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Orthodox bands of modules

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In this paper we shall consider orthodox bands of commutative groups, together with a ring of endomorphisms. We shall generalize the concept of a left module by introducing orthodox bands of left modules; we shall also deal with linear mappings, the transpose of a linear mapping and with the dual of an orthodox band of left modules.

We shall use the notations and terminology of [1] and [3].

1.

DEFINITION. Let $R, +, \circ$ be a ring with zero element 0 and identity 1. Let S be a semigroup and $R \times S \to S$, $(\alpha, x) \longmapsto \alpha x$ a mapping satisfying the following conditions:

(i) $\alpha(xy) = (\alpha x)(\alpha y)$ for every $\alpha \in R$ and every $x, y \in S$,

(ii) $(\alpha+\beta)x = (\alpha x)(\beta x)$ for every $\alpha, \beta \in R$ and every $x \in S$,

(iii) $(\alpha \circ \beta)x = \alpha(\beta x)$ for every $\alpha, \beta \in R$ and every $x \in S$,

(iv) $1x = x$ for every $x \in S$.

The structure defined this way will be called an orthodox band of left R-modules. The next theorem justifies our terminology.

2.

THEOREM 1. *Let R, S and the mapping $R \times S \to S$ be as in the definition of Section 1. Then S is an orthodox band of abelian groups and the maximal subgroups of S are left invariant by the elements of R.*

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Proof. Let x be any element of S, and α any element of R; we then have

$$(0x)(0x) = (0+0)x = 0x ,$$

$$(\alpha x)(0x) = (\alpha+0)x = \alpha x = (0+\alpha)x = (0x)(\alpha x) ,$$

$$(\alpha x)\big((-\alpha)x\big) = (\alpha-\alpha)x = 0x = (-\alpha+\alpha)x = \big((-\alpha)x\big)(\alpha x) .$$

This implies that for any $\alpha \in R$ and any $x \in S$, αx belongs to the maximal subgroup of S with identity $0x$; the inverse of αx in this maximal subgroup must be $(-\alpha)x$. More specifically $1x = x$ belongs to the maximal subgroup of S with identity $0x$, and its inverse in this maximal subgroup must be $(-1)x$. We conclude that S must be a completely regular semigroup and that all maximal subgroups of S are left invariant by the elements of R.

For every $x, y \in S$ we have

$$(xy)(xy) = (1+1)(xy) = \big((1+1)x\big)\big((1+1)y\big) = x^2y^2 .$$

Let e, f be any idempotents of S, then the foregoing implies that $(ef)^2 = e^2f^2 = ef$; hence $E_S = \{x \in S \mid x^2 = x\}$ must be a subsemigroup of S. Let x and y belong to the same maximal subgroup of S; then the foregoing implies

$$xy = \big((-1)x\big)x^2y^2\big((-1)y\big) = \big((-1)x\big)xyxy\big((-1)y\big) = yx ;$$

hence S is a union of abelian groups. We have yet to prove that S is an orthodox union of abelian groups [2].

Let e and f be any idempotents of S, and $x \in H_e$, $y \in H_f$. We put $(-1)x = x'$ and $(-1)y = y'$. Then

$$ef = (ef)^2 = (1+1)(ef) = (1+1)\big(x(x'f)\big) = x^2(x'f)^2 = x^2x'fx'f = (xf)(x'f)$$

and analogously

$$ef = (x'f)(xf) .$$

Since ef, $x'f$, and xf are elements of the rectangular group D_{ef} [2], the foregoing implies that xf and $x'f$ are mutually inverse elements of the maximal subgroup H_{ef}. Dually, ey and ey' are mutually inverse elements of the maximal subgroup H_{ef} . Since $(xy)y' = xf$ and $(xf)y = xy$ we have $xyRxf$; hence $xyRef$. Analogously, since $x'(xy) = ey$ and $x(ey) = xy$ we have $xyLey$; hence $xyLef$. We conclude that $xyHef$. Green's relation H must then be a congruence on S . Thus S is an orthodox band of commutative groups [2].

3.

REMARK. Let S be an orthodox band of abelian groups. Then, by Yamada's Theorem ([2] and [10]), there exists a band E and a semilattice of abelian groups Q , both having the same structure semilattice Y , such that S is the spined product of Q and E over Y : $S = Q \times_Y E$. Let $Q = \bigcup_{\kappa \in Y} G_\kappa$ and $E = \bigcup_{\kappa \in Y} E_\kappa$; then S consists of ordered pairs (x_κ, e_κ) , $\kappa \in Y$, $x_\kappa \in G_\kappa$, $e_\kappa \in E_\kappa$. Multiplication is defined by

$$(x_\lambda, e_\lambda)(y_\mu, f_\mu) = (x_\lambda y_\mu, e_\lambda f_\mu)$$

for any $\lambda, \mu \in Y$, $x_\lambda \in G_\lambda$, $y_\mu \in G_\mu$, $e_\lambda \in E_\lambda$, $f_\mu \in E_\mu$. The identity element of G_κ , $\kappa \in Y$, will be denoted by 1_κ .

The following result will generalize a theorem of [4] about semilattices of left modules. By combining the next theorem and Theorem 1, we obtain a characterization of orthodox bands of abelian groups.

4.

THEOREM 2. *Let S be any orthodox band of abelian groups, and let $\mathbb{Z}$ be the ring of integers. Let e be any idempotent of S , and x and x' mutually inverse elements of the maximal subgroup H_e . Define the mapping $\mathbb{Z} \times S \to S$, $(k, x) \mapsto kx$ by*

$$\begin{aligned} kx &= x^k \quad \text{if } k > 0 \\ &= e \quad \text{if } k = 0 \\ &= x'^{-k} \quad \text{if } k < 0 . \end{aligned}$$

Then S is an orthodox band of left $\mathbb{Z}$-modules.

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Proof. Conditions (i), (ii), (iii), and (iv) of the definition in Section 1 are checked by some easy calculations.

5.

DEFINITIONS and REMARKS. Let S be an orthodox band of left R-modules, and τ a congruence on the semigroup S. The natural homomorphism of S onto S/τ will be denoted by $\tau^{\#}$. τ will be called R-stable if and only if $x\tau y$ implies $(\alpha x)\tau(\alpha y)$ for every $x, y \in S$ and every $\alpha \in R$; we can then define a mapping $R \times (S/\tau) \to S/\tau$ by $(\alpha, \bar{x}) \mapsto \alpha\bar{x} = \overline{\alpha x}$; S/τ will then be an orthodox band of left R-modules.

