

# 11

## The dynamical analogues of the Lund model fragmentation formulas

### 11.1 Introduction

The Lund model fragmentation formulas are based upon general principles such as causality, Lorentz covariance and confinement. Only classical probability concepts and semi-classical dynamical considerations have been used in the derivations in the earlier chapters. Nevertheless, the resulting formula for the decay of a (color- and flavor-connected) cluster has an appealing simplicity, similar to those obtained in other dynamical situations. It is the product of the  $n$ -particle phase space factor and an *area suppression factor* written as the square of a ‘matrix element’  $\mathcal{M}$ :

$$dP_{int} = d\tau_n |\mathcal{M}|^2, \quad |\mathcal{M}|^2 \equiv \exp(-bA) \quad (11.1)$$

In this way we connect with formulas for multiparticle production cross sections in a quantum field theory and in Chapter 10 we have exhibited a wide class of models, the multiperipheral ladder models, with properties similar to the Lund model results. In this chapter we consider further dynamical analogies to the Lund model. We show that the *area suppression law* in Eq. (11.1) can be interpreted in at least two different ways stemming from field theory:

- in terms of a quantum mechanical tunnelling process (the decay of the vacuum in an external field); we will call this the *Schwinger way*,
- in terms of basic gauge-independent quantities, the Wilson phase operators, in a gauge field theory; in the same language this interpretation will be called the *Wilson way*.

Both these interpretations will lead to a discussion of the meaning of the parameter  $b$  in the Lund fragmentation model and of the behaviour of the color force field in the QCD vacuum. We will also show that the formulas can be interpreted in a statistical mechanics scenario as follows.

- The internal-part formula corresponds to the partition function for a gas with two-body interaction potentials. The coordinate space for the gas is then the longitudinal rapidity space of the cluster.

In this way another familiar phenomenological tool, the *Feynman-Wilson gas in rapidity space*, will occur in connection with the Lund model fragmentation formulas. Approximating the gas partition function according to the first nontrivial part of the virial expansion, we derive a relationship between the normalisation constant  $N$ , the parameters  $b$  and  $a$  and the particle density in rapidity space. The result can also be considered as an ideal gas law for the rapidity-space gas.

## 11.2 The decay of the vacuum in an external field

We now meet another example of the *law of the conservation of useful dynamics*. Again this is a case for which it is possible to obtain a closed expression in terms of elementary functions for a dynamically interesting situation. This time it is the reaction of the vacuum, defined as the state containing no quanta, to the onset of an external electromagnetic field. Quantum matter fields, such as an electron-positron ( $e^-e^+$ ) field, which are coupled to the electromagnetic field, will then start to fluctuate.

In Chapter 4 we have shown that that the vacuum in a quantum field theory is, due to quantum fluctuations, very similar to a dielectric medium. If there is an external field stretching over macroscopic regions then these polarisation charges will be driven by the field. Therefore the original no-particle vacuum state, existing before the onset of the field, will break down into a new state.

The problem of the reaction of the vacuum was considered by Heisenberg and Euler in the 1930s, [79], by Schwinger in the 1950s, [100], and by many authors in the 1970s and the 1980s. In this section we will formulate the problem as a *tunnelling process*. We will be satisfied with using semi-classical considerations (similar to the ones presented in [64]) to derive a formula for the *vacuum persistence*, i.e. the probability that the vacuum does not decay.

### 1 The tunnelling formulas

In Chapter 6 we found that a classical particle with mass  $m$  which experiences a constant force field (force constant  $\kappa$ ) will move in from far away towards the origin, ‘bounce’ at the classical turning point  $|x_c| = m/\kappa$  and then go back outwards again.

The origin  $x = 0$  is then defined by the requirement that the total energy of the particle, i.e. the sum of the kinetic and potential energies,

$$E = \sqrt{|\mathbf{p}|^2 + m^2} - \kappa x \quad (11.2)$$

vanishes, i.e. we choose the value  $E = 0$  (note that you may place the origin wherever you like by suitable choice of  $E$ ).

For a quantum mechanical particle, however, the wave function  $\psi$  will be oscillating for  $x > x_c$  but it will not vanish at the classical turning point  $x_c$ . There will be an exponential tail for smaller values of  $x$  and this tail can be approximately calculated by means of the well-known WKB approximation (for details on the WKB approximation, cf. Merzbacher). A WKB solution to the wave function inside the classically forbidden region is, with  $p_\ell$  (index  $\ell$  for the momentum component along the  $x$ -direction) chosen to fulfil Eq. (11.2),

$$\psi(x) = \psi_c \exp \left[ -i \int_{x_c}^x p_\ell(x') dx' \right], \quad x_c \geq x \geq 0 \quad (11.3)$$

$$p_\ell(x) = i\sqrt{E_\perp^2 - (\kappa x)^2}, \quad E_\perp = \sqrt{|\mathbf{p}_\perp|^2 + m^2} \quad (11.4)$$

The quantity  $\psi_c$  is the value of the wave function for  $x = x_c$ . We will come back to its significance later. The particle is assumed to move with a transverse momentum  $\mathbf{p}_\perp$  in the force field along the  $x$ -axis (transverse meaning orthogonal to the  $x$ -axis) so that the classical turning point is given by  $x_c = E_\perp/\kappa$ . Note that both classically and quantum mechanically the transverse momentum  $\mathbf{p}_\perp$  will be a conserved quantity.

We obtain for the integral in the exponent for the value  $x = 0$ :

$$R(p_\perp) = \frac{\psi(0)}{\psi_c} = \exp \left[ - \left( \frac{\pi E_\perp^2}{4\kappa} \right) \right] \quad (11.5)$$

We may then consider the following dynamical problem.

- Consider an incoming particle and antiparticle, a  $q\bar{q}$ -pair, each connected to the constant force field (the forces are, however, oppositely directed as the  $q$  and  $\bar{q}$  have opposite charges). Let us assume that they move in along the  $\mp$  directions, have opposite transverse momenta  $\pm\mathbf{p}_\perp$  and vanishing (total) energies. What is the probability  $P(p_\perp)$  that their wave functions will overlap, as required for them to annihilate each other?

