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NILPOTENTS AND UNITS IN SKEW POLYNOMIAL RINGS OVER COMMUTATIVE RINGS

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Abstract

Let R be a commutative ring with an automorphism α of finite order n. An element f of the skew polynomial ring $R[x, \alpha]$ is nilpotent if and only if all coefficients of f^n are nilpotent. (The case $n = 1$ is the well-known description of the nilpotent elements of the ordinary polynomial ring $R[x]$.) A characterization of the units in $R[x, \alpha]$ is also given.

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Let R be a commutative ring and suppose α is an automorphism of R with finite order n. We describe in Theorem 1 the nilpotent elements of $R[x,\alpha]$ in a way which is a generalization of the well-known characterization of the nilpotent elements in the ordinary polynomial ring $R[x]$ (the case $n = 1$). In Section 2 we characterize the units in $R[x,\alpha]$. The results are all obtained by embedding $R[x,\alpha]$ into an $n\times n$ matrix ring.

We write the elements of $R[x,\alpha]$ in the form

$$r_0+r_1x+\ldots+r_mx^m \quad (r_i\in R)$$

and multiplication is determined by $xr = r^{\alpha}x$ for $r\in R$.

These results appear in the first author's Ph.D. thesis (Rimmer (1978)), written under the supervision of the second author. Gilmer (1975) describes the related results for $R[x]$.

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1. Nilpotents

THEOREM 1. *Let R be a commutative ring with an automorphism α of order n. An element f of $R[x,\alpha]$ is nilpotent if and only if all coefficients of f^n are nilpotent in R.*

PROOF. When R is embedded in a ring with identity in the usual way, the automorphism α extends to an automorphism of the same order. Thus we may assume in what follows that R has an identity.

Denote x^n by y. Notice that y is central in $R[x,\alpha]$ since

$$yr = x^n r = r^{\alpha^n} x^n = rx^n = ry,$$

that the subring $R[y]$ generated by R and y is just the ordinary polynomial ring and that $R[x,\alpha]$ is a free (left) $R[y]$-module with basis $\mathscr{B} = \{1, x, \ldots, x^{n-1}\}$. The regular representation (f maps to right multiplication by f) embeds $R[x,\alpha]$ in $\mathrm{End}_{R[y]}(R[x,\alpha])$. If we replace elements of $\mathrm{End}_{R[y]}(R[x,\alpha])$ by their matrices with respect to $\mathscr{B}$ we have an embedding φ from $R[x,\alpha]$ into the ring $M(n,R[y])$ of $n\times n$ matrices over $R[y]$. It is easy to check that if $h = h_0 + h_1 x + \ldots + h_{n-1}x^{n-1}$ (with h_i in $R[y]$) then $h\varphi$ is the matrix

$$\begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \\ h_{n-1}^{\alpha} y & h_0^{\alpha} & h_1^{\alpha} & \cdots & h_{n-2}^{\alpha} \\ h_{n-2}^{\alpha^2} y & h_{n-1}^{\alpha^2} y & h_0^{\alpha^2} & \cdots & h_{n-3}^{\alpha^2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_1^{\alpha^{n-1}} y & h_2^{\alpha^{n-1}} y & h_3^{\alpha^{n-1}} y & \cdots & h_0^{\alpha^{n-1}} \end{pmatrix}. \tag{1}$$

Let P be any prime ideal of R. The natural map $R \to R/P$ extends to a homomorphism

$$\theta_P \colon M(n, R[y]) \to M(n, (R/P)[y]).$$

If $h = h_0 + h_1 x + \ldots + h_{n-1}x^{n-1}$ ($h_i \in R[y]$) is in the kernel of $\varphi\theta_P$ then we see from the first row of (1) that each $h_i \in P[y]$ and so $h \in P[x,\alpha]$.