Let S and T be orthodox bands of left R-modules. The mapping $\Phi : S \to T$ will be called R-linear if and only if

(i) $\Phi(xy) = (\Phi x)(\Phi y)$ for every $x, y \in S$,

(ii) $\Phi(\alpha x) = \alpha\Phi(x)$ for every $x \in S$ and every $\alpha \in R$.

$\Phi(S)$ will then be an orthodox band of left R-modules.

The subset A of S will be called R-stable if and only if $\alpha x \in A$ for every $x \in A$ and every $\alpha \in R$. If Φ is an R-linear mapping of S into T, $\Phi(S)$ will be an R-stable subsemigroup of T, and the kernel of Φ will be an R-stable subsemigroup of S. Any R-stable subsemigroup of an orthodox band of left R-modules must of course be an orthodox band of left R-modules. If τ is an R-stable congruence on S, the union of all τ-classes containing an idempotent will be an R-stable subsemigroup of S.

The mapping $\Phi : S \to T$ will be R-linear if and only if $\Phi^{-1}\Phi$ is an R-stable congruence on S. The equivalence relation τ on S is an R-stable congruence if and only if $\tau^{\#}$ is an R-linear mapping. The mapping $\Phi : S \to E_S$, $x \mapsto 0x$ is an R-linear mapping of S onto the band consisting of all idempotents of S; $\Phi^{-1}\Phi$ is then the R-stable congruence H.

Let S be the spined product of a semilattice of abelian groups Q and a band E; we shall use the same notation as in the remark of Section 3. Q is the greatest inverse semigroup homomorphic image of S, and the mapping $\Delta : S \to Q$, $(x_k, e_k) \mapsto x_k$ is a homomorphism of S onto Q ; we shall put $\Delta^{-1}\Delta = \sigma$; this congruence σ is the minimal inverse semigroup congruence on S , and we shall show that σ is R-stable. Let G be the greatest group homomorphic image of Q , and $\Gamma : Q \to G$, $x_\kappa \mapsto \tilde{x}_\kappa$ be a homomorphism of Q onto G , $\Gamma^{-1}\Gamma$ being the minimal group congruence on Q ; if x_λ and y_μ are any elements of Q , then $x_\lambda \Gamma^{-1}\Gamma y_\mu$ if and only if there exists a $\kappa \in Y$, $\kappa \leq \lambda \wedge \mu$, such that $x_\lambda 1_\kappa = y_\mu 1_\kappa$; we shall put $(\Gamma\Delta)^{-1}(\Gamma\Delta) = \rho$; this congruence ρ is the minimal group congruence on S , and we shall show that ρ is R-stable.

6.

THEOREM 3. *The minimal inverse semigroup congruence on an orthodox band of left R-modules is R-stable.*

Proof. Let x_κ be any element of Q , and let us take any two elements (x_κ, e_κ) and (x_κ, f_κ) in $\Delta^{-1}x_\kappa$. Let α be any element of R . Since H is an R-stable congruence on S , $\alpha(x_\kappa, e_\kappa)$ belongs to the H-class $G_\kappa \times e_\kappa$ of S containing (x_κ, e_κ) ; hence $\alpha(x_\kappa, e_\kappa) = (y_\kappa, e_\kappa)$ for some $y_\kappa \in G_\kappa$. Analogously, $\alpha(x_\kappa, f_\kappa) = (z_\kappa, f_\kappa)$ for some $z_\kappa \in G_\kappa$. Let $(1_\kappa, g_\kappa)$ be L-related with $(1_\kappa, e_\kappa)$ and R-related with $(1_\kappa, f_\kappa)$, and let $(1_\kappa, h_\kappa)$ be R-related with $(1_\kappa, e_\kappa)$ and L-related with $(1_\kappa, f_\kappa)$. Since by the restriction of $R \times S \to S$ to $R \times (G_\kappa \times g_\kappa)$ and $R \times (G_\kappa \times h_\kappa)$, respectively, $G_\kappa \times g_\kappa$ and $G_\kappa \times h_\kappa$ become left R-modules, we must have $\alpha(1_\kappa, g_\kappa) = (1_\kappa, g_\kappa)$ and $\alpha(1_\kappa, h_\kappa) = (1_\kappa, h_\kappa)$. Furthermore, we have

$$\begin{aligned}(z_\kappa, e_\kappa) &= (1_\kappa, h_\kappa)(z_\kappa, f_\kappa)(1_\kappa, g_\kappa) \\ &= (\alpha(1_\kappa, h_\kappa))(\alpha(x_\kappa, f_\kappa))(\alpha(1_\kappa, g_\kappa)) \\ &= \alpha((1_\kappa, h_\kappa)(x_\kappa, f_\kappa)(1_\kappa, g_\kappa)) \\ &= \alpha(x_\kappa, e_\kappa) = (y_\kappa, e_\kappa) ;\end{aligned}$$

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hence $z_\kappa = y_\kappa$, and $\Delta(\alpha(x_\kappa, e_\kappa)) = \Delta(\alpha(x_\kappa, f_\kappa))$.

7.

COROLLARY 1. *By the mapping* $R \times Q \to Q$,

$$(\alpha, x_\kappa) \mapsto \alpha x_\kappa = \Delta\left(\alpha\Delta^{-1}x_\kappa\right) ,$$

Q becomes a semilattice of left R-modules, and Δ an R-linear mapping of S onto Q .

8.

COROLLARY 2. *Let Q be any semilattice of left R-modules, and Y the structure semilattice of Q ; let E be a band with the same structure semilattice Y ; let $\bigcup_{\kappa\in Y} G_\kappa$ and $\bigcup_{\kappa\in Y} E_\kappa$ be the semilattice decompositions of Q and E respectively; let S be the spined product $Q \times_Y E$ of Q and E over Y . By the mapping $R \times S \to S$, $(\alpha, (x_\kappa, e_\kappa)) \mapsto (\alpha x_\kappa, e_\kappa)$ for every $\alpha \in R$, and every $\kappa \in Y$, $x_\kappa \in G_\kappa$, $e_\kappa \in E_\kappa$, S becomes an orthodox band of left R-modules. Conversely, any orthodox band of left R-modules can be so constructed.*

9.