We note that such an annihilation process is allowed (all quantum numbers of the pair are conserved) and that a reasonable answer is given by the square of the overlap of their wave functions at the origin:

$$P(p_\perp) = |R^2|^2 = \exp \left[ - \left( \frac{\pi E_\perp^2}{\kappa} \right) \right] \quad (11.6)$$

This turns out to be the right answer, and the question is investigated in more detail in e.g. [17]. If the wave function for  $q$  is  $\psi(x)$ , then that for  $\bar{q}$  is  $\psi(-x)$ , and the factor  $\psi_c$  can at this point be thought of as a flux factor, i.e. the density of incoming particles.

*This annihilation probability is equal to the production probability in a quantum field theory.* Thus we have deduced the probability that a  $q\bar{q}$ -pair with opposite transverse momenta will tunnel out in the constant force field  $\kappa$ . We note in passing that the result in Eq. (11.6) is intrinsically of a nonperturbative origin, i.e. we cannot expand the result as a power series in the force field constant  $\kappa$  for small values of  $\kappa$ .

The result in Eq. (11.6) can be compared to the results of Heisenberg's indeterminacy relation. Then one would ask, what is the probability of obtaining a vacuum fluctuation such that a  $q\bar{q}$ -pair occurs at a separation  $\Delta = 2x_c$  with transverse masses  $E_\perp$ ?

The answer is given by the square of the free coordinate-space propagator  $\Delta_F(x, m)$  evaluated for a spacelike value of  $x = \Delta$  and  $m = E_\perp$ :

$$[\Delta_F(\Delta, E_\perp)]^2 \sim [K_1(E_\perp \Delta)]^2 \simeq \exp[-2(E_\perp \Delta)] = \exp\left(-\frac{4E_\perp^2}{\kappa}\right) \quad (11.7)$$

The function  $K_1$  is the modified Bessel function of rank 1 (which is equal to  $\Delta_F$  for a spacelike argument, according to section 6.3, Eq. (6.39)) and here we use a simple approximation for it.

The results in Eqs. (11.6) and (11.7) are essentially the same except that the factor 4 in the exponential for the free (i.e. the no-field) case replaces the factor  $\pi$  in the previous case, where there is a field which pushes and therefore makes it easier for the pair to tunnel out.

## 2 The vacuum persistence probability

We will now consider the persistence probability of the vacuum as defined in [40]. It is the probability that no tunnelling will occur for any spin ( $s$ ), flavor ( $f$ ) and transverse momentum ( $\mathbf{p}_\perp$ ) at any place, i.e. for any value of  $0 < x < L_x$  and any time  $0 < t < T$ ,  $L_x$  and  $T$  being the extent of the field in longitudinal space and time. This quantity, which we will denote by  $\mathcal{P}$ , is evidently given by

$$\mathcal{P} = \prod_{s,f,\mathbf{p}_\perp,x,t} (1 - P) = \exp \left[ \sum_{s,f,\mathbf{p}_\perp,x,t} \ln(1 - P) \right] \quad (11.8)$$

We will start by considering the sums over the longitudinal ( $L_x$ ) and time ( $T$ ) extents of the field. We have repeatedly observed that it takes a spatial region of the size  $\Delta = 2E_\perp/\kappa$  to produce a pair. The lifetime  $\delta t$  of such a pair is evidently  $\delta t = 2\pi/(2E_\perp)$  according to the indeterminacy

principle. We conclude that each pair will need a space-time region of size  $\Delta\delta t = 2\pi/\kappa$  for its production to be possible.

As the probability  $P$  is independent of  $x$  and  $t$  we conclude that the summation over  $x$  and  $t$  in the exponent of Eq. (11.8) will give the factor

$$\frac{L_x T}{\Delta\delta t} = \frac{\kappa L_x T}{2\pi} \tag{11.9}$$

In this way the possible pairs are as ‘closely packed’ as possible.

Next we consider the transverse extent  $A_\perp$ . In accordance with the discussion in Chapter 3, the number of plane wave solutions that can be fitted into such a transverse box is

$$\frac{A_\perp d^2 p_\perp}{(2\pi)^2} \tag{11.10}$$

We conclude that the sum in the exponent of Eq. (11.8) can be written as

$$\sum_{s,f,\mathbf{p}_\perp,x,t} \ln(1 - P) = \frac{\kappa L_x T A_\perp}{(2\pi)^3} \sum_{s,f} \int d^2 p_\perp \ln(1 - P) \tag{11.11}$$

The integral over the transverse momenta is easily performed in terms of gaussian integrals if we expand the logarithm:

$$\begin{aligned} \ln(1 - P) &= - \sum_{n=1}^{\infty} \frac{1}{n} \exp \left[ \frac{-n\pi(m^2 + \mathbf{p}_\perp^2)}{\kappa} \right], \quad \mathcal{P} = \exp(-\kappa^2 L_x T A_\perp \Pi), \\ \Pi &= \frac{1}{4\pi^3} \sum_{s,f} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left( \frac{-n\pi m^2}{\kappa} \right) \end{aligned} \tag{11.12}$$

Comparing to Eq. (11.1) we find the Lund model area suppression law:

$$\mathcal{P} = \exp(-bA), \quad \kappa^2 L_x T = A/2. \tag{11.13}$$

This is the natural interpretation since the region  $A$  in the Lund formula is spanned in the longitudinal and time directions and is just the region over which the MRS persists, i.e. does not decay. Thus we identify  $|\mathcal{M}|^2$  in Eq. (11.1) with  $\mathcal{P}$  (but this obviously does not provide the phase of the matrix element  $\mathcal{M}$ ; cf. section 11.3). The factor 1/2 in Eq. (11.13) is due to our use of a lightcone metric  $dA = \kappa^2 dx_+ dx_-$ , which is double the usual metric  $\kappa^2 dx dt$  (cf. Chapter 7).

At the same time we have obtained a formula for the parameter  $b$  in terms of the transverse size of the force field,  $A_\perp$ :

$$b = A_\perp \frac{\Pi}{2} \tag{11.14}$$

The quantity  $\Pi$  in Eq. (11.12) is, for a number  $n_f$  of massless spin 1/2

particles (we will neglect the massive flavor contributions),

$$\frac{\Pi}{2} = \frac{n_f}{12\pi} \quad (11.15)$$

Although we have used semi-classical arguments for the evaluation of the persistence probability, the final result coincides with the one Schwinger wrote down for the production of  $e^+e^-$ -pairs in a constant electric field. He obtained the same formula as Eq. (11.12) with the quantity  $\kappa$  replaced by  $e\mathcal{E}$ , i.e. the electronic charge times the electric field strength, which evidently is the force acting upon the electrons or positrons in constant external field. We are instead applying the formula to the color fields in QCD and to the production of  $q\bar{q}$ -pairs along the constant string force field when we compare to the Lund model area law.

