



Iterated Galois towers, their associated martingales, and the p -adic Mandelbrot set

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ABSTRACT

We study the Galois tower generated by iterates of a quadratic polynomial f defined over an arbitrary field. One question of interest is to find the proportion a_n of elements at level n that fix at least one root; in the global field case these correspond to unramified primes in the base field that have a divisor at level n of residue class degree one. We thus define a stochastic process associated to the tower that encodes root-fixing information at each level. We develop a uniqueness result for certain permutation groups, and use this to show that for many f each level of the tower contains a certain central involution. It follows that the associated stochastic process is a martingale, and convergence theorems then allow us to establish a criterion for showing that a_n tends to 0. As an application, we study the dynamics of the family $x^2 + c \in \overline{\mathbb{F}}_p[x]$, and this in turn is used to establish a basic property of the p -adic Mandelbrot set.

1. Introduction

Let L be a field and $f \in L[x]$. Denote by f^{on} the n th iterate of f , that is $f^{o1} = f$ and $f^{on} = f \circ f^{o(n-1)}$ for $n \geq 2$. Let $L_n(f)$ be the splitting field over L of f^{on} , and let $G_n(f) = \text{Gal}(L_n(f)/L)$. The profinite group $G(f) = \varprojlim G_n(f)$ remains rather mysterious in general, having been studied broadly only by Odoni [Odo85a]. Even the case of f quadratic remains largely unresolved, although some progress has been made [Odo85a, Odo85b, Odo88, Sto92]. In this article we use tools from the theory of stochastic processes to study properties of $G(f)$ that have arithmetic applications.

The construction is as follows. Given any field L and a collection \mathcal{F} of separable polynomials f_1, f_2, \dots in $L[x]$, denote by $L(f_n)$ the splitting field of f_n over L , and let $G(f_n) = \text{Gal}(L(f_n)/L)$. Suppose that $L(f_n) \supseteq L(f_{n-1})$ for all $n \geq 2$, and let $G(\mathcal{F}) = \varprojlim G(f_n)$. Take \mathbf{P} to be the Haar measure on $G(\mathcal{F})$ with $\mathbf{P}(G(\mathcal{F})) = 1$, and ψ_n to be the natural projection $G(\mathcal{F}) \rightarrow G(f_n)$. We define random variables on $G(\mathcal{F})$ by setting $X_n(g)$ to be the number of roots of f_n fixed by $\psi_n(g)$. It follows that

$$\mathbf{P}(X_n > 0) = \frac{1}{\#G(f_n)} \cdot \#\{g \in G(f_n) : g \text{ fixes at least one root of } f_n\}. \quad (1)$$

Recall that a stochastic process is simply an infinite collection of random variables defined on a common probability space. We refer to the random variables in (1) as the *Galois process* of \mathcal{F} and denote it by $GP(\mathcal{F})$. In the case $f \in L[x]$ and $\mathcal{F} = \{f, f^{o2}, f^{o3}, \dots\}$, we write $GP(f)$ instead of $GP(\mathcal{F})$ and $G_n(f)$ instead of $G(f^{on})$. To state our main result on Galois processes, we require the following definition.

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DEFINITION 1.1. A stochastic process X_1, X_2, \dots taking values in \mathbb{Z} is a *martingale* if for all $n \geq 2$ and any $t_i \in \mathbb{Z}$,

$$E(X_n \mid X_1 = t_1, X_2 = t_2, \dots, X_{n-1} = t_{n-1}) = t_{n-1},$$

provided that $\mathbf{P}(X_1 = t_1, X_2 = t_2, \dots, X_{n-1} = t_{n-1}) > 0$.

We also define the *adjusted forward orbit* of a point $l \in L$ under a polynomial $f \in L[x]$ with leading coefficient a to be the set $\{-af(l)\} \cup \{f^{on}(l) : n = 2, 3, \dots\}$.

THEOREM 1.2. *Let L be a field of characteristic different from 2, take $f \in L[x]$ of degree two, and suppose that the adjusted forward orbit of the unique finite critical point of f contains no squares. Then $GP(f)$ is a martingale.*

Remark. Theorem 1.2 (and also Theorem 1.3) are true as long as, for all n , f^{on} is separable and irreducible and $\text{Disc } f^{on}$ is not a square. The hypothesis regarding the critical point ensures this in characteristic different from 2 (see Lemma 4.10) and is easy to check. A version of Theorem 1.2 (allowing the process to begin with X_k for a suitable $k \geq 1$) should remain true if one allows $\text{Disc } f^{on}$ to be a square for finitely many n and f^{on} to be reducible with a number of irreducible factors bounded independently of n .

To prove Theorem 1.2 we develop a uniqueness result on sets of fibers for certain morphisms of $G_n(f)$ -sets, where f satisfies the conditions of Theorem 1.2. This uniqueness property is used to show that for each n a certain involution must lie in the center of $G_n(f)$. The presence of this involution leads to the proof of Theorem 1.2.

If the hypotheses of Theorem 1.2 are verified, a basic martingale convergence theorem yields

$$\mathbf{P}(\{g \in G(f) : X_1(g), X_2(g), \dots \text{ is eventually constant}\}) = 1.$$

Let f satisfy the hypotheses of Theorem 1.2, let $L_n(f)$ be the splitting field of the n th iterate of f , and let $H_n(f) = \text{Gal}(L_n(f)/L_{n-1}(f))$. Then $H_n(f) \cong (\mathbb{Z}/2\mathbb{Z})^m$ for some $0 \leq m \leq 2^{n-1}$, and we call $H_n(f)$ *maximal* if $m = 2^{n-1}$. We show that if $H_n(f)$ is maximal, then for any $u > 0$ and $m < n$, we have

$$\mathbf{P}(X_n = u \mid X_{n-1} = u, \dots, X_m = u) \leq \frac{1}{2},$$

provided $\mathbf{P}(X_{n-1} = u, \dots, X_m = u) > 0$. This immediately gives the following theorem.

THEOREM 1.3. *Let L be a field of characteristic different from 2, take $f \in L[x]$ of degree two, and suppose that the adjusted forward orbit of the unique finite critical point of f contains no squares. Suppose also that $H_n(f)$ is maximal for infinitely many n . Then $GP(f)$ converges to 0, i.e.*

$$\lim_{n \rightarrow \infty} \mathbf{P}(X_n = 0) = 1.$$

As an application of Theorem 1.3, we establish a basic property of the p -adic Mandelbrot set. This requires the development of some background. Given a field K and an absolute value $|\cdot|$ on K , we define the *Mandelbrot set* of K to be

$$M(K) = \{c \in K : 0 \text{ has bounded orbit under iteration of } x^2 + c\}, \tag{2}$$

where we mean bounded with respect to $|\cdot|$.

We consider a subset of $M(K)$ that is motivated by the well-known case $K = \mathbb{C}$. Recall that $\phi \in \mathbb{C}(z)$ is said to be *hyperbolic* if each critical point of ϕ tends to an attracting cycle under iteration [McM94]. We therefore define the *hyperbolic Mandelbrot set* to be

$$\mathcal{H}(K) = \{c \in M(K) : 0 \text{ tends to a formally attracting cycle under iteration of } x^2 + c\}, \tag{3}$$

where by a formally attracting cycle of $f(x) = x^2 + c$ we mean that $|f'| < 1$ at all points of the cycle (see § 2 for more detailed definitions). When the topology on K induced by $|\cdot|$ gives rise to nontrivial geometry, e.g. $K = \mathbb{C}$ and $K = \mathbb{C}_p$, a formally attracting cycle is also geometrically attracting. We may decompose $\mathcal{H}(K)$ into a disjoint union of open components $\mathcal{H}(K)^{(i)}$ corresponding to c where 0 tends to a formally attracting i -cycle. In the complex case these components form some of the most visible features of $M(\mathbb{C})$. For instance, $\mathcal{H}(\mathbb{C})^{(1)}$ is the main cardioid, and $\mathcal{H}(\mathbb{C})^{(2)}$ is the circle tangent to the cardioid on the real axis. Conjecturally, $\mathcal{H}(\mathbb{C})$ is the interior of $M(\mathbb{C})$; this is the simplest case of the celebrated conjecture that hyperbolic rational maps are open and dense in the space of rational maps of given degree [McM94]. Moreover, both sets are Lebesgue measurable and the measure of $\mathcal{H}(\mathbb{C})$ exceeds 1.503 while the measure of $M(\mathbb{C})$ is less than 1.562 (see [FH93]).

We consider the size of $\mathcal{H}(K)$ relative to $M(K)$ for $K = \mathbb{C}_p$, the smallest complete, algebraically closed extension of \mathbb{Q}_p . The set $M(\mathbb{C}_p)$ proves far less topologically interesting than $M(\mathbb{C})$, as $M(\mathbb{C}_p)$ is just the closed unit disk \mathcal{O}_p in \mathbb{C}_p . However, $\mathcal{H}(\mathbb{C}_p)$ is not so simple. Letting $\phi : \mathcal{O}_p \rightarrow \overline{\mathbb{F}}_p$ be the reduction homomorphism, we show $\mathcal{H}(\mathbb{C}_p) = \phi^{-1}(\mathcal{H}(\overline{\mathbb{F}}_p))$ for $p \neq 2$. Note that since $\overline{\mathbb{F}}_p$ admits only the trivial absolute value, we have

$$\mathcal{H}(\overline{\mathbb{F}}_p) = \{c \in \overline{\mathbb{F}}_p : 0 \text{ is periodic under iteration of } x^2 + c\},$$

provided that $p \neq 2$. Given $\mathcal{C} \subseteq \overline{\mathbb{F}}_p$, we define its density to be

$$D(\mathcal{C}) = \lim_{s \rightarrow 1^+} \frac{\sum_{\alpha \in \mathcal{C}} (\deg \alpha)^{-1} N(\alpha)^{-s}}{\sum_{\alpha \in \overline{\mathbb{F}}_p} (\deg \alpha)^{-1} N(\alpha)^{-s}}, \tag{4}$$

where $\deg \alpha = [\mathbb{F}_p(\alpha) : \mathbb{F}_p]$, and $N(\alpha) = p^{\deg \alpha}$. In a natural sense, $D(\mathcal{H}(\overline{\mathbb{F}}_p))$ measures the density of $\mathcal{H}(\mathbb{C}_p)$. We use Theorem 1.3 to prove the following.