COROLLARY 3. *Let S be an orthodox normal band of left R-modules, and let $S = \bigcup_{\kappa\in Y} S_\kappa$ be the semilattice decomposition of S . For any λ , $\mu \in Y$, $\lambda \geq \mu$, the structure homomorphism $\Psi_{\lambda,\mu}$ is an R-linear mapping of the orthodox rectangular band of left R-modules S_λ into the orthodox rectangular band of left R-modules S_μ .*

Proof. In a semilattice of left R-modules the structure homomorphisms are R-linear [*6*]. The theorem now follows from Corollary 2 and from a result about normal bands [*11*].

10.

REMARK. Structure theorems for semilattices of left R-modules [*6*],

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together with Corollary 2 yield structure theorems for bands of left R-modules.

11.

THEOREM 4. *The minimal group congruence on an orthodox band of left R-modules is R-stable.*

Proof. Let $\tilde{x}_\lambda$ be any element of G, the greatest group homomorphic image of an orthodox band of left R-modules S. Let us take any two elements x_λ and y_μ in $\Gamma^{-1}\tilde{x}_\lambda$. There exists a $\kappa \in Y$, $\kappa \leq \lambda \wedge \mu$, such that $1_\kappa x_\lambda = 1_\kappa y_\mu$. Let α be any element of R. From

$$(\alpha x_\lambda)1_\kappa = (\alpha x_\lambda)(\alpha 1_\kappa) = \alpha(x_\lambda 1_\kappa) = \alpha(y_\mu 1_\kappa) = (\alpha y_\mu)(\alpha 1_\kappa) = (\alpha y_\mu)1_\kappa ,$$

and $\alpha x_\lambda \in G_\lambda$, $\alpha y_\mu \in G_\mu$, we conclude that $\alpha y_\mu \in \Gamma^{-1}\Gamma(\alpha x_\lambda)$, and thus $\widetilde{\alpha x}_\lambda = \widetilde{\alpha y}_\mu$. This implies that the minimal group congruence $\Gamma^{-1}\Gamma$ on Q must be R-stable; consequently, the minimal group congruence $(\Gamma\Delta)^{-1}\Gamma\Delta = \rho$ on S must be R-stable.

12.

COROLLARY 4. *By the mapping $R \times G \to G$, $(\alpha, \tilde{x}_\kappa) \mapsto \alpha\tilde{x}_\kappa = \widetilde{\alpha x}_\kappa$, G becomes a left R-module, and the mapping $\Gamma\Delta$ an R-linear mapping of S onto G.*

13.

DEFINITIONS. An orthodox band of right R-modules S can be defined in a way analogous to the way an orthodox band of left R-modules is defined. Condition (iii) of the definition in Section 1 must then be replaced by (iii)'; $(\alpha \circ \beta)x = \beta(\alpha x)$ for every $\alpha, \beta \in R$ and every $x \in S$. It will be more convenient to denote the mapping $R \times S \to S$ by $(\alpha, x) \mapsto x\alpha$; (iii)' then becomes

(iii)' $x(\alpha \circ \beta) = (x\alpha)\beta$ for every $\alpha, \beta \in R$ and every $x \in S$.

If S is at the same time an orthodox band of left R-modules, and an

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orthodox band of right R-modules, then we shall say that S is an orthodox band of R-bimodules.

Let $R^\infty = R \cup \{\infty\}$, and define addition in R^∞ as follows: for any $\alpha, \beta \in R$ we put

$$\alpha + \beta = \gamma \text{ in } R^\infty \text{ if and only if } \alpha + \beta = \gamma \text{ in } R ,$$

and we put

$$\alpha + \infty = \infty + \alpha = \infty .$$

R^∞ will be a group with "zero" ∞ . We next define the mapping $R \times R^\infty \to R^\infty$ by agreeing that for α, β in R ,

$$(\alpha, \beta) \mapsto \alpha\beta = \gamma \text{ if and only if } \alpha \circ \beta = \gamma \text{ in } R ,$$

and that

$$(\alpha, \infty) \mapsto \alpha\infty = \infty .$$

We also define the mapping $R \times R^\infty \to R^\infty$ by setting, for α, β in R ,

$$(\alpha, \beta) \mapsto \beta\alpha = \gamma \text{ if and only if } \beta \circ \alpha = \gamma \text{ in } R ,$$

and

$$(\alpha, \infty) \mapsto \infty\alpha = \infty .$$

By these two mappings R^∞ becomes a semilattice of R-bimodules, the structure semilattice being the two element semilattice. We shall use R^∞ later in this paper.

The next theorem generalizes a result of [9].

14.

THEOREM 5. *Let S be an orthodox band of left R-modules, and T an orthodox band of right R-modules. Let $I_{S,T}$ be the set of all partial mappings of S into T . Define a multiplication in $I_{S,T}$ as follows: for every $\Phi, \Psi \in I_{S,T}$, $\operatorname{dom} \Phi\Psi = \operatorname{dom} \Phi \cap \operatorname{dom} \Psi$, and for every $x \in \operatorname{dom} \Phi\Psi$ we put $\Phi\Psi(x) = (\Phi x)(\Psi x)$. Define the mapping $R \times I_{S,T} \to I_{S,T}$, $(\alpha, \Phi) \mapsto \Phi\alpha$ by $\operatorname{dom}(\Phi\alpha) = \operatorname{dom} \Phi$ and $(\Phi\alpha)x = (\Phi x)\alpha$, for every $x \in \operatorname{dom} \Phi$. $I_{S,T}$ will then be an orthodox band of right R-modules. $I_{S,T}$ will be a semilattice of right R-modules if and only if*

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T is a semilattice of right R-modules.

