ELLIPTIC INTEGRALS IN TERMS OF LEGENDRE POLYNOMIALS

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1. Consider the elliptic integral of the first kind

$$u = \operatorname{sn}^{-1} x = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

or, alternatively,

$$u = F(\phi, k) = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

where $x = \sin\phi$. It is customary in the theory of elliptic integrals to let $k = \sin\alpha$, $k' = \cos\alpha$, so that $k'^2 + k^2 = 1$. For convenience we shall also introduce the parameter

$$\lambda = k'^2 - k^2.$$

The substitution $x = \sin\phi$ is equivalent to

$$x = \frac{2v}{1+v^2}$$

where $v = \tan\frac{\phi}{2}$. By means of this substitution we find that

$$u = 2\int_0^v \frac{dv}{\sqrt{1+2\lambda v^2+v^4}}. \qquad \text{(1)}$$

We have

$$(1+2\lambda v^2+v^4)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(-\lambda)v^{2n} = \sum_{n=0}^{\infty} (-1)^n P_n(\lambda) v^{2n},$$

where $P_n(\lambda)$ denotes the Legendre polynomial of order n. This series converges uniformly with respect to v and λ when $|\lambda| \leqslant a$, $|v| \leqslant b$, a and b being positive constants such that $2ab^2 + b^4 \leqslant 1 - \delta\ (\delta > 0)$.

Hence it follows that

$$u = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda) v^{2n+1}$$

or

$$u = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda) \tan^{2n+1}\frac{\phi}{2}. \qquad \text{(2)}$$

For $\phi = \frac{\pi}{2}$ we find formally

$$K = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda). \qquad \text{(3)}$$

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a formula which we shall establish rigorously in (§ 2). If, in addition, $\lambda = 1$, the formula above gives

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} \cdots .$$

Thus (3) is an extension of the Gregory-Leibnitz series.

2. *Series expansions for K, K', E and E'.*

It is well known that the complete elliptic integral of the first kind K can be expressed in terms of Gauss's hypergeometric function as follows

$$K = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2). \qquad (4)$$

On the other hand, the Legendre function of degree n of the first kind is defined by means of the equation

$$P_n(\lambda) = F\left(-n, n+1, 1; \frac{1-\lambda}{2}\right)$$

which, for $n = -\frac{1}{2}$, gives

$$P_{-1/2}(\lambda) = F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{1-\lambda}{2}\right) \qquad (5)$$

Since $\dfrac{1-\lambda}{2} = k^2$, from (4) and (5) we obtain

$$K = \frac{\pi}{2} P_{-1/2}(\lambda). \qquad (6)$$

Similarly,

$$K' = \frac{\pi}{2} P_{-1/2}(-\lambda). \qquad (7)$$

Also, since

$$E = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2)$$

and

$$E' = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1; k'^2)$$

it can be shown that

$$E = \frac{\pi}{4}[P_{-1/2}(\lambda) + P_{1/2}(\lambda)] \qquad (8)$$

and that

$$E' = \frac{\pi}{4}[P_{-1/2}(-\lambda) + P_{1/2}(-\lambda)]. \qquad (9)$$

Now, if we expand the Legendre function $P_{-1/2}(\lambda)$ in a series of Legendre polynomials we obtain

$$P_{-1/2}(\lambda) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda). \qquad (10)$$

Similarly, by expanding $P_{1/2}(\lambda)$ in a series of Legendre polynomials we obtain

$$P_{1/2}(\lambda) = \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2n+1}{(2n-1)(2n+3)} P_n(\lambda). \qquad (11)$$

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Whence,

$$K = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda), \qquad \text{(12)}$$

$$K' = 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} P_n(\lambda), \qquad \text{(13)}$$

$$E = 4 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)} P_n(\lambda), \qquad \text{(14)}$$

$$E' = -4 \sum_{n=0}^{\infty} \frac{1}{(2n-1)(2n+1)(2n+3)} P_n(\lambda). \qquad \text{(15)}$$

For $\lambda = 1$ it follows from (14) that

$$\frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)}.$$

3. *Series expansions for* $\operatorname{cn}^{-1} x$ *and for* $\wp^{-1}(x)$.

By letting $x = (1-v^2)/(1+v^2)$ it can be shown that

$$\operatorname{cn}^{-1} x = \int_x^1 \frac{dx}{\sqrt{(1-x^2)(k'^2 + k^2 x^2)}} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda) \left(\frac{1-x}{1+x} \right)^{n+\frac{1}{2}}.$$

Also, by letting $x = e_1 + \gamma^2 u^{-2}$, where $3e_1 = 2\lambda\gamma^2$, $g_2 = 12 e_1{}^2 - 4\gamma^4$, it is easy to verify that

$$\wp^{-1}(x) = \int_x^{\infty} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \gamma^{2n} P_n(\lambda)(x - e_1)^{-(n+\frac{1}{2})}.$$

4. *Expansion for the elliptic integral of the second kind.*

Since

$$E(\phi, k) = \frac{k^2 \sin\phi \cos\phi}{\sqrt{1 - k^2 \sin^2\phi}} + k'^2 \left[k \frac{\partial F}{\partial k} + F \right]$$

we obtain, after introducing the parameter λ, and making use of (2):

$$E(\phi, \lambda) = \frac{(1-\lambda)\tan\phi}{\sqrt{4 + 2(1+\lambda)\tan^2\phi}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} [2(\lambda^2 - 1) P_n'(\lambda) + (\lambda + 1) P_n(\lambda)] \tan^{2n+1} \frac{\phi}{2},$$

or, alternatively,

$$E(\phi, \lambda) = \frac{(1-\lambda)\tan\phi}{\sqrt{4 + 2(1+\lambda)\tan^2\phi}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} [((2n+1)\lambda + 1) P_n(\lambda) - 2n P_{n-1}(\lambda)] \tan^{2n+1} \frac{\phi}{2}$$

since

$$(\lambda^2 - 1) P_n'(\lambda) = n\lambda P_n(\lambda) - n P_{n-1}(\lambda).$$

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