On the Envelope of the Directrices of a System of Similar Conics through three Points.

By R. E. Allardice, M.A.

(*Read and Received 13th January 1911*).

The problem was proposed by Steiner of finding certain loci and envelopes connected with a system of similar conics through three points. In previous papers I have found the locus of centres and the envelope of asymptotes of such a system of conics,* also the envelope of axes,† and the locus of foci.‡ I now propose to discuss the envelope of the directrices.

The equation of the conic may be written

$$\lambda\eta\zeta + \mu\zeta\xi + \nu\xi\eta = 0,$$

with the condition

$$\lambda^2 + \mu^2 + \nu^2 - 2\lambda\mu\cos C - 2\mu\nu\cos A - 2\nu\lambda\cos B$$
$$= s^2(\lambda\cos A + \mu\cos B + \nu\cos C)^2,$$

where s is the secant of the angle between the asymptotes (see the paper, *On the envelope of the axes, etc.*).

Transforming this equation to a triangle of reference that has for its vertices the circumcentre of the original triangle, and the

---

* *On some curves connected with a system of similar conics*, Annals of Mathematics, second series, vol. 3, No. 4 (1902).

† *On the envelope of the axes of a system of conics passing through three fixed points*, Transactions of the American Mathematical Society, vol. 4, No. 1 (1903).

‡ *On the locus of the foci of a system of similar conics through three points*, Proceedings of the Edinburgh Mathematical Society, vol. XXVII., Session 1908–09.

circular points at infinity $(-e^{Bi}, -e^{-Ai}, 1)$, $(-e^{-Bi}, -e^{Ai}, 1)$, by means of the equations

$$\xi = -e^{-Bi}x - e^{Bi}y + 2\cos A \,.\, z$$
$$\eta = -e^{Ai}x - e^{-Ai}y + 2\cos B \,.\, z$$
$$\zeta = x + y + 2\cos C \,.\, z$$

or, conversely,

$$x = -\sin A \,.\, e^{Ci}\xi - \sin B \,.\, e^{-Ci}\eta + \sin C \,.\, e^{-(A-B)i} \,.\, \zeta$$
$$y = -\sin A \,.\, e^{-Ci}\xi - \sin B \,.\, e^{Ci}\eta + \sin C \,.\, e^{(A-B)i} \,.\, \zeta$$
$$z = \sin A \,.\, \xi + \sin B \,.\, \eta + \sin C \,.\, \zeta,$$

we obtain the equation in the form

$$l[ax^2 + aqxz - yz - pz^2] + m[a_1y^2 + a_1pyz - xz - qz^2] + n[z^2 - xy] = 0,$$

where
$$a = e^{(A-Bi)} \qquad p = 2\cos C - e^{(A-B)i}$$
$$a_1 = e^{-(A-B)i} \qquad q = 2\cos C - e^{-(A-B)i}$$
$$l = -\lambda e^{Bi} - \mu e^{-Ai} + \nu$$
$$m = -\lambda e^{-Bi} - \mu e^{Ai} + \nu$$
$$n = 2\lambda\cos A + 2\mu\cos B + 2\nu\cos C.$$

Thus the condition for similarity becomes

$$4lm = s^2n^2,$$

and we may put $n = 1$.

Let (x_1, y_1, z_1) be the coordinates of a focus, and (u, v, w) the tangential coordinates of the corresponding directrix; then

$$l(2ax_1 + aqz_1) - mz_1 - y_1 = \rho u \qquad (1)$$
$$-lz_1 + m(2a_1y_1 + a_1pz_1) - x_1 = \rho v \qquad (2)$$
$$l(aqx_1 - y_1 - 2pz_1) + m(a_1py_1 - x_1 - 2qz_1) + 2z_1 = \rho w. \qquad (3)$$

Now the focus lies on the axis, which is the join of the intersections of the tangents from the circular points at infinity, viz. $(x', 0, 0)$, $(0, y'', 0)$.

The tangents from $(x', 0, 0)$ are

$$[x'U_x]^2 = 4UU'$$

or,
$$x'^2U_x^2 = 4lax'^2U$$

i.e.
$$U_x^2 = 4laU.$$

Similarly the tangents from $(0, y'', 0)$ are

$$U_y^2 = 4ma_1U.$$

Hence the axes are given by

$$ma_1 U_x^2 = la U_y^2$$

or, $$k^2 U_x^2 = l^2 a^2 U_y^2, \quad \text{where } s = 2k.$$

Thus we have the condition $ku = lav$, which may also be written $kv = ma_1 u$.