Proof. For any $\Phi, \Psi \in I_{S,T}$ and any $\alpha \in R$ we have

$$\mathrm{dom}\big((\Phi\Psi)\alpha\big) = \mathrm{dom}\ \Phi\Psi = \mathrm{dom}\ \Phi \cap \mathrm{dom}\ \Psi = \mathrm{dom}(\Phi\alpha) \cap \mathrm{dom}(\Psi\alpha) = \mathrm{dom}\big((\Phi\alpha)(\Psi\alpha)\big) ,$$

and for any $x \in \mathrm{dom}(\Phi\Psi)\alpha$ we have

$$\big((\Phi\Psi)\alpha\big)x = \big((\Phi\Psi)x\big)\alpha = \big((\Phi x)(\Psi x)\big)\alpha = \big((\Phi x)\alpha\big)\big((\Phi x)\alpha\big) = \\ = \big((\Phi\alpha)x\big)\big((\Phi\alpha)x\big) = \big((\Phi\alpha)(\Psi\alpha)\big)x ;$$

hence $(\Phi\Psi)\alpha = (\Phi\alpha)(\Psi\alpha)$. For any $\Phi \in I_{S,T}$ and any $\alpha, \beta \in R$ we have

$$\mathrm{dom}\big(\Phi(\alpha+\beta)\big) = \mathrm{dom}\ \Phi = \mathrm{dom}(\Phi\alpha) \cap \mathrm{dom}(\Phi\beta) = \mathrm{dom}\big((\Phi\alpha)(\Phi\beta)\big) ,$$

and for any $x \in \mathrm{dom}\big(\Phi(\alpha+\beta)\big)$ we have

$$\big(\Phi(\alpha+\beta)\big)x = (\Phi x)(\alpha+\beta) = \big((\Phi x)\alpha\big)\big((\Phi x)\beta\big) = \big((\Phi\alpha)x\big)\big((\Phi\beta)x\big) = \big((\Phi\alpha)(\Phi\beta)\big)x ;$$

hence $\Phi(\alpha+\beta) = (\Phi\alpha)(\Phi\beta)$. Furthermore,

$$\mathrm{dom}\big(\Phi(\alpha \circ \beta)\big) = \mathrm{dom}\ \Phi = \mathrm{dom}(\Phi\alpha) = \mathrm{dom}\big((\Phi\alpha)\beta\big) ,$$

and for any $x \in \mathrm{dom}\big(\Phi(\alpha \circ \beta)\big)$ we have

$$\big(\Phi(\alpha \circ \beta)\big)x = (\Phi x)(\alpha \circ \beta) = \big((\Phi x)\alpha\big)\beta = \big((\Phi\alpha)x\big)\beta = \big((\Phi\alpha)\beta\big)x ;$$

hence $\Phi(\alpha \circ \beta) = (\Phi\alpha)\beta$. Finally, $\mathrm{dom}(\Phi 1) = \mathrm{dom}\ \Phi$, and for any $x \in \mathrm{dom}(\Phi 1)$ we have

$$(\Phi 1)x = (\Phi x)1 = \Phi x ;$$

hence $\Phi 1 = \Phi$. We conclude that $I_{S,T}$ is an orthodox band of right R-modules.

From the definition of the multiplication in $I_{S,T}$ it follows that $I_{S,T}$ is commutative if and only if T is commutative. From this follows the last part of the theorem.

15.

THEOREM 6. *Let S be an orthodox band of left R-modules, S' the set of R-linear mappings of S into R, and S^* the set of R-linear mappings of S into R^∞. Then S' is an R-stable subsemigroup of $I_{S,R}$ and S^* is an R-stable subsemigroup of I_{S,R^∞}.*

Proof. We show that S^* is an R-stable subsemigroup of I_{S,R^∞}; the proof of the rest is similar. Let x^* and y^* be any elements of S^*; since R^∞ is a semilattice of commutative groups, x^*y^* must be a homomorphism of S into R^∞. For any $x \in S$ and any $x^* \in S^*$ we shall from now on put $x^*(x) = \langle x, x^*\rangle$. For any $x \in S$, any $\alpha \in R$, and any $x^*, y^* \in S^*$ we then have

$$\begin{aligned}\langle \alpha x, x^*y^*\rangle &= \langle \alpha x, x^*\rangle + \langle \alpha x, y^*\rangle \\ &= \alpha\langle x, x^*\rangle + \alpha\langle x, y^*\rangle \\ &= \alpha(\langle x, x^*\rangle + \langle x, y^*\rangle) \\ &= \alpha\langle x, x^*y^*\rangle .\end{aligned}$$

We conclude that for any $x^*, y^* \in S^*$, x^*y^* must be an R-linear mapping of S into R^∞; hence $x^*y^* \in S^*$. S^* is a subsemigroup of I_{S,R^∞}.

For any $x, y \in S$, any $x^* \in S^*$, and any $\alpha \in R$ we have

$$\begin{aligned}\langle xy, x^*\alpha\rangle &= \langle xy, x^*\rangle\alpha \\ &= (\langle x, x^*\rangle + \langle y, x^*\rangle)\alpha \\ &= \langle x, x^*\rangle\alpha + \langle y, x^*\rangle\alpha \\ &= \langle x, x^*\alpha\rangle + \langle y, x^*\alpha\rangle ;\end{aligned}$$

hence $x^*\alpha$ must be a homomorphism of S into R^∞. For any $x \in S$, any $x^* \in S^*$, and any $\alpha, \beta \in R$ we have

$$\begin{aligned}\langle \beta x, x^*\alpha\rangle &= \langle \beta x, x^*\rangle\alpha \\ &= \beta\langle x, x^*\rangle\alpha \\ &= \beta\langle x, x^*\alpha\rangle .\end{aligned}$$

We conclude that for any $x^* \in S^*$ and any $\alpha \in R$, $x^*\alpha$ must be an R-linear mapping of S into R^∞. Consequently S^* must be an R-stable subsemigroup of I_{S,R^∞}.

16.

COROLLARY 5. *S^* is a semilattice of right R-modules. The structure semilattice of S^* is isomorphic with the $\cup$-semilattice of prime ideals of S. The mapping $1^* : S \to R^\infty$, $x \mapsto 0$ is the identity of S^* and the mapping $0^* : S \to R^\infty$, $x \mapsto \infty$ is the zero of S^*.*

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Proof. R^{∞} is a semilattice of right R-modules; hence $I_{S,R^{\infty}}$ is a semilattice of right R-modules. Since S^* is R-stable in $I_{S,R^{\infty}}$, S^* must also be a semilattice of right R-modules.