THEOREM 1.4. *For $p \neq 2$, $D(\mathcal{H}(\overline{\mathbb{F}}_p)) = 0$.*

In the case $p = 2$, it is trivial to show $\mathcal{H}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p$, as all points are critical. We remark that there is another notion of density given by $\delta(\mathcal{C}) = \lim_{k \rightarrow \infty} (\#\mathcal{C} \cap \mathbb{F}_{p^k}/p^k)$. When $\delta(\mathcal{C})$ exists, then so does $D(\mathcal{C})$ and the two are equal; however, there are sets \mathcal{C} for which $D(\mathcal{C})$ exists and $\delta(\mathcal{C})$ does not. It is a consequence of Conjecture 6.7 that $\delta(\mathcal{H}(\overline{\mathbb{F}}_p)) = 0$ for $p \neq 2$, and this can be proven unconditionally if $p \equiv 3 \pmod{4}$ (see the discussion following Conjecture 6.7).

To prove Theorem 1.4, we put $f_c = x^2 + c$ and $f_c^{-on}(0) = \{b \in \overline{\mathbb{F}}_p : f_c^{on}(b) = 0\}$, and introduce sets

$$\mathcal{I}_n = \{c \in \overline{\mathbb{F}}_p : f_c^{-on}(0) \cap \mathbb{F}_p(c) \neq \emptyset\}. \tag{5}$$

We show that $\mathcal{I}_n \supseteq \mathcal{I}_{n+1}$ for all $n \geq 1$ and $\mathcal{H}(\overline{\mathbb{F}}_p) = \bigcap_{n \geq 1} \mathcal{I}_n$. It follows that if $D(\mathcal{I}_n)$ exists for all n and $\lim_{n \rightarrow \infty} D(\mathcal{I}_n) = 0$, then $D(\mathcal{H}(\overline{\mathbb{F}}_p)) = 0$. We then use the Tchebotarev density theorem for function fields to show the following.

THEOREM 1.5. *Set $L = \mathbb{F}_p(t)$ ($p \neq 2$), $f = x^2 + t \in L[x]$, and let $L_n(f)$ be the splitting field over L of f^{on} , the n th iterate of f . Put $G_n(f) = \text{Gal}(L_n(f)/L)$, and let \mathcal{I}_n be as in (5). Then*

$$D(\mathcal{I}_n) = \frac{1}{\#G_n(f)} \cdot \#\{g \in G_n(f) : g \text{ fixes at least one root of } f^{on}\}.$$

We end with an analysis of the Galois groups of iterates of $f = x^2 + t$ over $L = k(t)$, where k is a field of characteristic different from 2 and t is transcendental over k (cf. [Odo88, Sto92]). We prove that $G_n(f)$ is maximal when n is squarefree, and Theorem 1.3 applies to show Theorem 1.4.

The layout of the article follows the order in which the original work was done. In §§ 2 and 3 we give background on dynamics and \mathbb{C}_p and prove Theorem 1.5. In § 4 we introduce Galois processes and prove Theorem 1.2. In § 5 we prove Theorem 1.3 and discuss the behavior of $GP(f)$ when $G_n(f)$

is maximal for all n . In §6 we analyze the Galois groups of iterates of $x^2 + t$ and obtain the proof of Theorem 1.4.

2. Background on dynamics and \mathbb{C}_p

Let K be a field and $|\cdot|$ an absolute value on K . Let $R \in K(x)$. We recall that an n -cycle of R is a collection of distinct points c_1, \dots, c_n such that $R(c_i) = c_{i+1}$ for $1 \leq i \leq n - 1$ and $R(c_n) = c_1$. We refer to any point c in a cycle as *periodic* under R , and if c is contained in an n -cycle we say that c has *period* n . A cycle c_1, \dots, c_n is *formally attracting* if $|(R^{on})'(c_i)| < 1$ for any i (equivalently, for all i). We use this terminology rather than the more geometrically suggestive ‘attracting’ since we wish to work with fields where there is no nontrivial topology. We say that a point $b \in K$ tends to the cycle c_1, \dots, c_n under iteration of R if given $\epsilon > 0$, there exists M such that $m \geq M$ implies that $|R^{onm+i}(b) - c_i| < \epsilon$ for $i = 1, 2, \dots, n$, up to a relabeling of the c_i .

Let $c \in K$ and put $f_c = x^2 + c$. Recall from (2) and (3) the definitions of the Mandelbrot set and hyperbolic Mandelbrot set of K . We consider the case $K = \mathbb{C}_p$, where \mathbb{C}_p is the smallest complete, algebraically closed extension of \mathbb{Q}_p . We use two principal properties of \mathbb{C}_p ; see [Rob00] for details. First, there is a natural (non-archimedean) absolute value $|\cdot|$ on \mathbb{C}_p that extends the p -adic absolute value on \mathbb{Q}_p . Second, let $\mathcal{O}_p = \{c \in \mathbb{C}_p : |c| \leq 1\}$ and $m_p = \{c \in \mathbb{C}_p : |c| < 1\}$, and note that \mathcal{O}_p is a subring of \mathbb{C}_p and m_p its unique maximal ideal. Moreover, the quotient \mathcal{O}_p/m_p is isomorphic to $\overline{\mathbb{F}}_p$, the algebraic closure of the finite field with p elements. Denote the natural quotient homomorphism $\mathcal{O}_p \rightarrow \overline{\mathbb{F}}_p$ by ϕ . We call ϕ the *reduction homomorphism*.

PROPOSITION 2.1. *Let K be a field and $|\cdot|$ a non-archimedean absolute value on K . Then $M(K) = \{c \in K : |c| \leq 1\}$. In particular, $M(\mathbb{C}_p) = \mathcal{O}_p$ for all primes p .*

Proof. Let $f_c = x^2 + c$, and suppose $|c| > 1$. A consequence of the strong triangle inequality is that if $|x| \neq |y|$, then $|x + y| = \max\{|x|, |y|\}$. Using this, one easily shows by induction that $|f_c^{on}(0)| = |c|^{2^{n-1}}$, whence $c \notin M(K)$. On the other hand, if $|c| \leq 1$, it follows immediately from the strong triangle inequality that $|f_c^{on}(0)| \leq 1$ for all n , showing $c \in M(K)$. □

When $p = 2$ and $K = \mathbb{C}_p$, all cycles of f_c contained in \mathcal{O}_p are attracting. Since f_c has good reduction, it follows that cycles of $f_{\phi(c)}$ lift to cycles of f_c (contained in \mathcal{O}_p) and also that if $b \equiv a \pmod{m_p}$ and a is in an attracting cycle, then b tends to this cycle [Riv01, Proposition 4.32]. The orbit of 0 under $f_{\phi(c)}$ is finite and thus is periodic after a certain point, implying that 0 tends to an attracting cycle in \mathbb{C}_p . Hence, $\mathcal{H}(\mathbb{C}_p) = \mathcal{O}_p$, and clearly also $\mathcal{H}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p$. *For the remainder of this article we assume that $p \neq 2$.*

We wish to give a characterization of $\mathcal{H}(\mathbb{C}_p)$ via the reduction homomorphism. First we make a few remarks on $\mathcal{H}(K)$ when $K = \overline{\mathbb{F}}_p$. Since $\overline{\mathbb{F}}_p^*$ consists of roots of unity, the only absolute value K admits is trivial: $|c| = 1$ for all $c \in K^*$. Under the trivial absolute value, $c \in K$ tends to a formally attracting cycle if and only if c is, in fact, contained in a formally attracting cycle. For general K , we easily derive from the chain rule that c_1, \dots, c_n is a formally attracting cycle of $R \in K(x)$ if and only if

$$\prod_{i=1}^n |R'(c_i)| < 1. \tag{6}$$

In the case $K = \overline{\mathbb{F}}_p$, it follows that a cycle is formally attracting if and only if it contains a critical point. These observations show that

$$\mathcal{H}(\overline{\mathbb{F}}_p) = \{c \in \overline{\mathbb{F}}_p : 0 \text{ is periodic under iteration of } x^2 + c\}.$$

We now give the promised characterization of $\mathcal{H}(\mathbb{C}_p)$. This is a consequence of [Riv01, Proposition 4.32], but in our case the following more direct argument suffices.

PROPOSITION 2.2. *Let $\phi : \mathcal{O}_p \rightarrow \overline{\mathbb{F}}_p$ be the reduction homomorphism. Then $\mathcal{H}(\mathbb{C}_p) = \phi^{-1}(\mathcal{H}(\overline{\mathbb{F}}_p))$.*

Proof. Suppose first that $c \in \mathcal{H}(\mathbb{C}_p)$, so that 0 tends to the formally attracting cycle c_1, \dots, c_n under iteration of $f_c = x^2 + c$. Since $p \neq 2$, it follows from (6) that we must have $|c_i| < 1$ for some i , whence $\phi(c_i) = 0$. Since the forward orbit of 0 has points arbitrarily close to c_i in \mathbb{C}_p , we have that 0 is periodic under iteration of $x^2 + \phi(c)$, whence $\phi(c) \in \mathcal{H}(\overline{\mathbb{F}}_p)$.

Now suppose that 0 is periodic of period n under $x^2 + \phi(c)$. Then $f_{\phi(c)}^{\circ n}$ fixes 0, whence $|f_c^{\circ n}(0)| \leq 1$. Note that $f_c^{\circ n}$ is a polynomial in x^2 with coefficients in \mathcal{O}_p , and since $|f_c^{\circ n}(0)| \leq 1$, it follows by induction that $|f_c^{\circ in}(0)| \leq |f_c^{\circ n}(0)|$ for all $i \geq 1$. Now since $f_c^{\circ n}$ is a polynomial in x^2 , we have $(f_c^{\circ n}(x) - x)'(0) = -1$, so we can apply Hensel's lemma to obtain a fixed point d of $f_c^{\circ n}$ with $|d| < 1$. Consider $g(x, y) = (f_c^{\circ n}(x) - f_c^{\circ n}(y))/(x - y)$, which is a polynomial with coefficients in \mathcal{O}_p and without a constant term, since g is divisible by $x + y$. Thus, for $|a|, |b| < 1$ we have $|g(a, b)| \leq \max\{|a|, |b|\}$. Taking $m \geq 1$, $a = f_c^{\circ(m-1)n}(0)$, and $b = f_c^{\circ(m-1)n}(d) = d$, we have

$$|f_c^{\circ mn}(0) - d| \leq |f_c^{\circ(m-1)n}(0) - d| \cdot \max\{|f_c^{\circ(m-1)n}(0)|, |d|\}.$$

