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A NOTE ON THE WEISS CONJECTURE

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Abstract

Let G be a finite group acting vertex-transitively on a graph. We show that bounding the order of a vertex stabiliser is equivalent to bounding the second singular value of a particular bipartite graph. This yields an alternative formulation of the Weiss conjecture.

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1. Introduction

Throughout this note, G is a finite group acting vertex-transitively on a graph $\Gamma = (V, E)$ of valency k. We say that G is *locally*-P, for some property P, if G_v is P on $\Gamma(v)$. Here v is a vertex of Γ, and $\Gamma(v)$ is the set of neighbours of v. With this notation we can state the Weiss conjecture [9].

CONJECTURE 1 (The Weiss conjecture). There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if G is vertex-transitive and locally-primitive on a graph Γ of valency k, then $|G_v| < f(k)$.

A stronger version of this conjecture, in which 'primitive' is replaced by 'semiprimitive' has been recently proposed [6]. (A transitive permutation group is said to be *semiprimitive* if each of its normal subgroups is either transitive or semiregular.)

Our aim in this note is to connect the order of G_v to the singular value decomposition of the biadjacency matrix of a particular bipartite graph $\mathcal{G}$. This connection yields an alternative form of the Weiss conjecture (and its variants). Our main result is the following theorem (we write λ_2 for the second largest singular value of the biadjacency matrix of $\mathcal{G}$).

THEOREM 1.1. *For every function $f : \mathbb{N} \to \mathbb{N}$, there is a function $g : \mathbb{N} \to \mathbb{N}$ such that if G is a finite group acting vertex-transitively on a graph $\Gamma = (V, E)$ of valency k and $\lambda_2 < f(k)$, then $|G_v| < g(k)$.*

Conversely, for every function $g : \mathbb{N} \to \mathbb{N}$, *there is a function* $f : \mathbb{N} \to \mathbb{N}$ *such that if* G *is a finite group acting vertex-transitively on a graph* $\Gamma = (V, E)$ *of valency* k *and* $|G_v| < g(k)$, *then* $\lambda_2 < f(k)$.

All of the necessary definitions pertaining to Theorem 1.1 are discussed below. In particular, the bipartite graph $\mathcal{G}$ is defined in Section 1, and the singular value decomposition of its biadjacency matrix is discussed in Section 2.

Theorem 1.1 implies that, to any family of vertex-transitive graphs with bounded vertex stabiliser, we have an associated family of bipartite graphs with bounded second singular value, and *vice versa*. Proving the Weiss conjecture (or one of its variants) is, therefore, equivalent to bounding the second singular value for a particular family of bipartite graphs.

Gowers remarks that singular values are the 'correct analogue of eigenvalues for bipartite graphs' (see the preamble to Lemma 2.7 in [4])[1]. Thus bounding the second singular value of a bipartite graph is analogous to bounding the second eigenvalue of a graph; the latter task is a celebrated and much studied problem due to its connection to the expansion properties of a graph (see, for instance, [5]).

The fact that the Weiss conjecture has connections to expansion has already been recognised [7], we hope that this note adds to the evidence that it is a connection warranting a good deal more investigation.

2. The associated bipartite graph $\mathcal{G}$

Our first job is to describe $\mathcal{G}$, and for this we need the concept of a *coset graph*. Let H be a subgroup of G and let A be a union of double cosets of H in G such that $A = A^{-1}$. Define the coset graph $\mathrm{Cos}(G, H, A)$ as the graph with vertex set the left cosets of H in G and with edges the pairs $\{xH, yH\}$ such that $Hx^{-1}yH \subset A$. Observe that the action of G by left multiplication on the set of left cosets of H induces a vertex-transitive automorphism group of $\mathrm{Cos}(G, H, A)$.

The following result is due to Sabidussi [8].

Proposition 2.1. *Let* $\Gamma = (V, E)$ *be a* G*-vertex-transitive graph and* v *a vertex of* Γ. *Then there exists a union* S *of* G_v*-double cosets such that* $S = S^{-1}$, $\Gamma \cong \mathrm{Cos}(G, G_v, S)$ *and the action of* G *on* V *is equivalent to the action of* G *by left multiplication on the left cosets of* G_v *in* G.