Let e^* be any idempotent of S^*; then $V_{e^*} = \{x \in S \parallel \langle x, e^*\rangle = \infty\}$ is a prime ideal of S. For any $x \in S \backslash V_{e^*}$, $\langle x, e^*\rangle \in R$ and $\langle x, e^*\rangle = \langle x, e^{*2}\rangle = \langle x, e^*\rangle + \langle x, e^*\rangle$; hence $\langle x, e^*\rangle = 0$. Conversely, let P be any prime ideal of S; then we can define $e_P^* \in S^*$ by $\langle x, e_P^*\rangle = \infty$ for all $x \in P$, and $\langle x, e_P^*\rangle = 0$ for all $x \in S\backslash P$. Furthermore, if e^* and f^* are any two idempotents of S^*, we must have $V_{e^*f^*} = V_{e^*} \cup V_{f^*}$. Consequently, the semilattice E_{S^*} consisting of the idempotents of S^* is isomorphic with the $\cup$-semilattice of all prime ideals of S. Since E_{S^*} is isomorphic with the structure semilattice of S^*, the result stated in the corollary follows.

17.

COROLLARY 6. *S' is a right R-module which is an R-stable subgroup of S^*; S' is the maximal submodule of S^* containing the identity 1^* of S^*.*

Proof. All elements of S' are R-linear mappings of S into R; hence they can be considered as R-linear mappings of S into R^{∞}, and consequently $S' \subseteq S^*$. Since S' is R-stable in $I_{S,R}$, and since clearly $I_{S,R}$ is R-stable in $I_{S,R^{\infty}}$, S' must be R-stable in $I_{S,R^{\infty}}$; from this we infer that S' is R-stable in S^*.

It is evident that $1^* : S \to R$, $x \mapsto 0$ is the identity of S'. Let x^* be any element of S'; then $x^*(-1) \in S'$, and for any $x \in S$ we have

$$\begin{aligned}\langle x, x^*\big(x^*(-1)\big)\rangle &= \langle x, x^*\rangle + \langle x, x^*(-1)\rangle \\ &= \langle x, x^*\rangle + \langle x, x^*\rangle(-1) = 0 ,\end{aligned}$$

and analogously

$$\langle x, \big(x^*(-1)\big)x^*\rangle = 0 ;$$

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hence $x^*\big(x^*(-1)\big) = \big(x^*(-1)\big)x^* = 1^*$. This shows that x^* and $x^*(-1)$ are mutually inverse elements of the commutative group H_{1^*} , the maximal subgroup of S^* containing 1^* . For any element $y^* \in H_{1^*}$, we must have $V_{y^*} = \emptyset$; hence any element $y^* \in H_{1^*}$ belongs to S' . We can conclude that $H_{1^*} = S'$.

18.

THEOREM 7. *Let S be an orthodox band of left R-modules and τ any R-stable congruence on S . The mapping $\Phi : (S/\tau)^* \to S^*$, $\bar{x}^* \mapsto \Phi\bar{x}^*$ defined by $\langle x, \Phi\bar{x}^*\rangle = \langle \tau^{\#}x, \bar{x}^*\rangle$ for every $x \in S$ is an R-isomorphism of $(S/\tau)^*$ into S^* . Whenever $\iota_S \subseteq \tau \subseteq \sigma$, σ being the minimal inverse semigroup congruence on S , this mapping Φ is a surjective R-isomorphism of $(S/\tau)^*$ onto S^* .*

Proof. Let us suppose that $\bar{x}^*$, $\bar{y}^*$ are any elements of $(S/\tau)^*$, and x any element of S ; we then have

$$\begin{aligned}\langle x, \Phi(\bar{x}^*\bar{y}^*)\rangle &= \langle \tau^{\#}x, \bar{x}^*\bar{y}^*\rangle \\ &= \langle \tau^{\#}x, \bar{x}^*\rangle + \langle \tau^{\#}x, \bar{y}^*\rangle \\ &= \langle x, \Phi\bar{x}^*\rangle + \langle x, \Phi\bar{y}^*\rangle \\ &= \langle x, (\Phi\bar{x}^*)(\Phi\bar{y}^*)\rangle ;\end{aligned}$$

hence $\Phi(\bar{x}^*\bar{y}^*) = (\Phi\bar{x}^*)(\Phi\bar{y}^*)$. Let us suppose that $\bar{x}^*$ is any element of $(S/\tau)^*$, α any element of R , and x any element of S ; then

$$\begin{aligned}\langle x, \Phi(\bar{x}^*\alpha)\rangle &= \langle \tau^{\#}x, \bar{x}^*\alpha\rangle \\ &= \langle \tau^{\#}x, \bar{x}^*\rangle\alpha \\ &= \langle x, \Phi\bar{x}^*\rangle\alpha \\ &= \langle x, (\Phi\bar{x}^*)\alpha\rangle ;\end{aligned}$$

hence $\Phi(\bar{x}^*\alpha) = (\Phi\bar{x}^*)\alpha$. Since $\tau^{\#}$ is an R-linear mapping of S onto S/τ , $\Phi\bar{x}^* \in S^*$ for any $\bar{x}^* \in (S/\tau)^*$. We conclude that Φ is an R-linear mapping of $(S/\tau)^*$ into S^* . Let us now suppose that $\bar{x}^*, \bar{y}^* \in (S/\tau)^*$, and $\Phi\bar{x}^* = \Phi\bar{y}^*$; if for some $\bar{x} \in S/\tau$, $\langle \bar{x}, \bar{x}^*\rangle \neq \langle \bar{x}, \bar{y}^*\rangle$, then for any $x \in \left(\tau^{\#}\right)^{-1}\bar{x}$ we should have

$$\langle x, \Phi\bar{x}^*\rangle = \langle \tau^{\#}x, \bar{x}^*\rangle$$
$$= \langle \bar{x}, \bar{x}^*\rangle$$
$$\neq \langle \bar{x}, \bar{y}^*\rangle = \langle \tau^{\#}x, \bar{y}^*\rangle = \langle x, \Phi\bar{y}^*\rangle ,$$

and this is impossible. We conclude that $\Phi\bar{x}^* = \Phi\bar{y}^*$ implies $\bar{x}^* = \bar{y}^*$; hence Φ is an isomorphism of $(S/\tau)^*$ into S^* .