Repeating this $m - 2$ times, it follows that $|f_c^{\circ mn}(0) - d| < |d| \cdot \prod_{i=1}^{m-1} \max\{|f_c^{\circ in}(0)|, |d|\}$. By the remark at the beginning of this paragraph, the right-hand side of this expression is at most $|d| \cdot (\max\{|f_c^{\circ n}(0)|, |d|\})^{m-1}$. Thus, $f_c^{\circ mn}(0)$ tends to d as m grows. Now d is a fixed point of $f_c^{\circ n}$ and thus belongs to a cycle of f_c ; moreover, this cycle is attracting since $f_c^{\circ n}$ being a polynomial in x^2 implies $|(f_c^{\circ n})'(d)| \leq |d| < 1$. Given $\epsilon > 0$, the continuity of f_c allows us to choose $\delta > 0$ such that $|x - d| < \delta$ implies $|f_c^{\circ i}(x) - f_c^{\circ i}(d)| < \epsilon$ for all i with $1 \leq i \leq n$. Thus, for m large enough, $|f_c^{\circ mn+i}(0) - f_c^{\circ i}(d)| < \epsilon$ for all i with $1 \leq i \leq n$. Therefore, the orbit of 0 under f_c converges to the attracting cycle containing d , proving that $c \in \mathcal{H}(\mathbb{C}_p)$. \square

We discussed in §1 the decomposition of $\mathcal{H}(K)$ into a disjoint union of open components $\mathcal{H}(K)^{(i)}$ corresponding to c where 0 tends to a formally attracting i -cycle. In the case $K = \mathbb{C}_p$, these components are unions of open disks with radius 1. For instance, f_c has a formally attracting fixed point if and only if $f_c(x) - x = x^2 - x + c$ has a root in m_p . A quick exercise in Newton polygons shows that this happens if and only if $c \in m_p$, i.e. $\phi(c) = 0$ where ϕ is the reduction homomorphism. Thus, $\mathcal{H}(\mathbb{C}_p)^{(1)} = \phi^{-1}(0)$. A similar analysis shows that f_c has a formally attracting two-cycle if and only if $\phi(c) = -1$.

3. Applying the Tchebotarev density theorem

In order to prove Theorem 1.4, our overall strategy is to give an upper bound for $D(\mathcal{H}(\overline{\mathbb{F}}_p))$ and show that this upper bound is zero. In this section we prove Theorem 1.5, which uses the Tchebotarev density theorem for function fields to give a practical method for computing the upper bound.

Let $f_c = x^2 + c$, and note that for $c \in \overline{\mathbb{F}}_p$, the forward orbit $\{f_c^{\circ n}(0) : n = 1, 2, \dots\}$ of 0 is contained in $\mathbb{F}_p(c)$. Clearly 0 is periodic if and only if its backward orbit has points in common with its forward orbit. We thus let $f_c^{-\circ n}(0) = \{b \in \overline{\mathbb{F}}_p : f_c^{\circ n}(b) = 0\}$, and consider the sets

$$\mathcal{I}_n = \{c \in \overline{\mathbb{F}}_p : f_c^{-\circ n}(0) \cap \mathbb{F}_p(c) \neq \emptyset\}.$$

as defined in (5). These sets are useful because they furnish successively better ‘approximations’ of $\mathcal{H}(\overline{\mathbb{F}}_p)$, as we now show.

PROPOSITION 3.1. *For each $n \geq 1$, we have $\mathcal{I}_n \supseteq \mathcal{I}_{n+1}$. Moreover, $\mathcal{H}(\overline{\mathbb{F}}_p) = \bigcap_{n \geq 1} \mathcal{I}_n$.*

Proof. Let $c \in \mathcal{I}_{n+1}$, and take $b \in \mathbb{F}_p(c)$ such that $f_c^{on+1}(b) = 0$. Then $f_c^{on}(f_c(b)) = 0$ and $f_c(b) \in \mathbb{F}_p(c)$, whence $c \in \mathcal{I}_n$. To show the second statement, let $c \in \mathcal{H}(\overline{\mathbb{F}}_p)$, let $f_c^{om}(0) = 0$, and take $n \geq 1$. Write $n = im - j$ for some $0 < j \leq m$, and note that

$$f_c^{on}(f_c^{oj}(0)) = f_c^{im}(0) = 0.$$

Clearly $f_c^{oj}(0) \in \mathbb{F}_p(c)$, showing that $f_c^{-on}(0) \cap \mathbb{F}_p(c) \neq \emptyset$. Since n was arbitrary, this shows $c \in \bigcap_{n \geq 1} \mathcal{I}_n$. Now suppose $c \in \bigcap_{n \geq 1} \mathcal{I}_n$, and for each n , let $b_n \in f_c^{-on}(0) \cap \mathbb{F}_p(c)$. The finiteness of $\mathbb{F}_p(c)$ implies we must have $b_{n_1} = b_{n_2}$ for some $n_1 < n_2$. Therefore,

$$f_c^{on_2-n_1}(0) = f_c^{on_2-n_1}(f_c^{on_1}(b_{n_1})) = f_c^{on_2}(b_{n_1}) = f_c^{on_2}(b_{n_2}) = 0.$$

Hence, $c \in \mathcal{H}(\overline{\mathbb{F}}_p)$. □

Recall the definition of the density of $\mathcal{C} \subseteq \overline{\mathbb{F}}_p$ given in (4). The following proposition gives a method for showing $D(\mathcal{H}(\overline{\mathbb{F}}_p)) = 0$ using only information about $D(\mathcal{I}_n)$.

PROPOSITION 3.2. *Suppose that $D(\mathcal{I}_n)$ exists for all n and $\lim_{n \rightarrow \infty} D(\mathcal{I}_n) = 0$. Then $D(\mathcal{H}(\overline{\mathbb{F}}_p))$ exists and equals zero.*

Proof. Given $\mathcal{C} \subseteq \overline{\mathbb{F}}_p$, define

$$a_{\mathcal{C}}(s) = \frac{\sum_{\alpha \in \mathcal{C}} (\deg \alpha)^{-1} N(\alpha)^{-s}}{\sum_{\alpha \in \overline{\mathbb{F}}_p} (\deg \alpha)^{-1} N(\alpha)^{-s}}.$$

Since $\mathcal{H}(\overline{\mathbb{F}}_p) \subseteq \mathcal{I}_n$ for all n , we have $a_{\mathcal{H}(\overline{\mathbb{F}}_p)}(s) \leq a_{\mathcal{I}_n}(s)$ for $s > 1$. Taking lim sups and using the assumption that $D(\mathcal{I}_n)$ exists gives

$$\limsup_{s \rightarrow 1^+} a_{\mathcal{H}(\overline{\mathbb{F}}_p)}(s) \leq \limsup_{s \rightarrow 1^+} a_{\mathcal{I}_n}(s) = \lim_{s \rightarrow 1^+} a_{\mathcal{I}_n}(s) = D(\mathcal{I}_n).$$

Since $\lim_{n \rightarrow \infty} D(\mathcal{I}_n) = 0$ and $a_{\mathcal{H}(\overline{\mathbb{F}}_p)}(s) \geq 0$ for $s > 1$, it follows that $\limsup_{s \rightarrow 1^+} a_{\mathcal{H}(\overline{\mathbb{F}}_p)}(s) = 0$. Therefore, $\lim_{s \rightarrow 1^+} a_{\mathcal{H}(\overline{\mathbb{F}}_p)}(s) = 0$, proving that $D(\mathcal{H}(\overline{\mathbb{F}}_p)) = 0$. □

We now wish to use the Tchebotarev density theorem for function fields to prove Theorem 1.5, which shows that $D(\mathcal{I}_n)$ exists and gives a method for computing it. To do this, we must relate $D(\mathcal{I}_n)$ to the density of a set of primes in $\mathbb{F}_p[t]$.

Let P be the collection of primes in $\mathbb{F}_p[t]$. By the density of a set S of primes of $\mathbb{F}_p[t]$, we mean

$$D(S) = \lim_{s \rightarrow 1^+} \frac{\sum_{\mathfrak{p} \in S} N\mathfrak{p}^{-s}}{\sum_{\mathfrak{p} \in P} N\mathfrak{p}^{-s}}, \tag{7}$$

where $N\mathfrak{p} = p^{\deg \mathfrak{p}}$. Recall from (5) the definition of \mathcal{I}_n , and note that $f_c^{-on}(0) \cap \mathbb{F}_p(c) \neq \emptyset$ is equivalent to the factorization of $f_c^{on}(x)$ over $\mathbb{F}_p(c)$ having a linear factor. This, in turn, is equivalent to $f_t^{on}(x)$ having a linear factor modulo (π_c) , where $\pi_c \in \mathbb{F}_p[t]$ is the minimal polynomial of c . Hence,

$$\mathcal{I}_n = \{c \in \overline{\mathbb{F}}_p : f_t^{on} \text{ has a linear factor mod } (\pi_c)\}. \tag{8}$$

Since membership in \mathcal{I}_n depends only on properties of π_c , it follows that \mathcal{I}_n is invariant under the action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The following proposition relates the density of Galois-invariant subsets of $\overline{\mathbb{F}}_p$ to the density of related sets of primes in $\mathbb{F}_p[t]$.

PROPOSITION 3.3. *Suppose that $\mathcal{C} \subseteq \overline{\mathbb{F}}_p$ is invariant under the action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, and let $B \subseteq P$ be given by $\{(\pi_c) : c \in \mathcal{C}\}$, where π_c is the minimal polynomial of c . Suppose also that $D(B)$ exists. Then $D(\mathcal{C})$ exists and equals $D(B)$.*

Proof. Consider the map $\psi : \overline{\mathbb{F}}_p \rightarrow P$ that takes c to (π_c) . The Galois invariance of \mathcal{C} is equivalent to \mathcal{C} being the full inverse image of B under ψ . We thus have

$$\sum_{c \in \mathcal{C}} (\deg c)^{-1} N(c)^{-s} = \sum_{\mathfrak{p} \in B} \sum_{c \in \psi^{-1}(\mathfrak{p})} (\deg c)^{-1} N(c)^{-s}, \tag{9}$$

where we recall $N(c) = p^{\deg c}$. Now for any $c \in \psi^{-1}(\mathfrak{p})$ we have $\deg c = \deg \pi_c = \deg \mathfrak{p}$. Thus $N(c) = N\mathfrak{p}$. Hence, the inner sum in the right-hand side of (9) is repeated addition of the same quantity, and the right-hand side becomes $\sum_{\mathfrak{p} \in B} N\mathfrak{p}^{-s}$. Applying the same reasoning to $\sum_{c \in \overline{\mathbb{F}}_p} (\deg c)^{-1} N(c)^{-s}$ gives

$$\frac{\sum_{c \in \mathcal{C}} (\deg c)^{-1} N(c)^{-s}}{\sum_{c \in \overline{\mathbb{F}}_p} (\deg c)^{-1} N(c)^{-s}} = \frac{\sum_{\mathfrak{p} \in B} N\mathfrak{p}^{-s}}{\sum_{\mathfrak{p} \in P} N\mathfrak{p}^{-s}}.$$

Taking limits as $s \rightarrow 1^+$ and using the existence of $D(B)$ completes the proof. □

We now define the following set of primes in $\mathbb{F}_p[t]$:

$$I_n = \{\mathfrak{p} \in P : f_t^{on} \bmod \mathfrak{p} \text{ has at least one linear factor}\}. \tag{10}$$

The following corollary follows immediately from Proposition 3.3 and equation (8).