Note that G is locally-transitive if and only if S is equal to a single double coset of G_v. From here on we fix v to be a vertex in V and we set S to be the union of double cosets of G_v in G such that $\Gamma \cong \mathrm{Cos}(G, G_v, S)$. Observe that $S(\{v\}) = \Gamma(v)$.

[1] The mathematics behind this remark is set down in [2]. An elementary first observation is that the eigenvalues of the natural biadjacency matrix of a bipartite graph may be negative, in contrast to the eigenvalues of the (symmetric) adjacency matrix of a graph. This pathology is remedied by studying the singular values as we shall see.

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We are ready to define the regular bipartite graph $\mathcal{G}$. We define the two vertex sets, X and Y, to be copies of V. The number of edges between $x \in X$ and $y \in Y$ is defined to equal the number of elements $s \in S$ such that $s(x) = y$. Note that $\mathcal{G}$ is a multigraph.

3. The singular value decomposition

For V and W two real inner product spaces, we define a linear map

$$w \otimes v : V \to W, x \mapsto \langle x, v \rangle w.$$

With this notation we have the following result [4, Theorem 2.6].

Proposition 3.1. *Let $\alpha : V \to W$ be a linear map. Then α has a decomposition of the form $\sum_{i=1}^{k} \lambda_i w_i \otimes v_i$, where the sequences (v_i) and (w_i) are orthonormal in V and W, respectively, each λ_i is nonnegative, and k is the smaller of* $\dim V$ *and* $\dim W$.

The decomposition described in the proposition is called the *singular value decomposition*, and the values $\lambda_1, \lambda_2, \ldots$ are the *singular values* of α. In what follows we always assume that the singular values are written in nonincreasing order: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$.

Now write $\mathcal{A}$ for the biadjacency matrix of $\mathcal{G}$ *as a bipartite graph*, that is, the rows of $\mathcal{A}$ are indexed by X, the columns by Y and, for $x \in X$, $y \in Y$, the entry $\mathcal{A}(x, y)$ is equal to the number of edges between x and y. Then $\mathcal{A}$ can be thought of as a matrix for a linear map $\alpha : \mathbb{R}^X \to \mathbb{R}^Y$ and, as such, we may consider its singular value decomposition. From here on the variables $\lambda_1, \lambda_2, \ldots$ will denote the singular values of this particular map.

The next result gives information about this decomposition. (The result is [3, Lemma 3.3], although some of the statements must be extracted from the proof.)

Lemma 3.2. (1) *We have $\lambda_1 = t\sqrt{|V_1||V_2|}$, where t is the real number such that every vertex in V_1 has degree $t|V_2|$.*

(2) *If f is a function that sums to zero, then $\|\alpha(f)\|/\|f\| \leqslant \lambda_2$.*

Note that the only norm used in this note is the ℓ^2-norm.

4. Convolution

Consider two functions $\mu : G \to \mathbb{R}$ and $\nu : V \to \mathbb{R}$. We define the *convolution* of μ and ν to be

$$\mu * \nu : V \to \mathbb{R}, \quad v \mapsto \sum_{g \in G} \mu(g)\nu(g^{-1}v).$$

In the special case where $\mu = \chi_S$, the characteristic function of the set S defined above, $\chi_S * \nu$ takes on a particularly interesting form:

$$(\chi_S * f)(v) = \sum_{g \in G} \chi_S(g) f(g^{-1}v) = \sum_{w \in V} \mathcal{A}(v, w) f(v). \tag{1}$$

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Here, as before, $\mathcal{A}$ is the biadjacency matrix of the bipartite graph $\mathcal{G}$. Equation (1) implies that the linear map $\alpha : \mathbb{R}^X \to \mathbb{R}^Y$, for which $\mathcal{A}$ is a matrix, is given by $\alpha(f) = \chi_S * f$. This form is particularly convenient, as it allows us to use the following easy identities [3, Lemma 2.3].

LEMMA 4.1. *Let f be a function on V that sums to 0, p a probability distribution over V, q a probability distribution over G, and U the uniform probability distribution over V. Then:*

(1) $\|f + U\|^2 = \|f\|^2 + 1/|V|$;
(2) $\|p - U\|^2 = \|p\|^2 - 1/|V|$;
(3) $\|q * (p \pm U)\| = \|q * p \pm U\|$;
(4) *for k a real number,* $\|kp\| = k\|p\|$.

