Canad. Math. Bull. Vol. **29** (1), 1986

ON THE CARDINALITY OF SUBRINGS OF A COMMUTATIVE RING

BY
ROBERT GILMER AND WILLIAM HEINZER

ABSTRACT. If R is an uncountable commutative ring, it is shown that there exists a proper subring of R having the same cardinality as R. It is also shown that if $|R| = \omega$ is an uncountable regular cardinal, and if R_1 is a subring of R containing an identity of R and such that $|R_1| < \omega$, then there exists a proper R_1-subalgebra S of R such that $|S| = |R|$.

Let ω be an uncountable cardinal. In the terminology of universal algebra, an algebra A is said to be a *Jónsson ω-algebra* if $|A| = \omega$ while $|B| < \omega$ for each proper subalgebra B of A. Thus, a Jónsson ω-group is a group of cardinality ω such that each proper subgroup has smaller cardinality. Shelah in [16] answered an old question of Kurosh by proving existence of a Jónsson ω_1-group. On the other hand, Scott in ([15], Th. 9) proved that *abelian* Jónsson ω-groups do not exist. In this paper we investigate the question of existence of a commutative Jónsson ω-ring R. Because unitary rings are of primary importance in commutative ring theory, we are immediately faced with the question of whether R should be required to contain an identity element e. If so, should we restrict consideration to unitary subrings of R or, more restrictively, to subrings containing e?. Theorem 1.2 shows that for R unitary, the answer to the existence question is "no" and is independent of unitary restrictions placed on subrings. If we ignore existence of an identity element both in R and in its subrings, then Theorem 1.3 shows that again no commutative Jónsson ω-ring exists.

In the second part of the paper we generalize the considerations of Section 1 as follows. Suppose R is an uncountable commutative ring with identity element e and R_1 is a proper subring of R containing e. In view of Theorem 1.2 we ask whether there exists a proper R_1-subalgebra S of R such that $|S| = |R|$; note that Theorem 1.2 provides an affirmative answer in the case where $R_1 = Ze$ is the prime subring of R. We show in Theorem 2.4 that the answer to the question is "yes" if $|R|$ is a regular cardinal, and we note in Remark 2.5 that the answer is also "yes" if R_1 is a Noetherian ring. Using Theorem 2.4, we observe in Theorem 2.6 that if each proper R_1-subalgebra of R is countable, then R is countable.

Received by the editors October 2, 1984.
Research of the first author supported by NSF Grant MCS 8122095.
Research of the second author supported by NSF Grant DMS 8320558.
AMS Subject Classification: 13B02, 13C05, 13E05.

1. **Commutative Jónsson ω-rings do not exist**. We begin with a lemma in the domain case.

LEMMA 1.1. *Let E be an uncountable integral domain with identity element e and let D be a subring of E containing e such that $|D| < |E|$. There exists a proper subring E_1 of E containing D such that $|E_1| = |E|$.*

PROOF. Let K and F be the quotient fields of E and D, respectively. Choose a transcendence basis $B \subseteq E$ for K/F. An algebraic extension of a countable field is countable; hence $F(B)$ is uncountable, and in fact, $|F(B)| = |B| = |K| = |E|$ ([12], p. 143). Thus, if $b \in B$, then $D[B \backslash \{b\}]$ serves as a suitable choice for E_1.

THEOREM 1.2. *Let R be an uncountable commutative ring with identity element e. There exists a proper subring S of R containing e such that $|S| = |R|$.*

PROOF. Let $|R| = \omega$ and denote by Ze the prime subring of R. If there exists a prime ideal P of R such that $|R/P| = \omega$, then applying Lemma 1.1 to the domains R/P and $Ze/(Ze \cap P)$, we obtain a proper subring T of R/P containing $Ze/(Ze \cap P)$ such that $|T| = \omega$. If ϕ is the canonical homomorphism of R onto R/P, then we can take $S = \phi^{-1}(T)$. Thus, we assume that $|R/P| < \omega$ and $|P| = \omega$ for each prime ideal P of R. If I is any ideal of R such that $|I| = \omega$, then either $Ze + I$ is a suitable choice of S, or else $R = Ze + I$. We therefore assume that $R = Ze + I$ for each ideal I of R with $|I| = \omega$; in particular, $R = Ze + P$ for each prime ideal P of R. Note that if $|I| = |J| = \omega$ and if $I < J$, then $I \cap Ze < J \cap Ze$ because $J = J \cap R = J \cap (Ze + I) = I + (J \cap Ze) \nsubseteq I$.