COROLLARY 3.4. *For all $n \geq 1$, $D(\mathcal{I}_n) = D(I_n)$.*

We now give a version of a standard result that allows us to apply the Tchebotarev density theorem to compute $D(I_n)$.

PROPOSITION 3.5. *Let $R = \mathbb{F}_p[t]$, $L = \mathbb{F}_p(t)$, and $f \in L[x]$. Let $L_n(f)$ be the splitting field of f^{on} over L , and let $G_n(f) = \text{Gal}(L_n(f)/L)$. There exists a finite set S of primes in R (including the ramified primes) such that if \mathfrak{p} is not in S and $(\mathfrak{p}, L_n(f)/L) \subset G_n(f)$ is the Artin conjugacy class of \mathfrak{p} , then the following holds. If $f_1 f_2 \cdots f_r$ is the factorization into irreducibles of $f^{on} \bmod \mathfrak{p}$, then any element of $(\mathfrak{p}, L_n(f)/L)$ acts on the roots of f^{on} as a product $\sigma_1 \cdots \sigma_r$ of disjoint cycles, with σ_i having length $\deg f_i$.*

Proof. Let β be a root of f^{on} , set $L_\beta = L(\beta)$, and let R_β be the integral closure of R in L_β . A standard result in algebraic number theory [Nar04, Theorem 4.12, p. 177] states that for all primes \mathfrak{p} not contained in a finite set S' , we have

$$\mathfrak{p}R_\beta = \mathfrak{P}_1 \mathfrak{P}_2 \cdots \mathfrak{P}_r, \tag{11}$$

where \mathfrak{P}_i is a prime in R_β with residue class degree $d(\mathfrak{P}_i/\mathfrak{p}) = \deg f_i$. Another standard result (see [Nar04, Lemma 7.13, p. 391] for a proof easily adapted to the function field case) states that if \mathfrak{p} is in addition unramified, then (11) implies that any element of $(\mathfrak{p}, L_n(f)/L)$ acts on the roots of f^{on} as a product $\sigma_1 \cdots \sigma_r$ of disjoint cycles, with σ_i having length $d(\mathfrak{P}_i/\mathfrak{p})$. □

Using the notation of Proposition 3.5, let $f = f_t = x^2 + t$ and let U_n be the set of primes in R that are unramified in $L_n(f_t)$. Note that conjugacy preserves the lengths of the cycles in the disjoint cycle decomposition, so if one element of $(\mathfrak{p}, L_n(f_t)/L)$ fixes a root of f_t^{on} , then all do. Put

$$J_n = \{\mathfrak{p} \in U_n : \text{each } \sigma \in (\mathfrak{p}, L_n(f_t)/L) \text{ fixes at least one root of } f_t^{on}\}. \tag{12}$$

It follows immediately from Proposition 3.5 and (10) that $D(J_n) = D(I_n)$. To compute $D(J_n)$, we use the Tchebotarev density theorem, which we now state.

THEOREM 3.6 (Tchebotarev). *Let L/K be a Galois extension of function fields, and denote by U_K the set of primes of K unramified in L . For $\mathfrak{p} \in U_K$, let $(\mathfrak{p}, L/K)$ be the Artin conjugacy class of \mathfrak{p} .*

Fix a conjugacy class C of $G = \text{Gal}(L/K)$. Then for all $k \geq 1$,

$$D(\{\mathfrak{p} \in U_K : (\mathfrak{p}, L/K) = C\}) = \frac{\#C}{\#G}.$$

For a proof, see [Ros02, ch. 9].

Proof of Theorem 1.5. Recall the definition of J_n from (12). Using Propositions 3.2, 3.3, and $D(I_n) = D(J_n)$, it suffices to find $D(J_n)$. Let \mathfrak{C} denote the collection of conjugacy classes of $G_n(f)$ each of whose elements fixes at least one root of f_t^{on} . Using Theorem 3.6, we have

$$D(J_n) = \sum_{C \in \mathfrak{C}} \frac{\#C}{\#G_n(f)} = \frac{1}{\#G_n(f)} \sum_{C \in \mathfrak{C}} \#C = \frac{1}{\#G_n(f)} \#\left\{ \bigcup_{C \in \mathfrak{C}} C \right\}.$$

Since g fixes a root of f_t^{on} if and only if every element of its conjugacy class does the same, $\bigcup_{C \in \mathfrak{C}} C = \{g \in G_n(f) : g \text{ fixes at least one root of } f_t^{on}\}$. \square

Example 3.7. Consider the case $n = 2$. Label the roots of $f_t^{o2} = (x^2 + t)^2 + t$ as follows:

$$\begin{aligned} \sqrt{-t + \sqrt{-t}} &\longleftrightarrow 1, & -\sqrt{-t + \sqrt{-t}} &\longleftrightarrow 2 \\ \sqrt{-t - \sqrt{-t}} &\longleftrightarrow 3, & -\sqrt{-t - \sqrt{-t}} &\longleftrightarrow 4. \end{aligned}$$

We show (Corollary 6.6) that under this labeling $G_2(f)$ is a subgroup of S_4 of order 8 that contains $\{e, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$ as well as four elements that interchange the sets $\{1, 2\}$ and $\{3, 4\}$ and therefore have no fixed points. Hence, $D(\mathcal{I}_2) = 3/8$. In fact, we know more. Let k be large, choose $c \in \mathbb{F}_{p^k}$ at random, and let $i_c = \#\{f_c^{-o2}(0) \cap \mathbb{F}_p(c)\}$. Then $i_c = 2$ with probability $\frac{1}{4}$ and $i_c = 4$ with probability $\frac{1}{8}$.

Remark. There is a second version of Theorem 3.6 that gives the stronger conclusion $\#\{\mathfrak{p} \in U_K : \deg \mathfrak{p} = k \text{ and } (\mathfrak{p}, L/K) = C\} = (p^k/k)(\#C/\#G + O(p^{-k/2}))$ (see [Ros02, Theorem 9.13B]). With this conclusion, one can replace $D(\mathcal{I}_n)$ by $\delta(\mathcal{I}_n)$ (see §1) in Theorem 1.5, and thus also in Theorem 1.4. However, this stronger version of Theorem 3.6 requires the hypothesis that L/K be *geometric*, i.e. that if k is the field of constants in K , then $\bar{k} \cap L = k$. Determining the geometricity of the fields generated by roots of f_t^{on} appears to be a difficult problem. Indeed, the most natural approach may be to first prove Conjecture 6.7 (see the discussion following the conjecture).

4. Galois processes and Galois martingales

Let L be a field of characteristic different from 2, and for $f \in L[x]$, let $L(f)$ denote the splitting of f over L . By a *tower of polynomials over L* we mean a sequence f_1, f_2, \dots such that $L(f_n) \supseteq L(f_{n-1})$ for $n \geq 2$. We call the tower separable if f_n is separable over L for all $n \geq 1$.

Let $\mathcal{F} = f_1, f_2, \dots$ be a separable tower of polynomials, and put $L_\infty = \bigcup_{n=1}^\infty L(f_n)$. Denote $\text{Gal } L(f_n)/L$ by $G(f_n)$, and let

$$G(\mathcal{F}) = \text{Gal } L_\infty/L \cong \varprojlim G(f_n).$$

Let \mathbf{P} be the Haar measure on the compact group $G(\mathcal{F})$, normalized so that $\mathbf{P}(G(\mathcal{F})) = 1$. Letting \mathcal{B} be the Borel sigma algebra, the triple $(G(\mathcal{F}), \mathbf{P}, \mathcal{B})$ is then a probability space. Denote by ψ_n the natural projection $G(\mathcal{F}) \rightarrow G(f_n)$, and define random variables X_n on $G(\mathcal{F})$ as follows:

$$X_n(g) = \text{number of roots of } f_n \text{ fixed by } \psi_n(g).$$

The data $(G(\mathcal{F}), \mathbf{P}, \mathcal{B}, \{X_n\}_{n \geq 1})$ by definition give a stochastic process, which we call the *Galois process* of the tower \mathcal{F} , and denote by $GP(\mathcal{F})$. Intuitively, this process resembles a random walk

through successively higher levels of the group $G(\mathcal{F})$. Positions at each level are assigned a value based on the number of roots of f_n left fixed. Note that it follows from basic properties of Haar measure that

$$P(X_1 = t_1, \dots, X_n = t_n) = \frac{1}{\#G(f_n)} \#\{g \in G(f_n) : g \text{ fixes } t_i \text{ roots of } f_i \text{ for } i = 1, 2, \dots, n\}. \quad (13)$$

Example 4.1. Let $L = \mathbb{Q}$, $f_1 = x^2 + x + 1$, and $f_2 = x^3 - 2$. Clearly $L(f_2) \supseteq L(f_1)$. Define a separable tower \mathcal{F} by setting $f_n = f_2$ for all $n \geq 3$. Then $G(\mathcal{F}) \cong G(f_2)$, which is the full symmetric group on the roots of f_2 . Since the Haar measure \mathbf{P} is invariant under multiplication by an element of $G(\mathcal{F})$, it follows that $\mathbf{P}(\psi_n^{-1}(g)) = 1/\#G(f_n)$ for all $g \in G(f_n)$. Thus,

$$\mathbf{P}(X_2 = i) = \begin{cases} \frac{1}{6} & \text{if } i = 3 \\ \frac{1}{2} & \text{if } i = 1 \\ \frac{1}{3} & \text{if } i = 0. \end{cases}$$

One easily sees that the kernel of the quotient map $G(f_2) \rightarrow G(f_1)$ is isomorphic to the alternating group A_3 . Thus, if $g \in A_3$, $X_1(g) = 2$, while if $g \in G(f_2) \setminus A_3$, then $X_1(g) = 0$. Because $G(f_2) \setminus A_3$ is composed entirely of transpositions, we have $\mathbf{P}(X_2 = 1 \mid X_1 = 0) = 1$. Therefore, $GP(\mathcal{F})$ is not a martingale (see Definition 1.1).

Consider the case where $f \in L(x)$ has the property that all iterates f^{on} are separable over L and $\mathcal{F} = f, f^{\circ 2}, f^{\circ 3}, \dots$. This is the case of greatest interest to us. In this situation we write $GP(f)$ instead of $GP(\mathcal{F})$ and $G_n(f)$ instead of $G(f^{on})$. We now develop preparatory material for proving Theorem 1.2. Recall that a G -set is a set T on which G acts, and a map $\phi : T \rightarrow T'$ is a morphism of G -sets if $\phi(\sigma(t)) = \sigma(\phi(t))$ for all $\sigma \in G$ and $t \in T$. We define the following notion that we use throughout.

