ and we have the following lemma.

Lemma 3.2. *Let $d \le 3$.* (i) *There exists $C > 0$ such that $\|h\|_H^2 \le CT^{3-(d/2)}$ for every $T > 0$ large enough.*

(ii) *For every $0 < \delta < 1$, there exists $C = C(\delta) > 0$ such that*

$$\liminf_{T\to\infty} T^{(d/2)-2} \inf_{t\in[\delta T,T]} h_t^T \ge C.$$

Proof. At first, by (1.5) and (2.4)

$$\mathbb{E}[g_s g_t] = \frac{1}{2}\int_{t-s}^{t+s} \mathcal{P}(S_u = 0)\,\mathrm{d}u = \frac{1}{2(2\pi)^d}\int_{\mathcal{C}(\pi)} \frac{\mathrm{e}^{-(t-s)\phi(\theta)} - \mathrm{e}^{-(t+s)\phi(\theta)}}{\phi(\theta)}\,\mathrm{d}\theta, \tag{3.3}$$

for every $0 \le s \le t$ where $\mathcal{C}(r) := [-r, r]^d$, $r > 0$.

(i) By (3.3) and Fubini's theorem, we compute that

$$\begin{aligned}
\|h\|_H^2 &= \int_0^T \int_0^T \mathbb{E}[g_s g_t]\,\mathrm{d}s\,\mathrm{d}t \\
&= 2\int_0^T \left(\int_0^t \mathbb{E}[g_s g_t]\,\mathrm{d}s\right)\mathrm{d}t \\
&= \frac{1}{(2\pi)^d}\int_{\mathcal{C}(\pi)} \frac{1}{\phi(\theta)^2}\left(\int_0^T \mathrm{e}^{-t\phi(\theta)}(\mathrm{e}^{t\phi(\theta)} + \mathrm{e}^{-t\phi(\theta)} - 2)\,\mathrm{d}t\right)\mathrm{d}\theta \\
&= \frac{T^3}{(2\pi)^d}\int_{\mathcal{C}(\pi)} f(T\phi(\theta))\,\mathrm{d}\theta,
\end{aligned}$$

where $f(x) = 1/x^2(1 - (1/2x)\{(2 - \mathrm{e}^{-x})^2 - 1\})$ is a continuous function on $(0, \infty)$ which satisfies $\lim_{x\to 0} f(x) = \frac{1}{3}$ and $f(x) \le 1/x^2$ for every $x \in (0, \infty)$. Therefore,

$$\begin{aligned}
\|h\|_H^2 &= \frac{T^{3-(d/2)}}{(2\pi)^d}\int_{\mathcal{C}(\pi\sqrt{T})} f\left(T\phi\left(\frac{\theta}{\sqrt{T}}\right)\right)\mathrm{d}\theta \\
&= \frac{T^{3-(d/2)}}{(2\pi)^d}\left(\int_{\mathcal{C}(1)} + \int_{\mathcal{C}(\pi\sqrt{T})\setminus\mathcal{C}(1)}\right) f\left(T\phi\left(\frac{\theta}{\sqrt{T}}\right)\right)\mathrm{d}\theta.
\end{aligned}$$

By properties of f and ϕ, $f(T\phi(\cdot/\sqrt{T}))$ is bounded on $\mathcal{C}(1)$. Also,

$$\begin{aligned}
\int_{\mathcal{C}(\pi\sqrt{T})\setminus\mathcal{C}(1)} f\left(T\phi\left(\frac{\theta}{\sqrt{T}}\right)\right)\mathrm{d}\theta &\le \int_{\mathcal{C}(\pi\sqrt{T})\setminus\mathcal{C}(1)} \left(T\phi\left(\frac{\theta}{\sqrt{T}}\right)\right)^{-2}\mathrm{d}\theta \\
&\le \int_{\mathbb{R}^d\setminus\mathcal{C}(1)} \frac{1}{\varepsilon^2|\theta|^4}\,\mathrm{d}\theta \\
&\le \frac{C}{\varepsilon^2}\int_1^\infty \frac{r^{d-1}}{r^4}\,\mathrm{d}r.
\end{aligned}$$

This integral is finite if $d \le 3$.