If we use  $n_f = 2$  (i.e. consider the  $u$ - and  $d$ -flavors to be massless and neglect the rest) we obtain the following value for the transverse radius  $R_\perp$  ( $A_\perp = \pi R_\perp^2$ ) of the force field:

$$R_\perp = \sqrt{6b} \simeq 0.55 \text{ fm} \quad (11.16)$$

using the phenomenological value  $b \simeq 0.75 \text{ GeV}^{-2}$  which we have discussed before.

### 3 The relation between the parameter $b$ and the fields and charges

We will next relate the transverse area  $A_\perp$  to the charges of the  $q\bar{q}$ -pairs, which generate the fields. Although we will repeatedly make use of the analogous situation in the (abelian) QED field theory, we actually have in mind the more complicated color fields in the (nonabelian) QCD theory. We will therefore consider two different situations, one which we call the abelian setting and one that should be characteristic of a confining QCD vacuum.

In the abelian setting (where all fields and charges can be added in any order) the relation between the charge of a  $q$ -particle, which we will call  $g$ , and the electric field,  $\mathcal{E}_1$ , stemming from it is from Gauss's law (see Fig. 11.1)

$$2\mathcal{E}_1 A_\perp \equiv \mathcal{E} A_\perp = g \quad (11.17)$$

We note that the total electric field is  $\mathcal{E} = 2\mathcal{E}_1$  as it obtains contributions which add up in between the charges but subtract to zero elsewhere. This corresponds to confinement in this case, i.e. there is a field only in between the charges. Thus for the abelian scenario all the fields arise from the charges connected to the string force field and there is no influence from the vacuum.

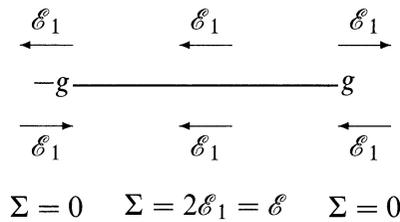


Fig. 11.1. The color electric field  $\mathcal{E} = 2\mathcal{E}_1$  from a connected charged pair  $\pm g$ . There is no field outside the connected region between the charges (a simple-minded approach to confinement). The field strengths from the charges are shown above and below the string, and the contributions are summed in the last line.

The force  $\kappa_a$  (index  $a$  for abelian) between the  $\bar{q}$  and the  $q$  is therefore

$$\kappa_a = g\mathcal{E}_1 = \frac{g\mathcal{E}}{2} = \frac{A_\perp \mathcal{E}^2}{2} \tag{11.18}$$

in accordance with the ordinary formulas for the energy density of an electric field. Note the difference from the Schwinger result where  $\kappa \rightarrow e\mathcal{E}$  for a truly external field  $\mathcal{E}$  ( $e$  the electric charge  $e$  being the electron). In our case we identify the external field with that spanned by the original  $q\bar{q}$ -pair at the endpoints of the MRS.

For consistency we note that if a  $q\bar{q}$ -pair is produced along the field then the same force and the same field relations are valid. In between the produced pair the two fields just compensate each other (to secure confinement) while the new endpoint fields take over in between the old and the new endpoints.

From these relations we obtain for the parameter  $b = b_a$ , introducing the ordinary coupling constant  $\alpha = g^2/(4\pi)$ ,

$$b_a \kappa_a = \frac{\Pi A_\perp}{2} \kappa_a = \frac{\Pi g^2}{4} = \frac{n_f \alpha}{6} \tag{11.19}$$

There is (at least) one point in this discussion which is disturbing. *There is no reason why such an abelian field should be kept inside a thin transverse region.* It is well known that the electromagnetic fields in the abelian QED field theory do not behave like that. We will therefore consider a different scenario in which confinement is actually enforced by the properties of the QCD vacuum.

If we consider the color dynamical fields in QCD it is not obvious how to treat Gauss's law in Eq. (11.17). The electric field in this case is a color-8 operator while the charge is a color-3 (for the  $q$ ) or a color- $\bar{3}$  (for the  $\bar{q}$ ). The energy density along the string due to the color electric field

should, however, have a color scalar, i.e. a color singlet meaning (although different ‘ideologies’ with respect to color dynamics may provide different numerical values).

We will now assume that for the QCD force fields the vacuum exerts a pressure on the fields and the charges (according to the bag model for hadrons mentioned in Chapter 6). There are, in the vacuum, gluonic field configurations, which compensate the fields from the charges outside the string region. In Chapter 6 we used the analogous picture of a color-superconducting QCD vacuum with a Meissner effect produced by these gluonic ‘Cooper pairs’. When the field is built up this vacuum pressure must be overcome and thus the total work done in creating the field configuration in the vacuum is twice as large as for the abelian situation.

In this case the force on a charge is  $\kappa \rightarrow \mathcal{E}^2 A_{\perp}/2 + BA_{\perp} = \mathcal{E}^2 A_{\perp} \equiv \kappa_b$ , with  $B = \mathcal{E}^2$  the bag pressure. The corresponding value for the parameter  $b \equiv b_b$  is then, in terms of the squared field flux  $\Xi$  divided by the squared  $q$ - or  $\bar{q}$ -charge, which should have a meaning also in color dynamics,

$$b_b \kappa_b = \frac{n_f \alpha}{6} \Xi, \quad \Xi = \frac{(\mathcal{E} A_{\perp})^2}{g^2} \quad (11.20)$$

One can argue in different ways at this point. One way, which is certainly not unreasonable, is to say that  $\Xi \alpha \equiv \bar{\alpha}$ , i.e. the effective coupling for gluonic emission along the field (cf. Chapters 16 and 17). Then the typical value, using  $\bar{\alpha} = n_c \alpha$  (with the QCD value  $n_c = 3$ ), for the strong coupling would be  $\alpha \simeq 0.3$  (once again using  $n_f = 2$ ,  $b = 0.75 \text{ GeV}^{-2}$  and  $\kappa \simeq 1 \text{ GeV/fm}$ ). There are other ways to interpret  $\Xi$  but all of them will, within a factor of 2, provide a similar ‘reasonable’ result for the strong coupling. We will come back to these formulas later on, both when we consider another field theoretical analogy to the Lund model fragmentation distributions (in the next section) and when we have learned more about the way gluonic radiation ‘resolves’ the color force field, i.e. how we can treat the massless relativistic string as a model for the color force field in accordance with the Lund interpretation (cf. Chapter 17).