5. The proof

Theorem 1.1 will follow from the next result, which shows that, provided k is not too large compared to $|V|$, the order of G_v is bounded in terms of λ_2 and k.

PROPOSITION 5.1. *Either $|G_v| < \sqrt{2}\lambda_2/k$ or $|V| < 2k$.*

PROOF. Let v be a vertex in V. We define two probability distributions, $p_S : G \to \mathbb{R}$ and $p_v : V \to \mathbb{R}$, as follows:

$$p_S(x) = \begin{cases} \frac{1}{|S|} & x \in S, \\ 0 & x \notin S, \end{cases} \qquad p_v(x) = \begin{cases} 1 & x = v, \\ 0 & \textit{otherwise}. \end{cases}$$

Observe that $\|p_S\| = 1/\sqrt{|S|} = 1/\sqrt{k|G_v|}$ and $\|p_v\| = 1$. Observe that $(p_S * p_v)(w) = 0$ except when $w \in S(\{v\}) = \Gamma(v)$. A simple application of the Cauchy–Schwarz inequality (or see [1, Observation 3.4]) gives

$$\frac{1}{k} = \frac{1}{|\Gamma(v)|} \leqslant \|p_S * p_v\|^2.$$

Define $f = p_v - U$ and observe that f is a function on V that sums to 0. Lemma 3.2 implies that $\|(\alpha f)\|/\|f\| \leqslant \lambda_2$. Using this fact, the identities in Lemma 4.1, and the fact that $\chi_S = |S| p_S$, we obtain

$$\begin{aligned} \frac{1}{k} &\leqslant \|p_S * p_v\|^2 \\ &= \|p_S * (f + U)\|^2 \\ &= \|p_S * f + U\|^2 \\ &= \|p_S * f\|^2 + \frac{1}{|V|} \\ &= \frac{1}{|S|^2}\|\chi_S * f\|^2 + \frac{1}{|V|} \end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{|S|^2}\|\alpha(f)\|^2 + \frac{1}{|V|}\\
&\leqslant \frac{1}{|S|^2}\lambda_2^2\|f\|^2 + \frac{1}{|V|}\\
&= \frac{1}{|S|^2}\lambda_2^2\|p_v - U\|^2 + \frac{1}{|V|}\\
&< \frac{\lambda_2^2}{|S|^2} + \frac{1}{|V|}.
\end{aligned}
$$

Since $|S| = k|G_v|$ we can rearrange to obtain

$$k > \frac{|V|}{1 + \frac{\lambda_2^2|V|}{k^2|G_v|^2}}.$$

Observe that if $\lambda_2^2|V|/k^2|G_v|^2 \leqslant 1$, then

$$k > \frac{|V|}{1 + \frac{\lambda_2^2|V|}{k^2|G_v|^2}} \geqslant \frac{|V|}{2}$$

and the result follows. On the other hand, if $\lambda_2^2|V|/k^2|G_v|^2 > 1$, then

$$k > \frac{|V|k^2|G_v|^2}{k^2|G_v|^2 + |V|\lambda_2^2} > \frac{|V|k^2|G_v|^2}{2|V|\lambda_2^2}$$

and we conclude that $|G_v|^2 < 2\lambda_2^2/k$ as required. □

Finally we can prove Theorem 1.1.

Proof of Theorem 1.1. The previous lemma implies that if $\lambda_2 < f(k)$ for some function $f : \mathbb{N} \to \mathbb{N}$ then $|G_v| < g(k)$ for some function $g : \mathbb{N} \to \mathbb{N}$. (Note that if $|V| \leqslant 2k$, then $|G_v| \leqslant |G| \leqslant (2k)!$.)

For the converse, Lemma 3.2 implies that $\lambda_1 = t\sqrt{|X| \cdot |Y|}$, where t is the real number such that every vertex in X has degree $t|Y|$. Now recall that $|X| = |Y| = |V|$ and observe that every vertex in X has degree $k|G_v|$. Thus we conclude that $\lambda_1 = k|G_v|$. Since $\lambda_2 \leqslant \lambda_1$ the result follows. □

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