The rest of the proof amounts to showing that the assumptions of the previous paragraph lead to a contradiction, thereby establishing the theorem. First, suppose that $J \cap Ze = (0)$ for some ideal J of R with $|J| = \omega$. Since $I \cap Ze < J \cap Ze$ is impossible for $I < J$, we conclude that $|I| < \omega$ for each ideal I of R properly contained in J—that is, J is Jónsson ω-ideal of R. By Corollary 2.6 of [9], it follows that either $J^2 = (0)$ or J is a field. If $J^2 = (0)$, then the ideals of $R = Ze + J$ contained in J coincide with the subgroups of J as an additive group, and this contradicts non-existence of abelian Jónsson ω-groups. We conclude that $J^2 \neq (0)$. If J is a field, then $R = J \oplus A$ for some maximal ideal A of R with $|R/A| = \omega$. No such maximal ideal exists by assumption. Therefore $J \cap Ze \neq (0)$ if $|J| = \omega$. Consider $(J \cap Ze)R$, the ideal of R generated by $J \cap Ze$. Since this ideal has the same intersection with Ze that J has, it follows that either $|(J \cap Ze)R| < \omega$, or else $(J \cap Ze)R = J$. If $|(J \cap Ze)R| < \omega$, then $|R/(J \cap Ze)R| = \omega$, the assumptions of the first paragraph carry over to the ring $R/(J \cap Ze)R$, and yet $J/(J \cap Ze)R$ is an ideal of cardinality ω in this ring that meets its prime subring $Ze/(J \cap Ze)$ in the zero ideal. This is impossible; hence $J = (J \cap Ze)R$ and $R/J' \simeq Ze/(J \cap Ze)$ for each J with $|J| = \omega$. Because Ze is a principal ideal ring, it follows that each prime ideal of R is principal and has finite associated residue class ring. This implies that to within isomorphism, R is a finite direct sum of rings of the form R/M^k, where M is maximal in R. If k is chosen so that $M^k < M^{k-1}$, then $|R/M^k| = |R/M|^k$ and hence R is finite. This contradiction completes the proof of Theorem 1.2.

The analogues of Lemma 1.1 and Theorem 1.2 fail in the case of countably infinite rings — that is, a unitary ring of cardinality ω_0 may not have a proper infinite unitary subring. In the characteristic-zero case, $E = Z$ is obviously the unique ring with this property and in the case of nonzero characteristic, all counter-examples are of the form $E = \bigcup_{i=1}^{\infty} GF(p^{g^i})$, where g and p are prime [13]. Laffey in [13] also determines in the non-unitary case that an infinite ring R whose proper subrings are finite is the zero ring on $Z(p^\infty)$, the p-quasicyclic group, for some prime p.

We turn to a consideration of the non-unitary version of Theorem 1.2. The main part of the proof of Theorem 1.3 amounts to an extension of Corollary 2.6 of [9] to rings without identity.

THEOREM 1.3. *If R is an uncountable commutative ring of cardinality ω, then there exists a proper subring S of R with $|S| = \omega$.*

PROOF. If a proper ideal of R has cardinality ω, then we're finished. Assume $|I| < \omega$ for each proper ideal I of R. Assume that $x \in R$ is such that $Rx \neq R$ and let $\mathrm{Ann}(x)$ denote the annihilator of x. We have $R/\mathrm{Ann}(x) \simeq Rx$ and $|Rx| < \omega$, so $|\mathrm{Ann}(x)| = \omega$ and $\mathrm{Ann}(x) = R$ — that is, $Rx = (0)$ if $Rx \neq R$. If $Rx = (0)$ for each $x \in R$, then R is the zero ring on its additive subgroup and the conclusion of the theorem follows from Scott's result cited in the introduction. If $Rx = R$ for some $x \in R$, then $x = ex$ for some $e \in R$, and e is an identity element for R. Then $y \in R \backslash \{0\}$ implies $Ry \neq (0)$ so $Ry = R$ and R is a field; in this case, Lemma 1.1 implies the desired conclusion.