DEFINITION 4.2. Let G be a group and T a G -set. By a *fiber system*¹ on T , we mean the collection of fibers of a surjective morphism $\phi : T \rightarrow T'$ of G -sets.

Note that a fiber system on T gives a partition of T , and the sets belonging to this partition are permuted by G ; indeed, these properties characterize fiber systems. By way of illustration, we offer the following.

PROPOSITION–DEFINITION 4.3. Let L be a field and $f \in L[x]$ a polynomial with all iterates separable. Let R_n denote the roots of f^{on} and R_{n-1} the roots of $f^{o(n-1)}$. Then $f : R_n \rightarrow R_{n-1}$ is a surjective morphism of $G_n(f)$ -sets. It defines a fiber system on R_n that we call the *fundamental fiber system*.

Proof. We need only check that $f(\sigma(\beta)) = \sigma(f(\beta))$ for any $\sigma \in G_n(f)$ and $\beta \in R_n$. This clearly holds since $\sigma \in \text{Gal}(L(f^{on})/L)$. □

For instance, let $L = k(t)$, $f(x) = x^2 + t$, and $n = 2$, and use the labelings of Example 3.7. Then the fundamental fiber system on R_2 is $\{\{1, 2\}, \{3, 4\}\}$. Note that the fundamental fiber system consists of sets each containing $\deg f$ elements.

The proof of Theorem 1.2 makes crucial use of a uniqueness result on fiber systems for G -sets when G is a certain kind of permutation group. Specifically, let f be quadratic with separable and irreducible iterates, and suppose that $G_n(f)$ contains at least one odd permutation. We wish to show that the fundamental fiber system is the only fiber system on R_n (considered as a $G_n(f)$ -set) that consists of two-element sets. The next few definitions and lemmas build up to this result (Corollary 4.9).

¹Some authors use the terminology ‘block system’.

Let T be a set and \mathfrak{S} a partition of T . Denote by $\text{Perm}(T, \mathfrak{S})$ the set of all permutations of T that act as permutations on \mathfrak{S} . Thus, if a group G acts on T and \mathfrak{S} is a fiber system for G , then $G \subseteq \text{Perm}(T, \mathfrak{S})$.

DEFINITION 4.4. Let T be a set and \mathfrak{S} a partition of T . A *permutation associated to \mathfrak{S}* is a permutation $\sigma \in \text{Perm}(T, \mathfrak{S})$ whose orbits are precisely the subsets belonging to \mathfrak{S} . In the case where \mathfrak{S} is composed entirely of two-element sets, we denote by $\sigma_{\mathfrak{S}}$ the unique permutation associated to \mathfrak{S} .

PROPOSITION 4.5. Let T be a set, \mathfrak{S} a partition of T , and σ a permutation associated to \mathfrak{S} . Then $\tau\sigma\tau^{-1}$ is a permutation associated to \mathfrak{S} for each $\tau \in \text{Perm}(T, \mathfrak{S})$. In particular, if \mathfrak{S} is composed of two-element subsets, then

$$\sigma_{\mathfrak{S}} \in Z(\text{Perm}(T, \mathfrak{S})),$$

the center of $\text{Perm}(T, \mathfrak{S})$.

Proof. Let $\tau \in \text{Perm}(T, \mathfrak{S})$, and $S \in \mathfrak{S}$ with $\#S = k$. Then $\tau\sigma\tau^{-1}$ must map S to itself. Note that σ acts on S as a k -cycle since by definition S is an orbit of σ . Since conjugation preserves cycle decomposition type and both σ and $\tau\sigma\tau^{-1}$ permute S , it follows that $\tau\sigma\tau^{-1}$ acts on S as a k -cycle. Therefore, S is an orbit of $\tau\sigma\tau^{-1}$. \square

LEMMA 4.6. Let G be a group acting on a set T . Suppose that $\mathfrak{S} = \{S_1, S_2, \dots, S_m\}$ is a fiber system for G with $\#S_i = 2j$ for $1 \leq i \leq m$. Then G is isomorphic to a subgroup of the wreath product $\text{Sym}(2j) \wr \text{Sym}(\mathfrak{S}) = \text{Sym}(2j)^{\mathfrak{S}} \rtimes \text{Sym}(\mathfrak{S})$ and if $g \mapsto ((\delta_1, \dots, \delta_m), \sigma)$ then the signature of the action of g on T is $\prod_{i=1}^m \text{sgn } \delta_i$.

Proof. Since \mathfrak{S} is a fiber system for G , we have $G \subseteq \text{Perm}(T, \mathfrak{S})$. It therefore suffices to prove the lemma for $\text{Perm}(T, \mathfrak{S})$. Each $\tau \in \text{Perm}(T, \mathfrak{S})$ induces a permutation τ' on \mathfrak{S} . Fix an ordering of the elements in each S_i , and suppose that $\tau(S_i) = S_k$. Say $S_i = \{s_1, \dots, s_{2j}\}$ and $S_k = \{t_1, \dots, t_{2j}\}$. Then $t_n \mapsto \tau(s_n)$ is a permutation of S_k , which we denote by δ_i . The map $\text{Perm}(T, \mathfrak{S}) \rightarrow \text{Sym}(2j) \wr \text{Sym}(\mathfrak{S}) : \tau \mapsto ((\delta_1, \dots, \delta_m), \tau')$ is readily seen to be an isomorphism.

Suppose that $\tau \in \text{Perm}(T, \mathfrak{S})$ satisfies $\delta_i = \text{id}$ for $i = 1, \dots, m$, and let $C = (S_{i_1} \cdots S_{i_l})$ be a cycle of τ' and $\Sigma = S_{i_1} \cup \cdots \cup S_{i_l}$. Consider the action of τ on Σ . Since C is an l -cycle, the orbit of any $s \in \Sigma$ must have length at least l , but clearly $\tau^l(s) = s$ for all s . Thus, τ acts on Σ as a product of $2j$ l -cycles, whence this action is even. The same holds for all cycles of τ' , implying that the signature of τ is 1.

If $\tau \in \text{Perm}(T, \mathfrak{S})$ satisfies $\tau' = \text{id}$, then τ has the same cycle structure as the product $\delta_1 \cdots \delta_m \in \text{Sym}(2jm)$. Thus, the signature of τ is $\prod_{i=1}^m \text{sgn } \delta_i$. Since any $((\delta_1, \dots, \delta_m), \sigma) \in \text{Sym}(2j) \wr \text{Sym}(\mathfrak{S})$ admits the decomposition $((\delta_1, \dots, \delta_m), \text{id}) \cdot ((\text{id}, \dots, \text{id}), \sigma)$, the lemma is proved. \square

THEOREM 4.7. Let T be a set with $2m$ elements, and let

$$\mathfrak{S} = \{S_1, \dots, S_m\} \quad \text{and} \quad \mathfrak{U} = \{U_1, \dots, U_m\}$$

be partitions of T with $\#S_i = \#U_i = 2$ for all i . Let $\sigma_{\mathfrak{S}}$ and $\sigma_{\mathfrak{U}}$ be the permutations associated to \mathfrak{S} and \mathfrak{U} , respectively, and suppose that $\sigma_{\mathfrak{U}} \in \text{Perm}(T, \mathfrak{S})$. If $\sigma_{\mathfrak{S}} \neq \sigma_{\mathfrak{U}}$, then any subgroup of $\text{Perm}(T, \mathfrak{S}) \cap \text{Perm}(T, \mathfrak{U})$ that acts transitively on T is alternating, i.e. composed entirely of even permutations.

Proof. Let $G \leq \text{Perm}(T, \mathfrak{S}) \cap \text{Perm}(T, \mathfrak{U})$ act transitively on T . From Proposition 4.5, we have that G centralizes $H = \langle \sigma_{\mathfrak{U}}, \sigma_{\mathfrak{S}} \rangle$ in $\text{Sym}(T)$ and also $\sigma_{\mathfrak{S}}$ commutes with $\sigma_{\mathfrak{U}}$ (the latter since $\sigma_{\mathfrak{U}} \in \text{Perm}(T, \mathfrak{S})$). Hence, H has order 4, because $\sigma_{\mathfrak{S}} \neq \sigma_{\mathfrak{U}}$. Note that by definition $\sigma_{\mathfrak{U}}$ and $\sigma_{\mathfrak{S}}$ have no fixed points in T , so that if an orbit of the action of H on T has fewer than four elements, then

we must have $\sigma_{\mathfrak{U}}(t) = \sigma_{\mathfrak{S}}(t)$ for some $t \in T$. However, G centralizes H and acts transitively on T , implying $\sigma_{\mathfrak{U}}(t) = \sigma_{\mathfrak{S}}(t)$ for all $t \in T$, which contradicts $\sigma_{\mathfrak{S}} \neq \sigma_{\mathfrak{U}}$.

Since G centralizes H , it follows immediately that the set $V = \{V_i\}$ of orbits of the action of H on T is a fiber system for G . From Lemma 4.6 we have that G injects into $\text{Sym}(4) \wr \text{Sym}(V)$. Let $g \in G$ and fix $\{t_i\}$ such that $V_i = \{t_i, \sigma_{\mathfrak{U}}(t_i), \sigma_{\mathfrak{S}}(t_i), \sigma_{\mathfrak{U}}\sigma_{\mathfrak{S}}(t_i)\}$ for each i . Suppose $g(V_i) = V_j$, and let δ_i be as in the proof of Lemma 4.6. Again since G centralizes H , one easily verifies that δ_i is the identity if $g(t_i) = t_j$ and is the product of two transpositions otherwise. Hence, the signature of δ_i is 1, and it follows from Lemma 4.6 that $\text{sgn } g = 1$. □

If the group G has nontrivial center, we have another source of nontrivial fiber systems.

PROPOSITION–DEFINITION 4.8. Let G be a group acting on a set T , suppose that $\sigma \in Z(G)$, and let \mathfrak{S} be the partition of T given by the orbits of σ . Then \mathfrak{S} is a fiber system for T and σ is a permutation associated to \mathfrak{S} . We call \mathfrak{S} a *central fiber system*.