It will be sufficient to show that the mapping $\Phi : (S/\sigma)^* \to S^*$, $\bar{x}^* \mapsto \Phi\bar{x}^*$ defined by $\langle x, \Phi x^*\rangle = \langle \sigma^{\#}x, \bar{x}^*\rangle$ for every $x \in S$, will be an R-isomorphism of $(S/\sigma)^*$ onto S^* . Let x^* be any element of S^* , and $\left(x_\kappa, e_\kappa\right)$ and $\left(x_\kappa, f_\kappa\right)$ any two σ-related elements of S . Since $\left(x_\kappa, e_\kappa\right)$ and $\left(x_\kappa, f_\kappa\right)$ are $\mathcal{D}$-related in S , they generate the same principal ideal of S , and thus $\langle \left(x_\kappa, e_\kappa\right), x^*\rangle = \infty$ if and only if $\langle \left(x_\kappa, f_\kappa\right), x^*\rangle = \infty$. Let us now suppose that $\left(x_\kappa, e_\kappa\right)$ and $\left(x_\kappa, f_\kappa\right)$ both belong to $S\backslash V_{x^*}$; let $\left(1_\kappa, g_\kappa\right)$ be $\mathcal{L}$-related with $\left(x_\kappa, e_\kappa\right)$ and $\mathcal{R}$-related with $\left(1_\kappa, f_\kappa\right)$, and $\left(1_\kappa, h_\kappa\right)$ $\mathcal{R}$-related with $\left(x_\kappa, e_\kappa\right)$ and $\mathcal{L}$-related with $\left(1_\kappa, f_\kappa\right)$; $\left(1_\kappa, g_\kappa\right)$ and $\left(1_\kappa, h_\kappa\right)$ are both $\mathcal{D}$-related with $\left(x_\kappa, e_\kappa\right)$ and $\left(x_\kappa, f_\kappa\right)$; hence $\left(1_\kappa, g_\kappa\right), \left(1_\kappa, h_\kappa\right) \in S\backslash V_{x^*}$. Since these two elements are idempotents of S , and since x^* is an homomorphism of $S\backslash V_{x^*}$ into R , we have

$$\langle \left(1_\kappa, g_\kappa\right), x^*\rangle = \langle \left(1_\kappa, h_\kappa\right), x^*\rangle = 0 .$$

From this **it follows that**

$$\langle \left(x_\kappa, e_\kappa\right), x^*\rangle = \langle \left(1_\kappa, h_\kappa\right)\left(x_\kappa, f_\kappa\right)\left(1_\kappa, g_\kappa\right), x^*\rangle$$
$$= \langle \left(1_\kappa, h_\kappa\right), x^*\rangle + \langle \left(x_\kappa, f_\kappa\right), x^*\rangle + \langle \left(1_\kappa, g_\kappa\right), x^*\rangle$$
$$= \langle \left(x_\kappa, f_\kappa\right), x^*\rangle .$$

In any case $(x^*)^{-1}x^* \supseteq \sigma$. Hence the mapping $\bar{x}^* \in (S/\sigma)^*$ defined by $\langle \sigma^{\#}x, \bar{x}^*\rangle = \langle x, x^*\rangle$ for all $x \in S$ is well-defined, and we shall have $\Phi\bar{x}^* = x^*$. Thus, in this case, Φ must be surjective.

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19.

COROLLARY 7. *If S is an orthodox band of left R-modules, and Q the greatest inverse homomorphic image of S, then S^* and Q^* are R-isomorphic.*

20.

THEOREM 8. *Let S be an orthodox band of left R-modules and τ any R-stable congruence on S. The mapping $\Psi : (S/\tau)' \to S'$, $\bar{x}^* \mapsto \Psi(\bar{x}^*)$ defined by $\langle x, \Psi\bar{x}^* \rangle = \langle \tau^{\#}x, \bar{x}^* \rangle$ for any $x \in S$ is an R-isomorphism of $(S/\tau)'$ into S'. Whenever $\iota_S \subseteq \tau \subseteq \rho$, ρ being the minimal group congruence on S, this mapping Ψ is a surjective R-isomorphism of $(S/\tau)'$ onto S'.*

Proof. It is clear that the mapping Ψ must be the restriction of mapping Φ (of Theorem 7) to the maximal submodule $(S/\tau)'$ of $(S/\tau)^*$; hence Ψ is an R-isomorphism of $(S/\tau)'$ into S^*. Since for every $x \in S$, and every $\bar{x}^* \in (S/\tau)'$, we must have $\langle \tau^{\#}x, \bar{x}^* \rangle \in R$, we conclude that $\Psi\bar{x}^* \in S'$ for every $\bar{x}^* \in (S/\tau)'$; thus Ψ is an R-isomorphism of $(S/\tau)'$ into S'.

It will be sufficient to show that the mapping $\Psi : (S/\rho)' \to S'$, $\bar{x}^* \mapsto \Psi\bar{x}^*$ defined by $\langle x, \Psi\bar{x}^* \rangle = \langle \rho^{\#}x, \bar{x}^* \rangle$ for every $x \in S$ will be an R-isomorphism of $(S/\rho)'$ onto S'. Let x^* be any element of S'. Since x^* must be a homomorphism of S into the additive group R, we have $(x^*)^{-1}x^* \supseteq \rho$. Hence the mapping $\bar{x}^* \in (S/\rho)'$ defined by $\langle \rho^{\#}x, \bar{x}^* \rangle = \langle x, x^* \rangle$ for every $x \in S$ is well-defined, and we shall have $\Psi\bar{x}^* = x^*$. Thus, in this case Ψ must be surjective.

21.

COROLLARY 8. *If S is an orthodox band of left R-modules, Q the greatest inverse homomorphic image of S, and G the greatest group homomorphic image of S, then S' and Q' are both R-isomorphic with right R-module G' which is the dual of left R-module G.*

22.

THEOREM 9. *Let* S *be an orthodox band of left* R*-modules, and* $S = \bigcup_{\kappa \in Y} S_\kappa = \bigcup_{\kappa \in Y} G_\kappa \times E_\kappa$ *its semilattice decomposition. For any* $\lambda \in Y$ *, the mapping* $1^*_\lambda : S \to R^\infty$ *defined by* $\langle x, 1^*_\lambda \rangle = 0$ *if and only if* $x \in \bigcup_{\kappa \geq \lambda} S_\kappa$ *, and* $\langle x, 1^*_\lambda \rangle = \infty$ *otherwise, is an idempotent of* S^* *. The maximal submodule* $H_{1^*_\lambda}$ *of* S^* *containing* 1^*_λ *is* R*-isomorphic with* $\left(\bigcup_{\kappa \geq \lambda} S_\kappa\right)'$ *and with the right* R*-module* G'_λ *, which is the dual of the left* R*-module* G_λ *.*