2. **Some results in the relative case**. We consider in this section the question of whether Theorem 1.2 extends to a result for commutative rings similar to that given in Lemma 1.1 for integral domains. In this section all rings and all modules are unitary. The general question we consider is the following.

QUESTION 2.1. *If R is an uncountable commutative ring with identity element e and R_1 is a subring of R containing e with $|R_1| < |R|$, does there exist a proper R_1-subalgebra S of R with $|S| = |R|$?*

In giving an affirmative answer to Question 2.1 for $R_1 = Ze$ in Theorem 1.2, we have used two special properties of Ze in the second paragraph of the proof of that result. The first of these is that Jónsson ω-groups do not exist; the second is that prime ideals of Ze are principal. The proof of Theorem 1.2 establishes the following.

REMARK 2.2. Suppose that R_1 and R are as in Question 2.1, and that for each proper R_1-subalgebra S of R, $|S| < |R| = \omega$. Then

(i) For each prime ideal P of R, $|R/P| < \omega$ and $|P| = \omega$.

(ii) If I is an ideal of R with $|I| = \omega$, then $R = R_1 + I$. Hence $R/I \cong R_1/(I \cap R_1)$, so $|R/I| < \omega$. In fact, $|R/I| \leq |R_1|$ for each ideal I of R with $|I| = \omega$.

(iii) If $I < J$ are ideals in R with $|I| = \omega$, then $I \cap R_1 < J \cap R_1$.

(iv) If J is an ideal of R with $|J| = \omega$ and $J \cap R_1 = (0)$, then $J^2 = (0)$ and J is a Jónsson ω-module over R_1.

(v) If the ring R_1 does not admit a Jónsson ω-module, then each ideal I of R with $|I| = \omega$ is extended from R_1 — that is, $I = (I \cap R_1)R$. In particular, $P = (P \cap R_1)R$ for each prime ideal P of R.

In view of Remark 2.2, we see that the answer to Question 2.1 is related to the question of whether there exists a ring R_1 with $|R_1| < \omega$ such that R_1 admits a Jónsson ω-module. Indeed, if such a ring R_1 exists and if M is a Jónsson ω-module, then R_1 and the idealization $R = R_1 \oplus M$ of R_1 and M ([14], p. 2) provide a negative answer to Question 2.1. We do not know, however, if such a ring R_1 exists. Our next result establishes non-existence of such rings for ω a regular cardinal. We recall that an infinite cardinal α is *regular* if $\alpha \neq \Sigma_{i \in I} \alpha_i$ for each nonempty family $\{\alpha_i\}_{i \in I}$ of cardinals with $|I| < \alpha$ and $\alpha_i < \alpha$ for each i ([2], p. 245; [3], p. 504).

PROPOSITION 2.3. *If α is an infinite regular cardinal and R_1 is a commutative ring with $|R_1| < \alpha$, then R_1 does not admit a Jónsson α-module.*

PROOF. Suppose that R_1 admits a Jónsson α-module M. By passing from R_1 to $R_1/\text{Ann}\, M$, we may assume that M is a faithful module, and Proposition 2.5 of [9] implies that $R_1/\text{Ann}\, M = D$ is an integral domain. Since $|D| < \alpha$, M is also a Jónsson α-generated module in the sense of [8]; that is, M is generated as a D-module by a set of cardinality α and has no generating set of cardinality less than α, while each proper D-submodule of M has a generating set of cardinality less than α. We claim that M is a torsion D-module. For suppose that M is not torsion and let N be the torsion submodule of M. Then M/N is a torsion-free Jónsson α-generated D-module. Hence by Theorem 1.4 of [8], M/N is isomorphic to the quotient field K of D. But $|K| = |D| < \alpha$. We conclude that M is a torsion D-module. It follows that

$$M = \cup\{\text{Ann}_M(a) \mid a \in D,\ a \neq 0\}.$$

And by Proposition 2.5 of [9], $M = aM$ so that $\text{Ann}_M(a) < M$ for each nonzero a in D. This contradicts the fact that α is a regular cardinal, and completes the proof of Propositions 2.3.