### 11.3 The Wilson loop exponential laws and gauge invariance

In chapter 2 we considered the invariance of the electromagnetic fields under gauge transformations. We will discuss the implications of gauge invariance for the matter fields in the first subsection below. We will then show how gauge invariance should constrain the production of  $q\bar{q}$ -pairs along the color force fields. We will find that the Lund model area suppression law is a natural consequence of these constraints.

These considerations will provide us with a possible phase for the matrix

element  $\mathcal{M}$  in Eq. (11.1), which we will later, in chapter 14, show to have significance in connection with the so-called Hanbury-Brown-Twiss (or Bose-Einstein) effect in multiparticle production.

### 1 The implications of gauge invariance for the matter fields

We have in chapter 2 considered the Maxwell equations and remarked that if we introduce the (four)vector potential  $A$  there is still a gauge degree of freedom. This means that we obtain the same electromagnetic fields if  $A_\mu$  is changed as follows (with  $\Lambda$  an arbitrary function in space and time):

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{\partial}{\partial x_\mu} \Lambda(x) = A_\mu(x) + \delta A_\mu(x) \quad (11.21)$$

We will now show that *the gauge degrees of freedom also have significance for the matter fields which couple to the electromagnetic fields.*

We firstly note that the motion of a nonrelativistic particle with mass  $m$  and coordinate  $\mathbf{x}$  will under the influence of a scalar (non-electromagnetic) potential  $V$  be described by the hamiltonian equations

$$\dot{p}_j = -\frac{\partial H}{\partial x_j}, \quad \dot{x}_j = \frac{\partial H}{\partial p_j} \quad (11.22)$$

The dotted variables mean derivatives with respect to time. If hamiltonian  $h$  is independent of one of the coordinates  $x_j$  (i.e. the derivatives of  $h$  with respect to that coordinate vanish) then the first line of Eq. (11.22) will provide us with a constant of motion,  $p_j$ , which in general is equal to  $m\dot{x}_j$ . In other words, *the (mechanical) momentum is conserved* if there is no force along that direction.

Next we consider a charged particle (charge  $g$ ) moving under the influence of a constant magnetic field  $\mathcal{B}$  along the 3-axis :  $\mathcal{B} = \mathcal{B}\mathbf{e}_3$ . Possible vector potentials  $\mathbf{A}$  to describe this field are given by e.g. the following two:

$$A_1^{(1)} = -\mathcal{B}x_2 \quad \text{or} \quad A_2^{(2)} = \mathcal{B}x_1 \quad (11.23)$$

with the components not exhibited in the two cases vanishing. The equation of motion for the particle is

$$\frac{d m \dot{\mathbf{x}}}{dt} = g \dot{\mathbf{x}} \times \mathcal{B} \quad (11.24)$$

from which we immediately obtain that

$$m\dot{x}_1 - g\mathcal{B}x_2 = c_1, \quad m\dot{x}_2 + g\mathcal{B}x_1 = c_2 \quad (11.25)$$

with  $c_1, c_2$  constants of motion. Consequently, in this case the mechanical linear momentum  $m\dot{\mathbf{x}}$  is not conserved *but there is a combination of it and*

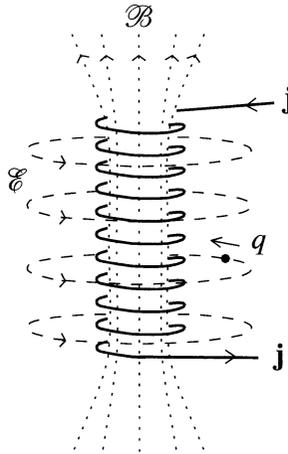


Fig. 11.2. A pointlike charge  $q$  is affected by the electric field  $\mathcal{E}$  arising when a sudden current  $\mathbf{j}$  causes a magnetic field  $\mathcal{B}$  through the solenoid.

a field quantity with this property. A little reflection using Eqs. (11.23) will lead us to guess the right answer:

- for a particle moving under the influence of an electromagnetic field described by a four-vector potential  $A_\mu$ , such that  $A_\mu$  can be chosen to be independent of a coordinate  $x_j$ , the combination  $m\dot{x}_j + gA_j \equiv p_j$  is a constant of motion.

As another example, which stems from the Feynman Lectures, consider a solenoid, Fig. 11.2. Suppose that we suddenly turn on a current through the wires and that there is a charged particle nearby. There will then be a sudden magnetic flux through the solenoid and correspondingly a *sudden buildup of the circumferential vector potential  $\mathbf{A}$* . Note that the relation between  $\mathbf{A}$  and the magnetic flux means that the line integral around the solenoid of  $\mathbf{A}$  is equal to the magnetic flux. There is also a sudden electric field

$$\mathcal{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad (11.26)$$

which provides a force on the particle. This force is equal to the charge times the field so that there is an impulse during the buildup of the magnetic field in the solenoid corresponding to  $-g\mathbf{A}$ . *The difference between the mechanical momentum  $m\dot{\mathbf{x}}$  and that added by the impulse does not change*, i.e. once again we find that  $\mathbf{p} = m\dot{\mathbf{x}} + g\mathbf{A}$  is a conserved quantity.

This principle of minimal coupling of the electromagnetic field, that everywhere the mechanical energy-momentum  $p_\mu$  is replaced by  $p_\mu - gA_\mu$ , i.e. a combination with the four-vector potential  $A_\mu$ , is of fundamental importance for the description of the interaction between the field and the charged particles. In quantum theory, where the canonical momentum and energy are operators  $\mathbf{p} \rightarrow -i\hbar\nabla$ ,  $E \rightarrow i\hbar\partial/\partial t$  acting on wave functions  $\psi$ , the occurrence of the particular combination  $p_\mu - gA_\mu$  means that the gauge transformations in Eq. (11.21) must be implemented as phase transformations on the wave functions:

$$\psi \rightarrow \psi \exp(ig\Lambda) \quad (11.27)$$

This means that the observable  $|\psi|^2$  is unaffected by gauge transformations. The same goes for many other observables. As an antiparticle has the opposite charge ( $g \rightarrow -g$ ) to a particle their wave function overlap at a single point (as in a current or charge density) will also be unaffected.