Proof. For any $\lambda \in Y$, $\bigcup_{\kappa \geq \lambda} S_\kappa$ is an R-stable subsemigroup of S , and G_λ will be the greatest group homomorphic image of $\bigcup_{\kappa \geq \lambda} S_\kappa$. From Corollary 8 it follows that $\left(\bigcup_{\kappa \geq \lambda} S_\kappa\right)'$ and G'_λ are R-isomorphic right R-modules. It is easy to show that $S \setminus \left(\bigcup_{\kappa \geq \lambda} S_\kappa\right)$ is a prime ideal of S . From results in the proof of Corollary 5, it then follows that 1^*_λ must be an idempotent of S^* . We remark that for any $x^* \in S^*$, $x^* \in H_{1^*_\lambda}$ if and only if $V_{x^*} = \{x \in S \parallel \langle x, x^* \rangle = \infty\} = S \setminus \left(\bigcup_{\kappa \geq \lambda} S_\kappa\right)$. Hence the mapping $H_{1^*_\lambda} \to \left(\bigcup_{\kappa \geq \lambda} S_\kappa\right)'$, $x^* \mapsto x^* \mid \bigcup_{\kappa \geq \lambda} S_\kappa$ is an R-isomorphism of $H_{1^*_\lambda}$ onto $\left(\bigcup_{\kappa \geq \lambda} S_\kappa\right)'$.

23.

COROLLARY 9. *We use the same notations as in Theorem 9. Let* Q *be the greatest inverse semigroup homomorphic image of* S *and* $Q = \bigcup_{\kappa \in Y} G_\kappa$ *its semilattice decomposition. For any* $\lambda, \mu \in Y$ *,* $\lambda \geq \mu$ *, let* $\Phi_{\lambda,\mu}$ *be the structure homomorphism of* Q *, and* ${}^t\Phi_{\lambda,\mu}$ *its transpose; then* $1^*_\mu \geq 1^*_\lambda$ *in* S^* *; let* $\Phi^*_{\mu,\lambda} : H_{1^*_\mu} \to H_{1^*_\lambda}$ *be the structure homomorphism of*

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S^*. *For any* $\lambda \in Y$ *the mapping* $\Psi_\lambda : H_{1^*_\lambda} \to G'_\lambda$, $x^* \mapsto \Psi_\lambda x^*$ *defined by* $\langle (x_\kappa, e_\kappa), x^* \rangle = \langle \Phi_{\kappa,\lambda} x_\kappa, \Psi_\lambda x^* \rangle$ *for all* $(x_\kappa, e_\kappa) \in \bigcup_{\kappa \geq \lambda} S_\kappa$ *is an* R*-isomorphism of* $H_{1^*_\lambda}$ *onto* G'_λ, *and the following diagram is commutative:*

$$\begin{array}{ccc} H_{1^*_\mu} & \xrightarrow{\Psi_\mu} & G'_\mu \\ {\scriptstyle \Phi^*_{\mu,\lambda}} \big\downarrow & & \big\downarrow {\scriptstyle {}^t\Phi_{\lambda,\mu}} \\ H_{1^*_\lambda} & \xrightarrow[\Psi_\lambda]{} & G'_\lambda \end{array} .$$

Proof. The mapping $\bigcup_{\kappa \geq \lambda} S_\kappa \to G_\lambda$, $(x_\kappa, e_\kappa) \mapsto \Phi_{\kappa,\lambda} x_\kappa$ is a homomorphism of $\bigcup_{\kappa \geq \lambda} S_\kappa$ onto its greatest group homomorphic image G_λ; Ψ_λ must then be an R-isomorphism of $H_{1^*_\lambda}$ onto G'_λ by Theorem 8.

Let x^* be any element of $H_{1^*_\mu}$, and x_λ any element of G_λ. We proceed to show that $\left\langle x_\lambda, {}^t\Phi_{\lambda,\mu} \Psi_\mu x^* \right\rangle = \langle x_\lambda, \Psi_\lambda \Phi^*_{\mu,\lambda} x^* \rangle$. Indeed,

$$\begin{aligned} \left\langle x_\lambda, {}^t\Phi_{\lambda,\mu} \Psi_\mu x^* \right\rangle &= \langle \Phi_{\lambda,\mu} x_\lambda, \Psi_\mu x^* \rangle \\ &= \langle x_\lambda 1_\mu, \Psi_\mu x^* \rangle \\ &= \langle (x_\kappa, e_\kappa), x^* \rangle \\ &\qquad \text{for all } \kappa \geq \mu, \ \Phi_{\kappa,\mu} x_\kappa = x_\lambda 1_\mu, \ e_\kappa \in E_\kappa, \\ &= \langle (x_\lambda, e_\lambda), x^* \rangle \ \text{ for all } e_\lambda \in E_\lambda, \\ &= \langle (x_\lambda, e_\lambda), x^* 1^*_\lambda \rangle \ \text{ for all } e_\lambda \in E_\lambda, \\ &= \langle (x_\lambda, e_\lambda), \Phi^*_{\mu,\lambda} \rangle \ \text{ for all } e_\lambda \in E_\lambda, \\ &= \langle x_\lambda, \Psi_\lambda \Phi^*_{\mu,\lambda} x^* \rangle . \end{aligned}$$

We conclude that ${}^t\Phi_{\lambda,\mu} \Psi_\mu = \Psi_\lambda \Phi^*_{\mu,\lambda}$.

24.

COROLLARY 10. *We use the same notations as in Theorem 9 and Corollary 9. Let the structure semilattice of* S *be a lattice. Consider* $V = \bigcup_{\kappa \in Y} G'_\kappa$ *, and define multiplication in* V *by the following: for any* $x', y' \in V$ *,* $x' \in G'_\lambda$ *,* $y' \in G'_\mu$ *, put* $x'y' = \left({}^t\Phi_{\lambda\vee\mu,\lambda}x'\right)\left({}^t\Phi_{\lambda\vee\mu,\mu}y'\right)$. *Define the mapping* $R \times V \to V$ *,* $(\alpha, x') \mapsto x'\alpha$ *in the usual way. Then* V *is a semilattice of right* R*-modules, and there exists an* R*-isomorphism of* V *into* S^* . *If* Y *satisfies the minimal condition,* V *must be* R*-isomorphic with* S^* .

25.