Using Proposition 2.3, we show that the answer to Question 2.1 is affirmative if $|R| = \omega$ is a regular cardinal.

THEOREM 2.4. *Let R be an uncountable commutative ring with identity e such that $|R| = \omega$ is a regular cardinal. If R_1 is a subring of R containing e such that $|R_1| < \omega$, then there exists a proper R_1-subalgebra S of R such that $|S| = |R|$.*

PROOF. Suppose there does not exist such an S. For each prime ideal P of R, we show that there exists $x \in P$ such that $|xR| = \omega$. By Proposition 2.3 and Remark 2.2v, $P = (P \cap R_1)R$. And by 2.2i, $|P| = \omega$. Since ω is a regular cardinal and $|P \cap R_1| < \omega$, it follows that there exists $x \in P \cap R_1$ such that $|xR| = \omega$. Also, the principal ideals xR of R such that $|xR| = \omega$ are closed under multiplication, for suppose $|xR| =$

$|yR| = \omega$, and consider the exact sequence

$$0 \to xyR \to yR \to yR/xyR \to 0.$$

To show $|xyR| = \omega$, it suffices to show that $|yR/xyR| < \omega$; this holds since $|R/xR| < \omega$ by Remark 2.2ii, and yR/xyR is a homomorphic image of R/xR. Let U be the multiplicative system of principal ideals xR of R such that $|xR| = \omega$. We have shown that U meets each prime ideal of R. It follows that $(0) \in U$. This yields the contradiction that $|(0)| = \omega$, which completes the proof of Theorem 2.4.

REMARK 2.5. In the proof of Theorem 2.4, the fact that the principal ideals xR of R such that $|xR| = |R|$ are closed under multiplication does not use the fact that $|R|$ is a regular cardinal. Therefore, for a given R_1 and R as in Question 2.1, if from the non-existence of a proper R_1-subalgebra S of R with $|S| = |R|$ one can deduce that each prime ideal of R contains a principal ideal xR with $|xR| = |R|$, then for this R_1 and R, such an S must exist. From this it follows that for R_1 Noetherian, the answer to Question 2.1 is affirmative. For if R_1 is Noetherian, then by a result of Enochs included as Theorem 3.1 of Section 3, it follows that R_1 does not admit a Jónsson ω-module. Therefore the nonexistence of a proper R_1-subalgebra S of R with $|S| = |R|$ implies that each prime ideal of R is extended from R_1 and thus is finitely generated. Since a finitely generated ideal of cardinality ω contains a principal ideal of cardinality ω, the result follows.

A question related to Question 2.1, but more in the spirit of the approach taken in [6], is the following. Suppose that R is a commutative ring with identity e and R_1 is a proper subring of R containing e. If α is an infinite cardinal and if each proper R_1-subalgebra of R has cardinality $\leq \alpha$, does it follow that $|R| \leq \alpha$? We can use Theorem 2.4 to provide an affirmative answer to this question by arguing as follows. Suppose $|R| > \alpha$. Let A be a generating set for R over R_1. Assuming that R_1 is infinite (as we may), we have $|R_1[A_i]| = |R_1|$ for each finite subset A_i of A. Now $R = R_1[A] = \cup\{R_1[A_i] | A_i \in F$, the family of finite subsets of $A\}$. Therefore $|R| \leq |F| \cdot |R_1|$. This implies that $|F| = |R|$, which implies $|A| = |R| \geq \alpha^+$, where α^+ is the successor cardinal of α. Choose a subset B of A of cardinality α^+. Since $|R_1[B]| > \alpha$, it follows that $R_1[B] = R$, but the argument just given shows that $|R_1[B]| = |B|$. Therefore $|R| = \alpha^+$. Since α^+ is a regular cardinal ([3], p. 505), Theorem 2.4 yields a contradiction to the assumption that $|R| > \alpha$. We record this result formally in the case where $\alpha = \omega_0$.