Proof. Let $S \in \mathfrak{S}$ and $\tau \in G$. Write $S = \{\sigma^n(s) : n \geq 1\}$, and note that $\tau(S) = \{\tau\sigma^n(s) : n \geq 1\} = \{\sigma^n(\tau(s)) : n \geq 1\}$, which is again an element of \mathfrak{S} . Thus, \mathfrak{S} is a G -set, and indeed is the fiber system associated to the natural morphism $T \rightarrow \mathfrak{S}$ of G -sets. Clearly σ is a permutation associated to \mathfrak{S} . □

A salient feature of central fiber systems is that at least one associated permutation must lie in the group G . This is not necessarily the case for the fundamental fiber system defined in Proposition–Definition 4.3. This feature of central fiber systems is precisely what we need to establish our uniqueness result on fiber systems consisting of two-element sets. In the process, we show that the central involution $\sigma_{\mathfrak{C}}$ associated to the fundamental fiber system must lie in $G_n(f)$. This provides a vital step in the proof of Theorem 1.2.

COROLLARY 4.9. Let L be a field, $f \in L[x]$ a quadratic polynomial with f^{on} separable and irreducible over L , and suppose that $\text{Disc } f^{on}$ is not a square in L . Then there is a unique fiber system of two-element sets on R_n , the set of roots of f^{on} (considered as a $G_n(f)$ -set). In particular, if \mathfrak{C} is the fundamental fiber system defined in Proposition–Definition 4.3, then the permutation $\sigma_{\mathfrak{C}}$ associated to \mathfrak{C} is contained in $G_n(f)$.

Proof. The splitting field $L(f^{on})$ of f^{on} is obtained from $L(f^{on-1})$ by adjoining roots of $f(x) - \alpha$ for each root α of f^{on-1} . Since $\deg f = 2$, it follows that $\text{Gal}(L(f^{on})/L(f^{on-1}))$ is an elementary abelian 2-group. Hence, $G_n(f)$ is a 2-group, and therefore has nontrivial center. Since $Z(G_n(f))$ is again a 2-group, there must be $\delta \in Z(G_n(f))$ of order two. Suppose that δ fixes $r \in R_n$. Since δ belongs to the center of $G_n(f)$, this gives $\delta\sigma(r) = \sigma(r)$ for all $\sigma \in G_n(f)$. The irreducibility of f^{on} implies that $G_n(f)$ acts transitively on R_n , whence δ is the identity, a contradiction. Therefore, δ has no fixed points, implying that the associated central fiber system \mathfrak{D} (see Proposition–Definition 4.8) consists of two-element sets.

Let \mathfrak{S} be another fiber system for $G_n(f)$ consisting of two element sets. Note that $\delta \in G_n(f) \subseteq \text{Perm}(T, \mathfrak{S})$. Finally, since $\text{Disc } f^{on}$ is not a square in L , basic Galois theory tells us $G_n(f)$ cannot be alternating. We then apply Theorem 4.7 to get $\mathfrak{S} = \mathfrak{D}$. In particular, $\mathfrak{C} = \mathfrak{D}$, implying that $\sigma_{\mathfrak{C}} = \delta \in G_n(f)$. □

Recall from § 1 that the adjusted forward orbit of a point l under $f \in L[x]$ with leading coefficient a is $\{-af(l)\} \cup \{f^{on}(l) : n = 2, 3, \dots\}$.

LEMMA 4.10. Let L be a field of characteristic different from 2, $f \in L[x]$ quadratic, and $\gamma \in L$ the unique finite critical point of f . Suppose that the adjusted forward orbit of γ contains no squares in L . Then for all n , f^{on} is separable and irreducible and $\text{Disc } f^{on}$ is not a square in L .

Proof. We first show that $\text{Disc } f^{on}$ is not a square; this implies that f^{on} is separable. Let $f(x) = ax^2 + bx + c$. For $n = 1$ we note that $-4af(\gamma) = \text{Disc } f$, so that the fact that $-af(\gamma)$ is not a square implies that $\text{Disc } f$ is not a square. For $n \geq 2$, it follows from [Odo85a, Lemma 3.1, part iv] that $\text{Disc } f^{on} = 2^{2n}(\text{Disc } f^{on-1})^2 \text{Res}(f', f^{on})$, where $\text{Res}(f', f^{on})$ denotes the resultant of f' and f^{on} . From the definition of resultant (see [Odo85a, p. 393]), $\text{Res}(f', f^{on}) = f^{on}(\gamma)$. Thus, $\text{Disc } f^{on}$ is not a square.

To show that f^{on} is irreducible, first note that the case $n = 1$ is covered by the previous paragraph. For $n \geq 2$ we use Capelli's lemma [Odo85a, p. 387], which implies that f^{on} is irreducible if and only if for any root α of f^{on-1} , we have $f(x) - \alpha$ irreducible over $L(\alpha)$. This is equivalent to $b^2 - 4ac + 4a\alpha$ not being a square in $L(\alpha)$, which must hold if $N_{L(\alpha)/L}(b^2 - 4ac + 4a\alpha)$ is not a square in L . However,

$$\begin{aligned} N_{L(\alpha)/L}(b^2 - 4ac + 4a\alpha) &= (-4a)^{2^{n-1}} \prod_{\alpha \text{ root of } f^{on-1}} \left(-\frac{b^2}{4a} + c \right) - \alpha \\ &= (4a)^{2^{n-1}} f^{on-1}(-b^2/4a + c) = (4a)^{2^{n-1}} f^{on-1}(f(\gamma)). \end{aligned}$$

Thus, $f^{on}(\gamma)$ not a square in L implies f^{on} is irreducible. □

Proof of Theorem 1.2. We must show that

$$E(X_n \mid X_1 = t_1, \dots, X_{n-1} = t_{n-1}) = t_{n-1}, \tag{14}$$

where t_1, \dots, t_{n-1} are integers with $\mathbf{P}(X_1 = t_1, \dots, X_{n-1} = t_{n-1}) > 0$. By definition, the left-hand side of (14) is

$$\sum_k k \cdot \frac{\mathbf{P}(X_1 = t_1, \dots, X_{n-1} = t_{n-1}, X_n = k)}{\mathbf{P}(X_1 = t_1, \dots, X_{n-1} = t_{n-1})}. \tag{15}$$

Put

$$\begin{aligned} S &= \{g \in G_n(f) : g \text{ fixes } t_i \text{ roots of } f^{oi} \text{ for } 1 \leq i \leq n-1\} \\ S_k &= \{g \in S : g \text{ fixes } k \text{ roots of } f^{on}\}. \end{aligned}$$

From the basic property of $GP(f)$ given in (13), the expression in (15) is equal to

$$\sum_k k \cdot \frac{\#S_k}{\#S}.$$

This, in turn, may be rewritten as

$$\frac{1}{\#S} \sum_{g \in S} (\text{number of roots of } f^{on} \text{ fixed by } g). \tag{16}$$

Note that if $h \in H_n(f) \stackrel{\text{def}}{=} \text{Gal}(L(f^{on})/L(f^{on-1}))$, then h fixes the roots of f^{oi} for $1 \leq i \leq n-1$. Thus, S is invariant under multiplication by $H_n(f)$, whence S is a union of cosets of $H_n(f)$. Recall from Proposition–Definition 4.3 the fundamental fiber system \mathfrak{C} for $G_n(f)$ defined by the morphism $f : R_n \rightarrow R_{n-1}$ of $G_n(f)$ -sets. Let $\sigma_{\mathfrak{C}}$ be the permutation associated to \mathfrak{C} . From Corollary 4.9 and Lemma 4.10 we have $\sigma_{\mathfrak{C}} \in G_n(f)$, and thus $\sigma_{\mathfrak{C}} \in H_n(f)$.

Now take $g_0 H_n(f) \subseteq S$. Note that the group $\{e, \sigma_{\mathfrak{C}}\}$ acts by right multiplication on the set $g_0 H_n(f)$, dividing it into two-element orbits. We analyze the number of roots of f^{on} fixed by the elements of such an orbit. Let $g \in g_0 H_n(f)$, let α be a root of f^{on-1} , and note that if $g(\alpha) \neq \alpha$, then neither g nor $g\sigma_{\mathfrak{C}}$ have any fixed points in $f^{-1}(\alpha)$. On the other hand, if $g(\alpha) = \alpha$, then $g(f^{-1}(\alpha)) = f^{-1}(\alpha)$. Since by definition $\sigma_{\mathfrak{C}}$ exchanges the elements of $f^{-1}(\alpha)$, we have that g fixes

the elements of $f^{-1}(\alpha)$ if and only if $g\sigma_{\mathcal{E}}$ exchanges them. It follows that

$$\#\{\text{roots of } f^{on} \text{ fixed by } g\} + \#\{\text{roots of } f^{on} \text{ fixed by } g\sigma_{\mathcal{E}}\} = 2 \cdot \#\{\text{roots of } f^{on-1} \text{ fixed by } g\}.$$

By the definition of S , all $g \in g_0H_n(f)$ fix t_{n-1} elements of f^{on-1} . Therefore, we have

$$\sum_{g \in g_0H_n(f)} \#\{\text{roots of } f^{on} \text{ fixed by } g\} = t_{n-1} \cdot \#H_n(f).$$

Since S is a union of cosets of $H_n(f)$, the expression in (16) equals t_{n-1} . □

Martingales are important chiefly because they often converge in the following sense.

DEFINITION 4.11. Let X_1, X_2, \dots be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The process *converges* if

$$\mathbf{P}\left(\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) \text{ exists}\right) = 1.$$

We give one simple martingale convergence theorem (see, e.g., [GS01, § 12.3] for a proof).

THEOREM 4.12. Let $M = (X_1, X_2, \dots)$ be a martingale whose random variables take nonnegative real values. Then M converges.

Since the random variables in $GP(f)$ take nonnegative integer values, we immediately have the following.

COROLLARY 4.13. Let L be a field and $f \in L[x]$ a quadratic polynomial satisfying the hypotheses of Theorem 1.2. Then

$$\mathbf{P}(\{g \in G(f) : X_1(g), X_2(g), \dots \text{ is eventually constant}\}) = 1.$$

5. Quadratic Galois processes under maximality assumptions

In this section, let L be a field and $f \in L[x]$ a quadratic polynomial with all iterates separable over L . Throughout the section, all quantities are assumed chosen so that conditional probabilities are defined.

Let $L(f^{on})$ be the splitting field of the n th iterate of f , and let $H_n(f) = \text{Gal}(L(f^{on})/L(f^{on-1}))$.

PROPOSITION–DEFINITION 5.1. For each $n \geq 1$, $H_n(f) \cong (\mathbb{Z}/2\mathbb{Z})^m$ for some $0 \leq m \leq 2^{n-1}$. We call $H_n(f)$ *maximal* if $m = 2^{n-1}$.