REMARKS. Corollaries 9 and 10 show that S^* could well be named the dual of S . If Y is a lattice, the structure semilattice of V is the $\vee$-semilattice Y . The results of [6] make the connections between the structure theorems for S and the structure theorems for V more explicit.

Theorem 7 is quite analogous with a result in [5], §5, about the character semigroup of a commutative semigroup, and Theorem 9, Corollary 9, and Corollary 10 are in a certain way analogous with results of [7] and [8] (see also [3], Chapter 5).

The next theorem generalizes the concept of the transpose of an R-linear mapping.

26.

THEOREM 10. *Let* S *and* T *be orthodox bands of left* R*-modules, and* $\Theta : S \to T$ *an* R*-linear mapping. The mapping* ${}^T\Theta : T^* \to S^*$ *,* $t^* \mapsto {}^T\Theta t^*$ *defined by* $\langle x, {}^T\Theta t^*\rangle = \langle \Theta x, t^*\rangle$ *for all* $x \in S$ *, must be an* R*-linear mapping of* T^* *into* S^* *, and* ${}^T\Theta(T^*)$ *is embeddable in* $(S/\Theta^{-1}\Theta)^* \cong (\Theta S)^*$.

Proof. It must be clear that for any $t^* \in T^*$, we must have ${}^T\Theta t^* \in S^*$, since Θ is R-linear; it is not difficult to show that

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${}^{T}\Theta$ is R-linear.

Let t^* and v^* be any elements of T^* ; then $t^*|\Theta S$ and $v^*|\Theta S$ are both elements of $(\Theta S)^*$ since ΘS is an R-stable subsemigroup of T . From the definition of ${}^{T}\Theta$ we have that ${}^{T}\Theta t^* = {}^{T}\Theta v^*$ if and only if $v^*|\Theta S = t^*|\Theta S$. This implies that the mapping ${}^{T}\Theta(T^*) \to (\Theta S)^*$, ${}^{T}\Theta t^* \mapsto t^*|\Theta S$ is an R-isomorphism of ${}^{T}\Theta(T^*)$ into $(\Theta S)^*$.

27.

COROLLARY 11. *Let* S, T *, and* Θ *be as in Theorem* 10. *The mapping* ${}^{t}\Theta : T' \to S'$, $t^* \mapsto {}^{t}\Theta t^*$ *defined by* $\langle x, {}^{t}\Theta t^* \rangle = \langle \Theta x, t^* \rangle$ *for all* $x \in S$ *, must be an* R*-linear mapping of* T' *into* S' *, and* ${}^{t}\Theta(T')$ *is embeddable in* $\left(S/\Theta^{-1}\Theta\right)' \cong (\Theta S)'$.

28.

COROLLARY 12. *We use the same notations as in Theorem* 10 *and Corollary* 11. *Let* ρ_S *and* ρ_T *be the minimal group congruences on* S *and* T *respectively. Let* $\Psi_S : \left(S/\rho_S\right)' \to S'$, $\bar{x}^* \mapsto \Psi_S\bar{x}^*$ *, be the* R*-isomorphism defined by* $\langle x, \Psi_S\bar{x}^* \rangle = \left\langle \rho_S^{\#}x, \bar{x}^* \right\rangle$ *for all* $x \in S$ *, and let* $\Psi_T : \left(T/\rho_T\right)' \to T'$, $\bar{t}^* \mapsto \Psi_T\bar{t}^*$ *be defined by* $\langle t, \Psi_T\bar{t}^* \rangle = \left\langle \rho_T^{\#}t, \bar{t}^* \right\rangle$ *for all* $t \in S$. *Then there exists an* R*-linear mapping* $\Lambda : S/\rho_S \to T/\rho_T$ *such that the following diagrams are commutative:*

$$
\begin{array}{ccc}
S & \xrightarrow{\Theta} & T \\
\downarrow{\scriptstyle \rho_S^{\#}} & & \downarrow{\scriptstyle \rho_T^{\#}} \\
S/\rho_S & \xrightarrow[\Lambda]{} & T/\rho_T
\end{array}
\quad , \quad
\begin{array}{ccc}
S' & \xleftarrow{{}^{t}\Theta} & T' \\
\uparrow{\scriptstyle \Psi_S} & & \uparrow{\scriptstyle \Psi_T} \\
\left(S/\rho_S\right)' & \xleftarrow[{}^{t}\Lambda]{} & \left(T/\rho_T\right)'
\end{array}
$$

Proof. Since $\rho_T^{\#}\Theta$ is an R-linear mapping of S into the left

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R-module T/ρ_T , $\left(\rho_T^{\#}\Theta\right)^{-1}\left(\rho_T^{\#}\Theta\right)$ must be an R-stable group congruence on S , and, since ρ_S is the minimal group congruence on S , we must have $\rho_S \subseteq \left(\rho_T^{\#}\Theta\right)^{-1}\left(\rho_T^{\#}\Theta\right)$; this implies that Λ is a well-defined R-linear mapping of S/ρ_S into T/ρ_T . ${}^t\Lambda$ is then an R-linear mapping of $\left(T/\rho_T\right)'$ into $\left(S/\rho_S\right)'$ which is defined by $\left\langle \rho_S^{\#}x,\ {}^t\Lambda\bar{t}^*\right\rangle = \left\langle \Lambda\rho_S^{\#}x,\ \bar{t}^*\right\rangle$ for all $x \in S$ and all $\bar{t}^* \in \left(T/\rho_T\right)'$; but since $\Lambda\rho_S^{\#} = \rho_T^{\#}\Theta$, we then have

$$\begin{aligned}
\left\langle \rho_S^{\#}x,\ {}^t\Lambda\bar{t}^*\right\rangle &= \left\langle \rho_T^{\#}\Theta x,\ \bar{t}^*\right\rangle \\
&= \langle \Theta x,\ \Psi_T\bar{t}^*\rangle \\
&= \left\langle x,\ \left({}^t\Theta\Psi_T\right)\bar{t}^*\right\rangle \\
&= \left\langle \rho_S^{\#}x,\ \left(\Psi_S^{-1}\ {}^t\Theta\Psi_T\right)\bar{t}^*\right\rangle
\end{aligned}$$

for all $x \in S$ and all $\bar{t}^* \in \left(T/\rho_T\right)'$; hence ${}^t\Lambda = \Psi_S^{-1}\,{}^t\Theta\Psi_T$.

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