THEOREM 2.6. *Suppose R is a commutative ring with identity e and R_1 is a subring of R containing e. If each proper R_1-subalgebra of R is countable, then R is countable.*

3. **Appendix**. In Remark 2.5 we referred to a result of E. Enochs; this result was also cited in ([1], p. 659). Enochs has supplied us with the statement and proof of this unpublished result, and has graciously allowed us to include them in this paper. We use the following notation. If L is a submodule of M, then $L \subset' M$ means that L is an essential submodule of M. $E(M)$ denotes an injective envelope of M. $\mathrm{Soc}(M)$ denotes

the socle of M. The proof makes use of a result of E. Matlis, Theorem 3.11 in the paper *Injective Modules Over Noetherian Rings*, Pacific J. Math. 8 (1958), 511-528, which states that if G is a simple module over a Noetherian ring R, then $E(G)$ is countably generated. Since R is Noetherian, it follows that any submodule of $E(G)$ is a countable union of finitely generated modules, and hence is countably generated. Also, from the fact that $E(\bigoplus M_i) \cong \bigoplus E(M_i)$, it follows that if G is semisimple and countably generated, then any submodule of $E(G)$ is countably generated.

THEOREM 3.1. *If a module M over a Noetherian ring is not finitely generated, then M has a factor module which is countably generated, but not finitely generated.*

PROOF. If $\mathrm{Soc}(M)$ is not finitely generated, let G be a countably generated submodule of $\mathrm{Soc}(M)$ that is not finitely generated. If $U < M$ is maximal with respect to $G \cap U = (0)$, then $G \subset' M/U$, so M/U is isomorphic to a submodule of $E(G)$. Since G is semi-simple, M/U is countably generated but not finitely generated since G is not finitely generated.

Thus to complete the proof of the theorem, it suffices to show that M has a homomorphic image for which the socle is not finitely generated. We may assume that M is not countably generated. If L_1 is any nonzero finitely generated submodule of M and if K_1 is a maximal proper submodule of L_1, then L_1/K_1 is a nonzero submodule of $\mathrm{Soc}(M/K_1)$, so $\mathrm{Soc}(M/K_1) \neq (0)$. If $\mathrm{Soc}(M/K_1)$ is not finitely generated, the proof is complete. If $\mathrm{Soc}(M/K_1) = S_1/K_1$ is finitely generated, let U_1/K_1 be maximal with respect to $S_1/K_1 \cap U_1/K_1 = (0)$. Then M/U_1 is isomorphic to a submodule of $E(S_1/K_1)$, and therefore is countably generated. It follows that U_1/K_1 is not countably generated. Let L_2/K_1 be a nonzero finitely generated submodule of U_1/K_1, and let K_2/K_1 be a maximal proper submodule of L_2/K_1. Then S_1/K_1 is isomorphic to a proper submodule of $\mathrm{Soc}(M/K_2) = S_2/K_2$. If S_2/K_2 is not finitely generated, the argument ends. If S_2/K_2 is finitely generated, let U_2/K_2 be maximal with respect to $S_2/K_2 \cap U_2/K_2 = (0)$, and let L_3/K_2 be a nonzero finitely generated submodule of U_2/K_2. A simple induction argument yields the existence of finitely generated submodules $K_1 < K_2 < \ldots$ of M such that either $\mathrm{Soc}(M/K_n)$ is not finitely generated for some n, in which case we are finished, or else $\mathrm{Soc}(M/K_n)$ is isomorphic to a proper submodule of $\mathrm{Soc}(M/K_{n+1})$ for each n. In the latter case, if $K = \bigcup_{n=1}^{\infty} K_n$, then $\mathrm{Soc}(M/K)$ is not finitely generated.