If we want to consider the overlap of the wave functions for a particle-antiparticle pair at different points, however, then the phase plays a role:

$$\rho(x_\mu, x'_\mu) = \psi(x_\mu)\bar{\psi}(x'_\mu) \rightarrow \rho \exp\{ig[\Lambda(x_\mu) - \Lambda(x'_\mu)]\} \quad (11.28)$$

This phase can evidently be expressed as a line integral of the vector  $\delta A_\mu$  in Eq. (11.21) between the two points  $x'_\mu, x_\mu$ :

$$\int_{x'_\mu}^{x_\mu} ig\delta A_\mu dx^\mu \quad (11.29)$$

A general prescription by Schwinger for handling these situations is to endow the wave function overlap with a phase as follows:

$$\rho(x_\mu, x'_\mu) \rightarrow \rho \exp\left(i \int_{x'}^x gA_\mu dx^\mu\right) \quad (11.30)$$

We note that as long as there are no singularities in the four-vector potential  $A_\mu$  then the line integral can be evaluated along any curve connecting the particle and antiparticle positions.

It is at this point worthwhile to note that the appearance of this phase difference, depending upon the vector potential  $A_\mu$ , was predicted to be an observable result, e.g. for interference effects in charged particle motion, by Aharanov and Bohm, [3] in 1956. It also turns out to be an observable effect, [23] when there is a *singular potential*, i.e. when the wave function difference is connected over a region inside which there is a nonvanishing magnetic field flux.

We have up to now considered *abelian* gauge transformations, i.e. those transformations in which it is possible to add and subtract charges and fields in any order. For *nonabelian* gauge transformations it is necessary to generalise our notions to take into proper account the order in which

the different quantities are added and multiplied just as for the exponent defining the  $S$ -operator in perturbation theory; cf. Chapter 3.

For QCD the quantity  $gA_\mu$  is a matrix in color space and the line integral contains the multiplication of all the matrices at every point. It is said to be *path-ordered*. With some care it is possible to carry through essentially the whole discussion above for the nonabelian case also.

## 2 The application of gauge invariance to the string decay process in the Lund model

The production of hadrons in the Lund model occurs by means of a  $q$ -particle from one vertex ( $V_1$ ) and a  $\bar{q}$  from an adjacent vertex ( $V_2$ ). Therefore the production matrix element should contain at least the factor

$$\gamma(V_1, V_2) = \exp\left(ig \int_{V_2}^{V_1} A_\mu dx^\mu\right) \quad (11.31)$$

in order to maintain gauge invariance. The next production will similarly involve  $\gamma(V_2, V_3)$  and so on. Therefore a minimal requirement for gauge invariance is that the matrix element contains a factor (with  $V_{n+1} = V_0$ ) that becomes an integral around the production area,

$$\prod_{j=0}^n \gamma(V_j, V_{j+1}) = \exp\left(ig \oint A_\mu dx^\mu\right) \quad (11.32)$$

This is a *Wilson loop operator* when it is evaluated in a quantum field theory state. Wilson's condition for confinement is that it should behave as

$$\langle s | \exp\left(ig \oint A_\mu dx^\mu\right) | s \rangle = \exp(i\xi \tilde{A}) \quad (11.33)$$

where  $|s\rangle$  is the state and  $\tilde{A}$  is the (space-time) area enclosed by the integration contour. The quantity  $\xi$  is a parameter whose real part is equal to the string constant  $\kappa$ , i.e. the force on the charges in the confining force field.

Wilson's suggestion has been studied by means of approximative calculations on a lattice and his area law has been confirmed for a number of confining situations. These calculations are, however, outside the scope of this book.

In order to understand the reason why one should obtain this result for the loop integral and also to see why the loop integrals are gauge-invariant we will consider the integral around a connected curve rearranged as in Fig. 11.3. In this way the integral is seen to be equivalent to a large number of integrals around smaller curves, which together make up the larger one.

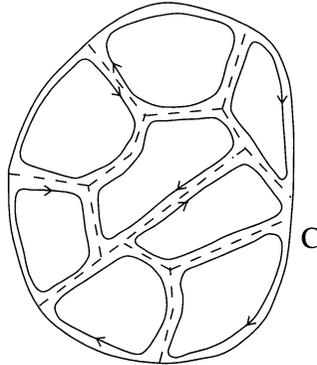


Fig. 11.3. The area inside the boundary curve  $C$  is subdivided into smaller areas defined by closed curves. If one sums over the contributions from them all only the original curve will be left since each of the internal parts of the curves is traversed twice, in opposite directions.

This is the construction used to prove Stoke's theorem: the line integral of a vector field around a connected curve equals the area integral of the rotation of the vector field. The area integration vector  $d\mathbf{a}$  is defined by the right-hand screw rule from the direction of the curve  $C$ :

$$\oint_C \mathbf{A} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} \quad (11.34)$$

If we identify  $\mathbf{A}$  with the vector potential we obtain the gauge-independent magnetic field  $\mathcal{B}$  in the integral and in this way the integral corresponds to the magnetic flux through the region surrounded by the curve  $C$ . There will consequently be a phase difference between wave functions describing motion on one side of the field and on the other side, according to Eq. (11.30), which is just the Aharonov-Bohm prediction mentioned above. Stoke's theorem can be extended to surfaces in a longitudinal space-time plane; one obtains

$$\oint_C g A_\mu dx^\mu = \int_{\tilde{A}} g \mathcal{E}_\ell dx_\ell dt \quad (11.35)$$

from the relationship given in Chapter 2 between the four-vector potential  $A_\mu$  and the electric field  $\mathcal{E}$  (the index  $\ell$  stands for the direction along the field). Wilson's criterion for confinement implies that the surface integral of the electric force over  $\tilde{A}$  should be proportional to the area of  $\tilde{A}$ . This is fulfilled when (the real part of)  $g\mathcal{E}$  is a constant.