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(ii) By (3.3) and Fubini's theorem again, we compute that

$$\begin{aligned} h_t^T &= \int_0^T \mathbb{E}[g_s g_t]\,\mathrm{d}s \\ &\geq \frac{1}{2(2\pi)^d}\int_{\mathcal{C}(\pi)}\left(\int_0^t \frac{\mathrm{e}^{-(t-s)\phi(\theta)} - \mathrm{e}^{-(t+s)\phi(\theta)}}{\phi(\theta)}\,\mathrm{d}s\right)\mathrm{d}\theta \\ &= \frac{1}{2(2\pi)^d}\int_{\mathcal{C}(\pi)} \frac{1}{\phi(\theta)^2}\mathrm{e}^{-t\phi(\theta)}(\mathrm{e}^{t\phi(\theta)} + \mathrm{e}^{-t\phi(\theta)} - 2)\,\mathrm{d}\theta \\ &\geq \frac{1}{2(2\pi)^d}\int_{\mathcal{C}(\pi)} \frac{1}{\phi(\theta)^2}\mathrm{e}^{-t\phi(\theta)}\frac{1}{2}(t\phi(\theta))^2\,\mathrm{d}\theta \\ &= \frac{t^2 T^{-(d/2)}}{4(2\pi)^d}\int_{\mathcal{C}(\pi\sqrt{T})} \mathrm{e}^{-t\phi(\theta/\sqrt{T})}\,\mathrm{d}\theta, \end{aligned}$$

where the second inequality follows from the fact that $\mathrm{e}^x + \mathrm{e}^{-x} - 2 \geq x^2/2$ for every $x \in \mathbb{R}$. Finally, by Fatou's lemma we have

$$\liminf_{T\to\infty} T^{(d/2)-2} \inf_{t\in[\delta T, T]} h_t^T \geq \frac{\delta^2}{4(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-(1/2)\theta\cdot A\theta}\,\mathrm{d}\theta > 0.$$

Remark 3.1. Asymptotics of $\|h\|_H^2$ as $T \to \infty$ for $d \geq 3$ have been studied in [11] by using SDE (1.2).

By using Lemma 3.2, we first show the lower bound of the persistence probability in time interval $[\delta T, T],\ 0 < \delta < 1$.

Proposition 3.1. *Let $0 < \delta < 1$ be fixed. There exists $C > 0$ such that*

$$\mathbb{P}\{g_t \leq 1 \text{ for every } t \in [\delta T, T]\} \geq \begin{cases} \mathrm{e}^{-C} & \text{if } d = 1, \\ \mathrm{e}^{-C(\log T)^2} & \text{if } d = 2, \\ \mathrm{e}^{-C\sqrt{T}\log T} & \text{if } d = 3, \end{cases}$$

for every $T > 0$ large enough.

Proof. Set $\widetilde{h} = \{\widetilde{h}_t^T\}_{t\in[0,T]}$, $\widetilde{h}_t^T := \beta a(T) h_t,\ 0 \leq t \leq T$ where $\beta > 0$ and

$$a(T) = \begin{cases} T^{-(5/4)} & \text{if } d = 1, \\ T^{-1}\log T & \text{if } d = 2, \\ T^{-(1/2)}\sqrt{\log T} & \text{if } d = 3. \end{cases}$$

By the Cameron–Martin formula and Schwarz's inequality, we have

$$\begin{aligned} &\mathbb{P}\{g_t - \widetilde{h}_t^T \leq 1 \text{ for every } t \in [\delta T, T]\} \\ &\quad = \mathbb{E}[1_{\{g_t \leq 1 \text{ for every } t\in[\delta T, T]\}}\mathrm{e}^{\widetilde{\eta}-(1/2)\|\widetilde{h}\|_H^2}] \\ &\quad \leq \mathbb{P}\{g_t \leq 1 \text{ for every } t \in [\delta T, T]\}^{1/2}\mathrm{e}^{-(1/2)\|\widetilde{h}\|_H^2}\mathbb{E}[\mathrm{e}^{2\widetilde{\eta}}]^{1/2}, \end{aligned}$$

where $\widetilde{\eta}$ is a centered Gaussian random variable with variance $\|\widetilde{h}\|_H^2$. This yields that

$$\mathbb{P}\{g_t \leq 1 \text{ for every } t \in [\delta T, T]\} \geq \mathrm{e}^{-\|\widetilde{h}\|_H^2}\mathbb{P}\{g_t - \widetilde{h}_t^T \leq 1 \text{ for every } t \in [\delta T, T]\}^2.$$