COROLLARY. If ω is an uncountable cardinal and if R_1 is a Noetherian ring with $|R_1| < \omega$, then R_1 admits no Jónsson ω-module.

PROOF. Assume, to the contrary, that M is a Jónsson ω-module over R_1. A countably generated module over R_1 has cardinality at most $\sup\{|R_1|, \omega_0\} < \omega$, and hence M is not countably generated. Choose, by Theorem 3.1, a submodule N of M such that M/N is countably generated, but not finitely generated. Then $\omega = |M| = |N| \cdot |M/N|$; this is a contradiction, for $|N| < \omega$ by hypothesis, and $|M/N| < \omega$ by the argument above. This completes the proof.

https://doi.org/10.4153/CMB-1986-019-0 Published online by Cambridge University Press

References

1. J. T. Arnold, R. Gilmer, and W. Heinzer, *Some countability conditions in a commutative ring*, Illinois J. Math. **21** (1977), pp. 648–665.

2. N. Bourbaki, *Elements of Mathematics*, Theory of Sets, Addison-Wesley, Reading, Mass., 1968.

3. C. C. Chang and H. J. Keisler, *Model Theory (Studies in Logic and the Foundations of Mathematics 73)*, North-Holland Publ. Co., Amsterdam, 1973.

4. P. Erdös and A. Hajnal, *On a problem of B*. Jónsson, Bull. Acad. Polon, Sci. Math. Astronom, Phys. **14** (1966), pp. 19–23.

5. R. Gilmer, *Multiplicative Ideal Theory*, Marcel-Dekker, New York, 1972.

6. R. Gilmer and W. Heinzer, *Cardinality of generating sets for ideals of a commutative ring*, Indiana Univ. Math. J. **26** (1977), pp. 791–798.

7. R. Gilmer and W. Heinzer, *Some countability conditions on commutative ring extensions*, Trans. Amer. Math. Soc. **264** (1981), pp. 217–234.

8. R. Gilmer and W. Heinzer, *Cardinality of generating sets for modules over a commutative ring*, Math. Scand. **52** (1983), pp. 41–57.

9. R. Gilmer and W. Heinzer, *On Jónsson modules over a commutative ring*, Acta Scient. Math. Szeged **46** (1983), pp. 3–15.

10. R. Gilmer, R. Lea, and M. O'Malley, *Rings whose proper subrings have property P*, Acta. Scient. Math. Szeged, **33** (1972), pp. 69–75.

11. R. Gilmer and M. O'Malley, *Non-Noetherian rings for which each proper subring is Noetherian*, Math. Scand. **31** (1972), pp. 118–122.

12. N. Jacobson, *Lectures in Abstract Algebra*, Vol. III, Van Nostrand, Princeton, N.J., 1964.

13. T. J. Laffey, *Infinite rings all of whose proper subrings are finite*, Amer. Math. Monthly **81** (1974), pp. 270–272.

14. M. Nagata, *Local Rings*, Interscience, New York, 1962.

15. W. Scott, *Groups and cardinal numbers*, Amer. J. Math. **74** (1952), pp. 187–197.

16. S. Shelah, *On a problem of Kurosh, Jónsson groups and applications; in Word Problems II*, (Proc. Conf. Oxford, 1976), North-Holland, Amsterdam, 1980; pp. 373–394.

17. A Simis, *On N_α-Noetherian modules*, Canad. Math. Bull. **13** (1970), pp. 245–247.

18. S. P. Strunkor, *Subgroups of periodic groups*, Soviet Math. Strunkov/Dokl. **7** (1966), pp. 1201–1203.

Department of Mathematics
Florida State University
Tallahassee, Florida 32306
U.S.A.

Department of Mathematics
Purdue University
W. Lafayette, Indiana 47907
U.S.A.

https://doi.org/10.4153/CMB-1986-019-0 Published online by Cambridge University Press