The description in Fig. 11.4 of Lund model fragmentation by a subdivision of the string persistence region in different ways is a direct realisation

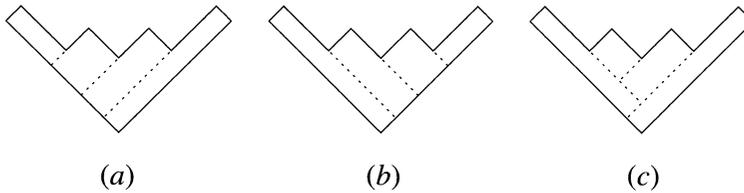


Fig. 11.4. Different ways to subdivide the string region in the fragmentation process, corresponding to different gauge choices: (a) the gauge  $A_- = 0$ ; (b) the gauge  $A_+ = 0$ ; (c) mixed gauge.

of choosing a gauge in different ways and should be compared with the Stoke's construction in Fig. 11.3.

For the case when the hadron yoyos are produced along the positive lightcone (Fig. 11.4(a)) the gluonic field can be chosen in the lightcone gauge  $A_- = 0$ . This means that there is no gluon-field transmission along the negative lightcone. The pairs can be thought of as produced by the gluons emitted along the positive axis from the original  $\bar{q}$ . In the case in Fig. 11.4(b) the corresponding gauge is  $A_+ = 0$ . The case exhibited in Fig. 11.4(c) corresponds to a mixed gauge condition, which is different in different parts of the string region.

### 3 The possible relationship to the Lund area law

At this point it is necessary to make a few clarifying remarks. We have already pointed out that the phase integral in Eq. (11.30) is independent of the curve chosen between the particle and antiparticle positions. This is, however, only true for non-singular fields and *the field we are working with is singular*. It is constant inside the string region and vanishes outside. The sudden change occurs along the curve along which we are integrating.

For the abelian case described above the force on a charge is  $g\mathcal{E}/2$ . It seems reasonable to identify the constant in the integral in Eq. (11.35) with this value. The field from the particle itself (i.e. from the particle on the contour) should not be counted in order to avoid self-interactions.

For the second case, in which the external vacuum fields contribute, we again expect to identify the constant with the true force on the charge. We will therefore use  $\kappa_b$  times the area  $\tilde{A} = A/(2\kappa_b^2)$ , where we have introduced the (lightcone metric) area  $A$  used in the Lund model. Then the real part of  $\xi\tilde{A}$  equals  $A/2\kappa$ .

There should, however, also be imaginary contributions to  $\xi$ . If there is absorption the dielectricity  $\epsilon$  changes from its vacuum value (which in our case is unity); cf. Chapters 2 and 4. This occurs because the vacuum

itself is polarisable in a quantum theory and consequently we expect in accordance with the results in Chapter 4 that

$$\epsilon = 1 + i\eta \quad (11.36)$$

where the imaginary part  $\eta$  equals the absorption rate. In a quantum field theory this absorption rate is  $\pi$  times the pair production rate in the vacuum in the presence of an external field. For QCD with  $n_f$  massless flavors this becomes  $\eta = n_f \alpha_s / 6$  (cf. the calculations of the imaginary part of the vacuum polarisation tensor in Chapter 4).

Therefore we expect that the quantity  $\xi$  in the Wilson area law contains both a real and an imaginary part,

$$\xi = \kappa\epsilon \quad (11.37)$$

We are, however, again in trouble with regard to the interpretation of this. For the abelian case there are no problems in relating the imaginary part of  $\epsilon$  to the production of pairs because we are then only discussing a field stemming from the true charges. The energy density is  $\kappa_a = e^2/2$ , the electric field being generated by the  $q$ - and  $\bar{q}$ -charges.

For the nonabelian case the force  $\kappa$  contains contributions not only from the true charges but also (the same amount) from the vacuum field pressure. On the one hand *the vacuum field should not be allowed to produce pairs, because that would mean that the vacuum state is not stable.* On the other hand when the pairs have been produced (by the true field) then the vacuum pressure may well push them apart during the tunnelling process.

We will then write for  $\xi = \xi_b$ ,

$$\xi_b = \kappa_b \left( 1 + i\frac{\eta}{2} \right) \quad (11.38)$$

The matrix element  $\mathcal{M}$  in Eq. (11.1) can be identified, in the expression in Eq. (11.33), with the ‘true’ area  $A$  used in the Lund model fragmentation functions:

$$\mathcal{M} = \exp \left[ \left( i\kappa - \frac{\kappa l}{2} \right) \tilde{A} \right] = \exp \left[ \left( \frac{i}{2\kappa} - \frac{b'}{2} \right) A \right] \quad (11.39)$$

The parameter  $b'$  is then equal to the  $b$  we obtained in the last section from Schwinger’s persistence probability:

$$b' = \frac{\eta}{2\kappa} = \frac{n_f \alpha}{12\kappa} \quad (11.40)$$

Evidently the matrix element  $\mathcal{M}$  will then fulfil the area law

$$|\mathcal{M}|^2 = \exp(-bA) \quad (11.41)$$

We now have, accepting the considerations above, no ‘fudge-factor’ in the  $b$ -value and we have also a well-defined phase for  $\mathcal{M}$  from Eq. (11.39).

**11.4 The fragmentation formulas and the partition functions for the Feynman-Wilson gas in rapidity space**

*1 Preliminaries and definitions*

We will now rewrite the decay distributions for a finite-energy cluster in terms of the partition functions of statistical physics. In this way we may identify the area suppression law with the well-known Boltzman factor, i.e. the negative exponential of a state energy divided by the temperature.

To this end we introduce in the cms system the rapidity variables instead of the hadron momenta,  $p_{oj}$  (with the index  $o$  for ‘observable’; for simplicity we consider only a single species of hadron with mass  $m$ ):

$$(p_{oj+}, p_{oj-}) = m(\exp y_j, \exp(-y_j)), \quad W_{\pm} = \sqrt{s} = W \quad (11.42)$$

Then we obtain for the hadronic phase space volume element in Eq. (11.1)

$$\prod_{j=1}^n N dy_j \delta \left( W - m \sum_{j=1}^n \exp y_j \right) \delta \left( W - m \sum_{j=1}^n \exp(-y_j) \right) \quad (11.43)$$

The area  $A$  in the exponent in Eq. (11.1) can be expressed in terms of the rapidities in the following way (see Fig. 11.5):

$$A = \left( \sum_{j=1}^n p_{oj-} \right) \left( \sum_{k=j}^n p_{ok+} \right) = m^2 \sum_{j=1}^n \sum_{k=j}^n \exp(y_k - y_j) \quad (11.44)$$

This corresponds to summing systematically the partial areas corresponding to the different particles starting from the positive and going towards the negative lightcone. (The same result is of course obtained by going the opposite way.)