Proof. Let R_{n-1} denote the set of roots of f^{on-1} . Over $L(f^{on-1})$, f^{on} factors as $\prod_{\alpha \in R_{n-1}} f(x) - \alpha$. Since $\#R_{n-1} = 2^{n-1}$, the extension $L(f^{on})/L(f^{on-1})$ is the compositum of at most 2^{n-1} quadratic extensions. □

The next result gives, for n with $H_n(f)$ maximal, an explicit expression of the probability distribution of X_n given past behavior. However, the lemma does not hold for all possible past behaviors: we must assume that the value of X_{n-1} is known.

LEMMA 5.2. Let L be a field, $f \in L[x]$ a quadratic polynomial with all iterates separable over L , $H_n(f) = \text{Gal}(L(f^{on})/L(f^{on-1}))$, and $(\Omega, \mathcal{F}, \mathbf{P}, (X_n)_{n \geq 0}) = GP(f)$. Suppose that $H_n(f)$ is maximal, and let $m_1 < m_2 < \dots < m_k$, be positive integers with $m_k = n - 1$. Then for any positive integers t_1, \dots, t_k we have

$$\mathbf{P}(X_n = u \mid X_{m_1} = t_1, \dots, X_{m_k} = t_k) = \begin{cases} \binom{t_k}{w} \frac{1}{2^{t_k}} & \text{if } u = 2w \text{ for some } 0 \leq w \leq t_k \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Proof. To prove the lemma, we must compute

$$\frac{\mathbf{P}(X_{m_1} = t_1, \dots, X_{m_k} = t_k, X_n = u)}{\mathbf{P}(X_{m_1} = t_1, \dots, X_{m_k} = t_k)}. \tag{18}$$

Note that by our standing assumption that all quantities are chosen so that conditional probabilities are defined, t_1, \dots, t_k are such that the denominator of (18) is nonzero. Put

$$T = \{g \in G_n(f) : g \text{ fixes } t_i \text{ roots of } f^{o_{m_i}} \text{ for } 1 \leq i \leq k\}$$

$$T_u = \{g \in T : g \text{ fixes } u \text{ roots of } f^{o_n}\}.$$

From (13), we see that we must compute $\#T_u/\#T$. Since the denominator of (18) is nonzero, $\#T \neq 0$ as well. As in the proof of Theorem 1.2, note that T is invariant under multiplication by $H_n(f)$, whence it is a union of cosets of $H_n(f)$. Let R_n be the set of roots of f^{o_n} , and recall from Proposition–Definition 4.3 that the sets $f^{-1}(\alpha)$, where α is a root of $f^{o_{n-1}}$, form a partition of R_n .

Consider a coset $g_0H_n(f) \subset T$. By the maximality of $H_n(f)$, there exists $h_\alpha \in H_n(f)$ that exchanges the elements of $f^{-1}(\alpha)$ and fixes the elements of $f^{-1}(\alpha')$ for all $\alpha' \neq \alpha$. Let Q be the set of roots of $f^{o_{n-1}}$ fixed by g_0 , and put $M = f^{-1}(Q) \subseteq R_n$. Since $g_0 \in T$, g_0 fixes t_k roots of $f^{o_{n-1}}$, whence $\#M = 2t_k$.

Now let J be the subgroup of $H_n(f)$ that fixes each element of M . The maximality of $H_n(f)$ shows

$$\#J = 2^{(2^{n-1}-t_k)}. \tag{19}$$

Take $h \in H_n(f)$. Since h fixes all roots of $f^{o_{n-1}}$, we have $g_0h(f^{-1}(\alpha)) = f^{-1}(g_0(\alpha))$ for all α . Thus, g_0h cannot fix any element of $R_n - M$. On the other hand, elements of J fix all members of M , so it follows that every element of a set of the form g_0hJ has the same number of fixed points in R_n .

Since $M = \bigcup_{\alpha \in Q} f^{-1}(\alpha)$, we can write any $h \in H_n(f)$ as

$$j \prod_{\alpha \in Q} (h_\alpha)^{e_\alpha},$$

where $j \in J$ and $e_\alpha = 0$ or 1 for each α . Thus, any coset g_0hJ may be written uniquely as $g_0J \prod_{\alpha \in Q} (h_\alpha)^{e_\alpha}$. Moreover, all elements of this coset have exactly

$$2t_k - \sum_{\alpha \in Q} 2e_\alpha \tag{20}$$

fixed points in R_n (recall $\#Q = t_k$). The number of ways (20) can equal u is precisely $\binom{t_k}{w}$ if $u = 2w$ for some $0 \leq w \leq t_k$ and zero otherwise. Note that from (19) and the maximality of $H_n(f)$ we have $\#J/\#H_n(f) = 2^{-t_k}$. Hence, the proportion of elements of $g_0H_n(f)$ contained in T_u is $\binom{t_k}{w}2^{-t_k}$. The lemma now follows from the fact that T is a union of cosets of $H_n(f)$. \square

Note that Lemma 5.2 remains valid if the t_i are allowed to be 0. Indeed it is easy to see directly that $X_m = 0$ implies $X_n = 0$ for all $n > m$, and in the case $t_k = 0$, the lemma gives $P(X_n = 0) = 1$.

We give two consequences of Lemma 5.2. The first requires the Markov property, which a stochastic process X_1, X_2, \dots satisfies if

$$\mathbf{P}(X_n = u \mid X_{m_1} = t_1, \dots, X_{m_k} = t_k) = \mathbf{P}(X_n = u \mid X_{m_k} = t_k) \tag{21}$$

for n , any $m_1 < \dots < m_k < n$ and any u, t_i . Such a stochastic process is called a *Markov chain*. Lemma 5.2 shows that, for n with $H_n(f)$ maximal, $GP(f)$ obeys a restricted version of the Markov property at stage n (since $m_k = n - 1$ is required). However, if $H_n(f)$ is maximal for all n , it is a straightforward exercise to show that $GP(f)$ is a Markov chain.

The second consequence of Lemma 5.2 is that when H_n is maximal and for any $m < n$ and $1 \leq w \leq 2^{m-1}$, we have

$$\mathbf{P}(X_n = 2w \mid X_m = 2w, \dots, X_{n-1} = 2w) = \binom{2w}{w} \frac{1}{4^w}. \tag{22}$$

We now give an upper bound for the right-hand side of (22).

LEMMA 5.3. *Suppose that $H_n(f)$ is maximal. Then for any $m < n$ and $u > 0$ we have*

$$\mathbf{P}(X_n = u \mid X_m = u, \dots, X_{n-1} = u) \leq \frac{1}{2}.$$

Proof. First note that if u is not of the form $2w$ for some $1 \leq w \leq 2^{m-1}$, then $P(X_n = u) = 0$ from Lemma 5.2 and we are done. Thus, we assume that u is of this form. From (22) we need only show that

$$c_w \stackrel{\text{def}}{=} \binom{2w}{w} \frac{1}{4^w} \leq \frac{1}{2}$$

for all $w \geq 1$. Note that

$$\frac{c_{w+1}}{c_w} = \frac{1}{4} \frac{(2w+2)(2w+1)}{(w+1)^2} = \frac{4w^2 + 6w + 2}{4w^2 + 8w + 4}.$$

The right-hand side of this equation is less than 1 for $w \geq 1$. Since $c_1 = \frac{1}{2}$, the lemma follows. \square

Proof of Theorem 1.3. By Theorem 1.2, $GP(f)$ is a martingale, and thus is eventually constant with probability 1 (see Corollary 4.13). Therefore, it remains only to show that for any $m \geq 0$ and $u > 0$,

$$\mathbf{P}\left(\bigcap_{i=m}^{\infty} X_i = u\right) = 0.$$

Clearly

$$\mathbf{P}\left(\bigcap_{i=m}^{\infty} X_i = u\right) \leq \lim_{j \rightarrow \infty} \mathbf{P}\left(\bigcap_{i=m}^j X_i = u\right),$$

and note that the sequence on the right-hand side is decreasing. Let $C_i = \{X_i = u\}$, and suppose $\mathbf{P}(C_{j-1} \cap \dots \cap C_m) \neq 0$ (otherwise we are done). We have

$$\mathbf{P}\left(\bigcap_{i=m}^j C_i\right) = \mathbf{P}(C_m)\mathbf{P}(C_{m+1} \mid C_m) \cdots \mathbf{P}(C_j \mid C_m \cap \dots \cap C_{j-1}). \tag{23}$$

By Lemma 5.3, if $H_n(f)$ is maximal, then

$$\mathbf{P}(C_n \mid C_m \cap \dots \cap C_{n-1}) \leq \frac{1}{2}.$$

Let $S = \{n \in \mathbb{N} : H_n(f) \text{ maximal}\}$. Then (23) yields

$$\mathbf{P}\left(\bigcap_{i=m}^j X_i = u\right) \leq \left(\frac{1}{2}\right)^{\#\{S \cap \{m, \dots, j\}\}}.$$

The infinitude of S now gives

$$\lim_{j \rightarrow \infty} \mathbf{P}\left(\bigcap_{i=m}^j X_i = u\right) = 0. \tag{24} \quad \square$$

Note that it follows from Theorem 1.3 that $\lim_{m \rightarrow \infty} \mathbf{P}(\{X_m > 0\}) = 0$.

We close this section with an examination of $GP(f)$ under the assumption that $H_n(f)$ is maximal for all n . This situation arises rather frequently, for example when $L = \mathbb{Q}$ and $f = x^2 + a$ for many values of a (see [Sto92]). It appears likely that $H_n(f)$ is maximal for all n also in the case that concerns us, namely $L = k(t)$, $\text{char } k \neq 2$, and $f = x^2 + t$; see Conjecture 6.7. The following definition is adapted from [Har63]. Statements about conditional probabilities apply only when the conditional probabilities are well-defined, and the sum of zero random variables is taken to be zero with probability 1.

DEFINITION 5.4. A Markov chain X_1, X_2, \dots is *time-homogeneous* if $\mathbf{P}(X_n = u \mid X_{n-1} = t)$ depends only on u and t . By a *branching process* we mean a time-homogenous Markov chain X_1, X_2, \dots taking nonnegative integer values such that the random variable $(X_n \mid X_{n-1} = t)$ has the same distribution as the sum of t independent copies of X_1 .

PROPOSITION 5.5. *Suppose that $H_n(f)$ is maximal for all n . Then $GP(f)$ is a branching process with $\mathbf{P}(X_1 = 0) = \frac{1}{2}$ and $\mathbf{P}(X_1 = 2) = \frac{1}{2}$.*

Proof. This is clear from Lemma 5.2 and the discussion of the Markov property immediately following. □

It is interesting to note that Odoni observes in [Odo85a, p. 398] that branching processes share many properties with iterated wreath products. This observation is a forerunner of Proposition 5.5, since it follows from [Odo88, Lemma 1.1] that $H_n(f)$ maximal for all n implies $G_n(f)$ is the n -fold iterated wreath product of $\mathbb{Z}/2\mathbb{Z}$.