But the coordinates of the foci satisfy the equation

$$4la\mathrm{U} = \mathrm{U}_x^2,$$

whence, $$2la(x_1 u + y_1 v + z_1 w) = \rho u^2. \qquad (4)$$

The equations (1), (2), (3), (4) may be written in the form

$$2al \,.\, x_1 - y_1 + (laq - m) \,.\, z_1 = \rho u,$$
$$-x_1 + 2a_1 m \,.\, y_1 + (ma_1 p - l) \,.\, z_1 = \rho v,$$
$$(laq - m) \,.\, x_1 + (ma_1 p - l) \,.\, y_1 + (-2pl - 2qm + 2) \,.\, z_1 = \rho w,$$
$$2alu \,.\, x_1 + 2alv \,.\, y_1 + 2alw \,.\, z_1 = \rho u^2;$$

whence $$\begin{vmatrix} 2al & -1 & laq - m & u \\ -1 & 2a_1 m & ma_1 p - l & v \\ laq - m & ma_1 p - l & 2 - 2pl - 2qm & w \\ 2alu & 2alv & 2alw & u^2 \end{vmatrix} = 0,$$

or, putting $lv = ka_1 u$, $mu = kav$

$$\begin{vmatrix} 2ku^2 & -uv & qku^2 - akv^2 & u^2 v \\ -uv & 2kv^2 & pkv^2 - a_1 ku^2 & uv^2 \\ qku^2 - akv^2 & pkv^2 - a_1 ku^2 & 2uv - 2a_1 pku^2 - 2aqkv^2 & uvw \\ 2ku^2 & 2kuv & 2kuw & u^2 v \end{vmatrix} = 0.$$

The factor $u^2 v$ divides out, and the equation may easily be reduced to the form

$$k^2(a_1 u^2 - pv^2 + 2vw)(av^2 - qu^2 + 2uw)$$
$$- (2k+1)uv(a_1 pku^2 + aqkv^2 + kw^2 - uv) = 0.$$

Reduction of the Equation.

Let now (x, y, z) denote tangential coordinates with respect to the original triangle of reference, then we have

$$u = -e^{-\mathrm{B}i} \,.\, x - e^{\mathrm{A}i} \,.\, y + z$$
$$v = -e^{\mathrm{B}i} \,.\, x - e^{-\mathrm{A}i} \,.\, y + z$$
$$w = 2(\cos\mathrm{A} \,.\, x + \cos\mathrm{B} \,.\, y + \cos\mathrm{C} \,.\, z).$$

By direct calculation we may show that

$$a_1u^2 - pv^2 + 2vw = 4[-e^{Bi}\sin B\sin Cx^2 - e^{-Ai}\sin C\sin Ay^2 + \sin A\sin Bz^2]$$

$$av^2 - qu^2 + 2uw = 4[-e^{-Bi}\sin B\sin Cx^2 - e^{Ai}\sin C\sin Ay^2 + \sin A\sin Bz^2]$$

$$uv = x^2 + y^2 + z^2 - 2xy\cos C - 2yz\cos A - 2zx\cos B.$$

$$a_1pu^2 + aqv^2 + w^2 = (8\sin B\sin C\cos A - 2)x^2 + (8\sin C\sin A\cos B - 2)y^2$$
$$+ (8\sin A\sin B\cos C - 2)z^2 + 4\cos Ayz + 4\cos Bzx + 4\cos Cxy.$$

Let $P = \Sigma\sin^2 B\sin^2 Cx^4 - 2\sin A\sin B\sin C\,.\,\Sigma\sin A\cos Ay^2z^2$

$Q = \Sigma\sin B\sin C\cos A\,.\,x^2$

$I = \Sigma(x^2 - 2yz\cos A)$,

then the equation is

$$16k^2P - (2k+1)I\,.\,[2k(4Q - I) - I] = 0,$$

or, putting $s = 2k$

$$4s^2P - (s+1)I\,.\,[4sQ - (s+1)I] = 0.$$

Now
$$P - Q^2 = \sin^2 A\sin^2 B\sin^2 C\,.\,\Sigma(x^4 - 2y^2z^2)$$
$$= \sigma^2 S \quad \text{(say)},$$

$$\therefore\ 4s^2(Q^2 + \sigma^2 S) - 4s(s+1)QI + (s+1)^2I^2 = 0$$

or
$$[2sQ - (s+1)I]^2 + 4s^2\sigma^2 S = 0.$$

Thus the envelope is a curve of the fourth class. From the form of the equation it is obvious that the four factors of S give the equations of four double points on the curve; these are $x \pm y \pm z = 0$, the equations of the in-centre and the three ex-centres.

It may be shown from the equation that the curve touches one of the bisectors of the angle between the two lines into which the conic degenerates when the vertex lies on one of the sides of the triangle of reference.

It may also be shown (most easily from the original form of the equation), that there is only one double tangent, the straight line at infinity, the circular points being the points of contact. Hence the deficiency of the curve is 2; it is of the tenth degree and has 16 double points and 18 cusps.