Suppose $f \in R[x,\alpha]$ is nilpotent. Since any nilpotent $n\times n$ matrix A over a field (or integral domain) satisfies $A^n = 0$, and since $(R/P)[y]$ is an integral domain, we see that $(f\varphi\theta_P)^n = 0$. Hence f^n is in the kernel of $\varphi\theta_P$ which means that $f^n \in P[x,\alpha]$. Because P was arbitrary, all coefficients of f^n are in the prime radical of R and so are nilpotent.

Conversely, consider any polynomial $r = r_0 + r_1 x + \ldots + r_m x^m$ ($r_i \in R$) such that all r_i are nilpotent. If $r_i^t = 0$ for all i, it follows that $r^{nt(m+1)} = 0$, since a typical term in this is a product of $nt(m+1)$ terms $r_i x^i$ and so has coefficient a product of $nt(m+1)$ terms of the form $r_i^{\alpha^j}$ $(0 \leqslant j \leqslant n-1)$, some t of which must be equal. Thus if all coefficients of f^n are nilpotent, f^n is nilpotent and hence f is nilpotent.

NOTE. It follows from the second part of the above proof that f is nilpotent if all its coefficients are nilpotent. That the converse is not true can be seen by taking $R = \mathbf{Z} \oplus \mathbf{Z}$ with α of order 2 given by $(a, b)^\alpha = (b, a)$. If

$$f = (1,0)\,x + (1,-1)\,x^2 + (0,-1)\,x^3$$

then $f^2 = 0$ but the coefficient of x^2 in f is a unit.

Nilpotent $n \times n$ matrices over a field are those with characteristic polynomial λ^n and over a commutative ring are those with characteristic polynomial

$$\lambda^n + d_{n-1}\lambda^{n-1} + \dots + d_1\lambda + d_0,$$

where each d_i is nilpotent. Hence the embedding φ gives another description of the nilpotents.

PROPOSITION 2. *Let R be a commutative ring with an automorphism α of order n. Then $f \in R[x, \alpha]$ is nilpotent if and only if $f\varphi$ has characteristic polynomial*

$$\lambda^n + d_{n-1}\lambda^{n-1} + \dots + d_1\lambda + d_0,$$

where each d_i is a nilpotent polynomial.

2. Units

THEOREM 3. *Let R be a commutative ring with identity and let α be an automorphism of R with order n. A polynomial f is a unit in $R[x, \alpha]$ precisely when the matrix $f\varphi$ has determinant $r_0 + r_1 y + \dots + r_m y^m$ with r_0 a unit in R and $r_1, \dots, r_m$ nilpotent.*

PROOF. If f is a unit in $R[x, \alpha]$ then clearly $f\varphi$ is a unit in $M(n, R[y])$. Since $R[y]$ is commutative, $\det(f\varphi)$ is a unit in $R[y]$ and hence has the appropriate form.

Conversely, if $\det(f\varphi)$ is as described, $\det(f\varphi)$ is a unit in $R[y]$ and so $f\varphi$ has an inverse $(f\varphi)^{-1}$ in $M(n, R[y])$. Because $f\varphi$ satisfies its characteristic polynomial,

$$(f\varphi)^{-1} = (-1)^{n-1}(\det f\varphi)^{-1}((f\varphi)^{n-1} + c_{n-1}(f\varphi)^{n-2} + \dots + c_1)$$

for some $c_1, \dots, c_{n-1} \in R[y]$. If

$$g = (-1)^{n-1}(\det f\varphi)^{-1}(f^{n-1} + c_{n-1}f^{n-2} + \dots + c_1)$$

then $g\varphi$ has the same first row as $(f\varphi)^{-1}$ and so, since $(f\varphi)^{-1}(f\varphi) = I$, $g\varphi . f\varphi$ has first row $(1, 0, \dots, 0)$. Since $g\varphi . f\varphi = (gf)\varphi \in R[x, \alpha]\varphi$, it follows from (1) that $g\varphi . f\varphi = I$ and hence $g\varphi = (f\varphi)^{-1}$. Thus, since φ is injective, $gf = fg = 1$ and f is a unit.

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