Branching processes are very well-understood; see [Hai02, §7.1] for a readable introduction and [Har63] for a detailed account. Here we merely state some results of interest in our case.

PROPOSITION 5.6. *Let X_1, X_2, \dots be the branching process of Proposition 5.5. Let $a_n = \mathbf{P}(X_n = 0)$ and $b_n = \mathbf{P}(X_n > 0) = 1 - a_n$. Then:*

- (i) a_n is given by the evaluation at $z = 0$ of the n th iterate of $\frac{1}{2} + \frac{1}{2}z^2$;
- (ii) as $n \rightarrow \infty$, we have

$$b_n = \frac{2}{n} \left\{ 1 - \frac{\log n}{n} - \frac{\alpha}{n} + O((\log n)^2/n^2) \right\}$$

for some constant α . In particular, $b_n \downarrow 0$.

Proof. For part (i), see [Hai02, Theorem 7.2]. A proof of a much more general theorem than part (ii) can be found in [Har63, p. 21]. For a simpler, direct proof of part (ii) see [Odo85b, p. 5]. □

6. The Galois groups of iterates of $x^2 + t$

In this section we use the same notation as in §5, only with the following specializations: let k be a field with $\text{char } k \neq 2$, let t be transcendental over k , let $L = k(t)$, and let $f(x) = x^2 + t$. We also put $A = k[t]$. As f is fixed throughout this section, we write G_n , H_n , and L_n in place of $G_n(f)$, $H_n(f)$ and $L(f^{on})$, respectively. Our goal is an in-depth examination of H_n , along the lines of that found for the characteristic 0 case in [Odo88, Sto92]. At the end of the section we apply our results to give a proof of Theorem 1.4.

Let $\{p_n : n = 1, 2, 3\}$ be the adjusted forward orbit of the critical point 0, i.e. $p_1 = -t$ and $p_n = f^{on}(0)$ for $n \geq 2$. Note that p_n is a square in L_n for all n ; this is clear for $n = 1$, and follows for $n \geq 2$ because p_n is the product of the roots of f^{on} , which occur in an even number of \pm pairs. We define a related sequence Φ_n :

$$\Phi_n = \prod_{d|n} (p_d)^{\mu(n/d)} \in L. \tag{24}$$

We show that p_n and Φ_n have much to do with the maximality of H_n . First we establish some divisibility properties of these sequences.

LEMMA 6.1. *Let $q \in A$ be irreducible, let v_q be the valuation corresponding to q , and suppose that $v_q(p_n) = e \geq 1$. Then for all $m \geq 1$, we have $v_q(p_{mn}) = e$.*

Proof. Induction on m . The case $m = 1$ is trivial. Suppose inductively that $v_q(p_{(m-1)n}) = e$. Note that $p_{mn} = f^{\circ(m-1)n}(p_n)$, and also $f^{\circ(m-1)n}$ is a polynomial in x^2 . Thus, we can write

$$f^{\circ(m-1)n}(x) = x^2g(x) + f^{\circ(m-1)n}(0) = x^2g(x) + p_{(m-1)n},$$

for some $g \in L[x]$. Hence, putting $x = p_n$ we have

$$p_{mn} = p_n^2g(p_n) + p_{(m-1)n}.$$

Now $v_q[(p_n)^2(g(p_n))] \geq 2e$, and by our inductive hypothesis $v_q(p_{(m-1)n}) = e$. Since $e \geq 1$, the first summand vanishes to higher order at q than the second, so we conclude $v_q(p_{mn}) = e$. □

PROPOSITION 6.2. *For each n , Φ_n is a polynomial, and the Φ_n are pairwise relatively prime.*

Proof. Let $q \in A$ be irreducible, and let $m = \min\{n \geq 1 : q \mid p_n\}$. By Lemma 6.1, we have $v_q(p_n) = e$ if $m \mid n$ and one easily sees $v_q(p_n) = 0$ otherwise. Thus,

$$v_q(\Phi_n) = \sum_{d \mid n} v_q(p_d) \cdot \mu(n/d) = e \cdot \sum_{dm \mid n} \mu(n/dm),$$

and this last expression is e if $n = m$ and 0 otherwise. Hence, Φ_n is a polynomial and, moreover, $v_q(\Phi_n) > 0$ for only one n . Thus, the Φ_n are pairwise relatively prime. □

PROPOSITION 6.3. *For each n , $\text{Disc } f^{\circ n} = a^2p_n$ for some $a \in A$.*

Proof. See the proof of Lemma 4.10, first paragraph. □

Our main result in this section has to do with the maximality of H_n (see Proposition–Definition 5.1). We first give two preparatory results.

LEMMA 6.4. *Let $n \geq 1$. Then H_n is maximal if and only if p_n is not a square in L_{n-1} .*

Proof. Identical to the argument in [Sto92, Lemma 1.6]. □

THEOREM 6.5. *Let $n \geq 1$. Then H_n is maximal if and only if Φ_n is not a square in L .*

Proof. The case $n = 1$ is clear, so we take $n \geq 2$. Suppose that Φ_n is a square in L , and note that it follows from (24) that

$$p_n = \prod_{d \mid n} \Phi_d. \tag{25}$$

Now p_m is a square in L_m for all $m \geq 1$, and a quick induction allows one to deduce that Φ_m is also a square in L_m for all $m \geq 1$. Thus, from (25) we have that p_n is a square in L_{n-1} , whence by Lemma 6.4 H_n is not maximal.

Now suppose that Φ_n is not a square in L . We claim the squarefree part of Φ_n has positive degree. To see this, note that p_n is monic and of even degree for $n \geq 2$ while p_1 has odd degree and leading coefficient -1 . Thus, from (24) we have that Φ_n is monic and of even degree if $\mu(n) = 0$ and has leading coefficient -1 and of odd degree otherwise. If $\mu(n) = 0$, then Φ_n is a monic non-square in L and thus its squarefree part has positive degree. If $\mu(n) \neq 0$, then Φ_n has odd degree and thus its squarefree part has positive degree as well.

Now let $q \in A$ be an irreducible dividing the squarefree part of Φ_n . Since Φ_n is relatively prime to Φ_m for $m < n$, q cannot divide $\text{Disc } f^{\circ n-1}$ by Proposition 6.3 and (25). Now a prime $\mathfrak{p} \subset A$

not dividing $(\text{Disc } f^{on-1})$ cannot be ramified in $L(\alpha)$, where α is a root of f^{on-1} . From [Nar04, Corollary 2, p. 157] it follows that \mathfrak{p} is unramified in L_{n-1} . Therefore, (q) is unramified in L_{n-1} . Thus, the squarefree part of p_n has an irreducible factor unramified in L_{n-1} , whence p_n cannot be a square in L_{n-1} . By Lemma 6.4, H_n is therefore maximal. \square

Remark. Recall $L = k(t)$, and let F be the prime subfield of k . Since $\Phi_n \in F[t]$, the roots of Φ_n in \bar{k} must lie in \bar{F} , whence all factors of Φ_n in $k[t]$ must have coefficients in \bar{F} . Thus, if Φ_n is a square in $k[t]$, then, in fact, it is a square in $\bar{F}[t]$, and since F is perfect it follows that Φ_n must be a square in $F[t]$. To show the last assertion, note that if Φ_n is not a square in $F[t]$, then the squarefree part of Φ_n has positive degree (using the same argument as in the proof of Theorem 6.5), and thus is divisible by an irreducible polynomial in $F[t]$. Since F is perfect, this irreducible polynomial is separable, and thus cannot become a square in $\bar{F}[t]$, showing that Φ_n is not a square in $\bar{F}[t]$. We have now shown that Φ_n is a square in L if and only if it is a square in $F[t]$, so that only the characteristic of L is relevant in this matter. In particular, if Φ_n is not a square in L , then H_n remains maximal if we replace L by $\bar{k}(t)$. Therefore, if Φ_n is not a square for all $n \leq m$, then $[L_n : k(t)] = [\bar{k}L_n : \bar{k}(t)]$, whence L_n/L is geometric.

COROLLARY 6.6. *If n is squarefree, then H_n is maximal.*

Proof. From (24), n squarefree implies $\deg \Phi_n$ odd. The corollary now follows from Theorem 6.5. \square

Proof of Theorem 1.4. Let $k = \mathbb{F}_p$ with $p \neq 2$, $L = k(t)$, and $f(x) = x^2 + t$. By Proposition 3.2 it is enough to show $\lim_{n \rightarrow \infty} D(\mathcal{I}_n) = 0$. Let X_1, X_2, \dots be the Galois process of f , and note that by Theorem 1.5 and (13) we have $P(X_n > 0) = D(\mathcal{I}_n)$. From Lemma 6.1 and the fact that $v_t(p_1) = 1$, we have $v_t(p_n) = 1$ for all n . Hence, the adjusted forward critical orbit of f contains no squares. Finally, by Corollary 6.6 we have $H_n(f)$ maximal for infinitely many n . Theorem 1.3 then applies to show $\lim_{n \rightarrow \infty} \mathbf{P}(X_n = 0) = 1$, which implies $\lim_{n \rightarrow \infty} \mathbf{P}(X_n > 0) = 0$. \square

We close with a conjecture.

CONJECTURE 6.7. Let $\text{char } k \neq 2$, $L = k(t)$, and $f(x) = x^2 + t$. Then H_n is maximal for all $n \geq 1$.

Thanks to Propositions 5.5 and 5.6, Conjecture 6.7 would give a simpler proof of Theorem 1.4. It would also give near-complete information about $GP(f)$ and provide very precise estimates for $D(\mathcal{I}_n)$ for large n (see part (ii) of Proposition 5.6). Moreover, if Conjecture 6.7 is true, then it follows from the remark just before Corollary 6.6 that L_n/L is geometric for all n . Thus, the strong form of the Tchebotarev density theorem for function fields [Ros02, Theorem 9.13B] applies to show that $\delta(\mathcal{H}(\bar{\mathbb{F}}_p)) = 0$ for $p \neq 2$ (see the discussion in the remark at the end of § 3).

One approach to proving Conjecture 6.7 is to use Theorem 6.5 and show that Φ_n is not a square in L for all $n \geq 1$. In the characteristic zero case, one can show that Φ_n is separable for all n by reducing mod 2. In the case $\text{char } k \equiv 3 \pmod{4}$, one can show that Φ_n is not a square for all n by adapting the argument in [Sto92, §§ 2 and 3] (see [Jon05, § 3.5] for details). The remaining cases are still open, although calculations for several small primes $p \equiv 1 \pmod{4}$ have shown that Φ_n is not a square for $n \leq 2000$.

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