In this way we have exhibited the two-particle correlations explicitly. We note that the area looks very much like a sum of ‘two-body potentials’,  $V(y_j - y_l)$ , in the rapidity differences. This is the way we are going to treat the expression and we define a ‘partition function’  $Z$  by

$$Z = \sum_n Z_n = \sum_n \prod_{j=1}^n N dy_j \delta(\dots) \delta(\dots) \exp \left[ -\frac{\sum V(y_j - y_k)}{kT} \right] \quad (11.45)$$

the exponential factor being given by  $bA$  with  $A$  written as in Eq. (11.44).  $Z$  essentially has the properties of a partition function if the particles are imagined as making up a gas in rapidity space and interacting via the exponential two-body potentials. The hamiltonian in that case is

$$H = \sum T_j(\pi_j) + \sum V(y_j - y_k) \quad (11.46)$$

and the phase space volume element is  $\prod dy_j d\pi_j$ , the quantities  $\pi_j$  being the ‘momenta’ canonically conjugate to the ‘coordinates’  $y_j$ .

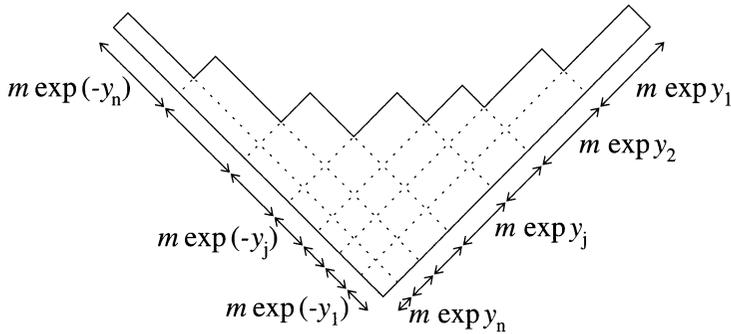


Fig. 11.5. The fragmentation area partitioned into two-particle regions in order to understand how the correlations are produced.

The kinetic energy factors  $T_j$  (which correspond to gaussian integrals for a nonrelativistic gas and can be expressed in terms of Bessel functions for the relativistic case) are integrated out in Eq. (11.45) and incorporated into the constants  $N$ . These factors then play the role of (the exponentials of) ‘chemical potentials’ or ‘fugacities’.

We note that the potentials  $V$  correspond to interactions also between particles distant in rank. Therefore the interaction term is principally of a long-range character. But, owing to the exponential falloff ( $y_j \gg y_k$  if  $j \ll k$ ), it is in practice a good approximation to keep only a few of the near-neighbor terms if the gas is dilute.

The two  $\delta$ -distributions contain the requirement that the ‘gas volume’ should be of the order of  $\log s$ . To see this we may integrate out the rapidities of the first and the last particles in rank to obtain

$$dy_1 dy_n \delta(\dots) \delta(\dots) \simeq \frac{1}{s}, \quad y_1 \simeq -y_n \simeq \log(\sqrt{s}) \tag{11.47}$$

We may in this approximation choose a number  $s_0$  in such a way that

$$\Delta Y \equiv y_1 - y_n = \log(s/s_0) \tag{11.48}$$

and assume that all the particles are kept inside this rapidity ‘volume’.

If we order the particles in rank,  $y_1 > y_2 > \dots > y_n$ , the phase space volume is (the result is most easily obtained by iteration, using the translational invariance and the similarity to the symmetrical integral in Eq. (9.19))

$$\int_{y_n}^{y_1} N dy_2 \int_{y_n}^{y_2} N dy_3 \dots \int_{y_n}^{y_{n-2}} N dy_{n-1} = \frac{(N \Delta Y)^{n-2}}{(n-2)!} \tag{11.49}$$

This approximation of ‘well-orderedness’ in rank and rapidity may seem

drastic, in particular when we remember the results in Chapter 9 that two rank neighbors may well have different rapidity order.

The latter result is true as a local statement, i.e. it may well happen that a few,  $m \ll n$ , pairs are not well-ordered. But if many pairs,  $m \sim n$ , are not well-ordered then many of the exponential potentials will be strongly increasing, i.e. the area suppression in the Lund model will make these contributions small. We are evidently invoking the same arguments as those which were used for the multiperipheral models in Chapter 10.

We next turn to the exponential and the sum of the potentials. We will momentarily go back to the usual Lund model notions and observe that in order to be on the mass shell a particle  $p_j$  produced between two vertices (or rather between two momentum transfers  $q_j, q_{j-1}$  as in the multiperipheral chains in Fig. 10.2) must fulfil

$$m^2 = p_j^2 = (q_j - q_{j-1})^2 = 2Q_j Q_{j-1} \cosh(y_j - y_{j-1}) - Q_j^2 - Q_{j-1}^2 \quad (11.50)$$

in terms of the momentum transfer sizes  $\Gamma_j \equiv Q_j^2 = -q_j^2$  and the rapidities  $y_j$  of the vertices. Expanding these formulas up to second order in the differences  $\delta Q_j = Q_j - Q_{j-1}$  and  $\delta y_j = y_j - y_{j-1}$  we obtain

$$m^2 = (\delta Q_j)^2 + Q^2 (\delta y_j)^2, \quad Q = \frac{Q_j + Q_{j-1}}{2} \quad (11.51)$$

Such a formula occurs in statistical analysis when there are independent variations in the quantities  $\delta Q_j$  and  $Q \delta y_j$ . We interpret the result to imply that the particles gather along a hyperbola with size parameter  $Q$ . They may ‘twist and turn’ independently along the hyperbola ( $Q \delta y$ ) and transversely to it ( $\delta Q$ ); cf. also the results in Chapter 18.