Therefore, by definition of $\widetilde{h}$ and Lemma 3.2(i) it is sufficient to prove that

$$\liminf_{T\to\infty} \mathbb{P}\{g_t - \widetilde{h}_t^T \le 0 \text{ for every } t \in [\delta T, T]\} \ge C > 0. \tag{3.4}$$

By Lemma 3.1 and Lemma 3.2(ii), if we choose $\beta > 0$ large enough then $\inf_{\delta T \le t \le T} \widetilde{h}_t^T \ge 2b(T)$ where we set the right-hand side of (3.1) as $b(T)$. Then,

$$\begin{aligned}\mathbb{P}\{g_t - \widetilde{h}_t^T \ge 0 \text{ for some } t \in [\delta T, T]\} &\le \mathbb{P}\Big\{\sup_{0\le t\le T} g_t - \mathbb{E}[\sup_{0\le t\le T} g_t] \ge b(T)\Big\} \\ &\le \exp\Big\{-\frac{b(T)^2}{2\sigma_T^2}\Big\},\end{aligned}$$

where

$$\sigma_T^2 := \sup_{0\le t\le T} \mathbb{E}[g_t^2] \le \begin{cases} C\sqrt{T} & \text{if } d = 1, \\ C\log T & \text{if } d = 2, \\ C & \text{if } d = 3, \end{cases}$$

and the last inequality follows from Borell's inequality (see [1, Theorem 2.1.1]). Hence, we obtain (3.4) and complete the proof.

Proof of Theorem 1.1 lower bound; the case $d \le 3$. By Proposition 3.1, there exist $T_0 > 0$ and $C_0 > 0$ such that

$$\mathbb{P}\{g_t \le 1 \text{ for every } t \in [T, 2T]\} \ge \mathrm{e}^{-C_0 a(T)},$$

for every $T \ge T_0$ where

$$a(T) = \begin{cases} 1 & \text{if } d = 1, \\ (\log T)^2 & \text{if } d = 2, \\ \sqrt{T}\log T & \text{if } d = 3. \end{cases}$$

For every $T \ge T_0$, we have $[T_0, T] \subset \bigcup_{l=0}^{\bar{l}} [2^l T_0, 2^{l+1} T_0]$ where $\bar{l} = [\log(T/T_0)/\log 2]$. Slepian's lemma yields that

$$\begin{aligned}&\mathbb{P}\{g_t \le 1 \text{ for every } t \in [0, T]\} \\ &\quad \ge \mathbb{P}\{g_t \le 1 \text{ for every } t \in [0, T_0]\} \prod_{l=0}^{\bar{l}} \mathbb{P}\{g_t \le 1 \text{ for every } t \in [2^l T_0, 2^{l+1} T_0]\} \\ &\quad \ge C_1 \exp\Big\{-C_0 \sum_{l=0}^{\bar{l}} a(2^l T_0)\Big\},\end{aligned}$$

for some $C_1 = C_1(T_0) > 0$. Then, by elementary computation we have

$$\sum_{l=0}^{\bar{l}} a(2^l T_0) \le \begin{cases} C_2 \log T & \text{if } d = 1, \\ C_2 (\log T)^3 & \text{if } d = 2, \\ C_2 \sqrt{T} \log T & \text{if } d = 3, \end{cases}$$

for some $C_2 > 0$ and we complete the proof.

Proof of Theorem 1.1 lower bound; the case $d \geq 4$. By (1.5) and (1.8), there exist $T_0, C_1, C_2 > 0$ such that $C_1 \leq \mathbb{E}[g_t g_{t+s}] \leq C_2$ for every $t \geq T_0, 0 \leq s \leq 1$. Therefore, there exists $C_3 > 0$ such that $\mathbb{P}\{g_s \leq 1 \text{ for every } s \in [t, t+1]\} \geq C_3$ for every $t \geq T_0$. Then, Slepian's lemma yields that

$$\begin{aligned}
&\mathbb{P}\{g_t \leq 1 \text{ for every } t \in [0, T]\} \\
&\qquad \geq \mathbb{P}\{g_t \leq 1 \text{ for every } t \in [0, T_0]\} \prod_{l=0}^{[T-T_0]} \mathbb{P}\{g_t \leq 1 \text{ for every } t \in [T_0 + l, T_0 + l + 1]\} \\
&\qquad \geq \mathrm{e}^{-CT},
\end{aligned}$$

for some $C > 0$ and we complete the proof.

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