If the particles are distributed along this hyperbola there will be a relation between  $Q$  and the number of particles ( $c$  describes the relative weight between the transverse and longitudinal variations along the hyperbola, with  $c = \sqrt{2}$  if they are equal):

$$m \simeq cQ(y_j - y_{j-1}), \quad nm \simeq cQ\Delta Y \quad (11.52)$$

The area below such a hyperbola is

$$A = Q^2 \Delta Y \simeq \frac{n^2 m^2}{c^2 \Delta Y} \quad (11.53)$$

### 2 An equation of state for an (almost) ideal rapidity gas

We will now use the results for the phase space factor and the area suppression we derived in the last subsection. Thus we obtain for the term  $Z_n$  in Eq. (11.45), using the Stirling approximation for the factorial in the

denominator of Eq. (11.49) and assuming  $n \gg 1$ :

$$Z_n = \frac{\exp \Phi_n}{s} \quad (11.54)$$

$$\Phi_n = n \log(N\Delta Y) - n \log n + n - \frac{n^2 b m^2}{c^2 \Delta Y}$$

As a function of  $n$  the the exponent  $\Phi_n$  has a maximum for

$$n \equiv \bar{n} = R\Delta Y \quad \text{with} \quad \Phi_{\bar{n}} = \left( R + \frac{R^2 b m^2}{c^2} \right) \Delta Y \quad (11.55)$$

where the parameter  $R$  is determined from

$$\frac{2R b m^2}{c^2} = \log \left( \frac{N}{R} \right) \quad (11.56)$$

In this way we obtain as a result for  $Z \simeq Z_{\bar{n}}$  that as a function of the squared cms energy  $s$  it is

$$Z \sim s^a, \quad a = R - 1 + \frac{R^2 b m^2}{c^2} \quad (11.57)$$

The parameter  $R$  evidently corresponds to the density of particles in rapidity and it is worthwhile to relate it to our earlier results in Chapter 9, cf. Eq. (9.15). In that case we have derived that the inverse density is given by the average value of  $\log 1/(1-z)$ , the average taken over the fragmentation function  $f(z)$ . We remember from Eq. (9.11) that  $\langle \log 1/(1-z) \rangle$  corresponds to a typical rapidity difference in the cascade. For the Lund model fragmentation function it is straightforward to derive a formula for this rapidity difference:

$$-N \frac{\partial}{\partial a} \left( \frac{1}{N} \right) = N \int \frac{dz}{z} \log \left( \frac{1}{1-z} \right) (1-z)^a \exp(-b m^2/z) \equiv \langle \log 1/(1-z) \rangle \quad (11.58)$$

where we have used the normalisation condition

$$\frac{1}{N} = \int \frac{dz}{z} (1-z)^a \exp(-b m^2/z) \quad (11.59)$$

Therefore we need the logarithmic derivative of the normalisation constant with respect to the parameter  $a$  with the parameter  $b$  fixed. From Eqs. (11.56) and (11.57) it is straightforward to show that

$$\frac{\partial R}{\partial a} + 2R \frac{\partial R}{\partial a} \frac{b m^2}{c^2} = 1 \quad (11.60)$$

$$\frac{\partial \log N}{\partial a} = \frac{\partial \log R}{\partial a} + 2 \frac{\partial R}{\partial a} \frac{b m^2}{c^2} = \frac{1}{R}$$

where we have used the result of the first equation to obtain the expected

result in the second one. Note that the introduction of  $R$  in Eqs. (11.55) and (11.56) stems from a ‘global’ result, i.e. it is obtained from (an approximation of) the total area law for the average state, while the particle density as defined by Eq. (11.58) is a local result, defined from the fragmentation function for a single particle in the Lund model.

The identification of  $Z$  with the maximal term in the sum is a good approximation when  $n$  is very large (as in an ordinary gas). In our case we can estimate not only the largest contribution to  $Z$  but also the width of the ‘multiplicity distribution’, i.e.  $P_n = Z_n/Z$ .

This is obtained if we expand the exponential  $\Phi_n$  in a Taylor series to second order in  $n$  (treating  $n$  as a continuous variable):

$$\Phi_n \simeq \Phi_{\bar{n}} + (n - \bar{n}) \left. \frac{d\Phi}{dn} \right|_{n=\bar{n}} + \frac{(n - \bar{n})^2}{2} \left. \frac{d^2\Phi}{dn^2} \right|_{n=\bar{n}} \tag{11.61}$$

The first-order term vanishes due to the choice of  $\bar{n}$  as the maximum value and we obtain as an approximation for  $\Phi_n$  to second order:

$$\Phi_n = \Phi_{\bar{n}} - \frac{(n - \bar{n})^2}{2\mathcal{V}} \tag{11.62}$$

In this gaussian approximation we can identify the width  $\mathcal{V}$  of the distribution, i.e. the inverse of the coefficient of  $(n - \bar{n})^2$ , to be  $(n - \bar{n})^2$ , the variance in  $n$ :

$$\mathcal{V} = \overline{n^2} - (\bar{n})^2 = \frac{\bar{n}c^2}{c^2 + 2bm^2R} \tag{11.63}$$

We conclude that the multiplicity width in the Lund model should be somewhat more narrow than the predictions from a Poissonian distribution (where the width is  $\mathcal{V}_p = \bar{n}$ ). This is also true if we produce a single species of hadron and neglect transverse momentum fluctuations.

If these are taken into account together with the various hadrons, resonance decays and gluon radiation etc. occurring in the experiments then the multiplicity width (for the decay of a single string) behaves rather like  $\bar{n}^2$  as we will see later. The most essential contribution for large energies is that of the gluon radiation (but we are then no longer in a single space-dimensional setting).

There is, according to statistical mechanics, a simple relationship between the grand canonical partition function and the properties of the gas:

$$\log Z \simeq \Phi_{\bar{n}} = \frac{PV}{kT} \tag{11.64}$$

Our treatment in Eq. (11.55) basically corresponds to the first two terms in the virial expansion in the particle density  $R = \bar{n}/V$ , where  $V = \log(s/s_0)$

(although our treatment of the fluctuations around the mean hyperbola by means of the single parameter  $c$  is probably too cavalier).

It is of course possible to consider the virial expansion to higher orders and to calculate different quantities for the rapidity gas, such as its entropy etc., but that will be left for the interested reader.