NON-DESARGUESIAN PROJECTIVE PLANE GEOMETRIES WHICH SATISFY THE HARMONIC POINT AXIOM

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1. Introduction and summary. In her papers **(12)** and **(13)** R. Moufang discusses projective plane geometries which satisfy the axiom of the uniqueness of the fourth harmonic point. Her main result is that in such geometries non-homogeneous co-ordinates may be assigned to the points of the plane (except for the "line at infinity") in such a way that straight lines have equations of the forms $\alpha x + y + \beta = 0$, or $x + \gamma = 0$. It is shown that the co-ordinates form an alternative field and furthermore, given an alternative field, it is shown that a non-Desarguesian geometry can be constructed for which the axiom of the uniqueness of the fourth harmonic point is satisfied.

In the present paper the author attacks the same problem from a radically different, and which seems to him a more natural, point of view. The theory of nets is completely avoided. No use is made of algebraic arguments since it turns out that all the rules of operation of the algebraic symbols have a natural geometric interpretation.

The basic idea of the argument used is the following. Projective collineations are defined for non-Desarguesian geometries and a projective collineation group as well as a unimodular subgroup are obtained. The axiom of the fourth harmonic point immediately leads to the result that the harmonic relation for four points in line is invariant under projection. This in turn leads to the facts that in such geometries a "full" unimodular projective collineation group exists and that any projective transformation between two lines can be extended to a unimodular projective collineation of the whole plane. With these results as background the construction of co-ordinate systems in a line and in the plane are readily carried out and the usual rules of operations on the symbols follow quite smoothly.

In this development the reason for the failure of the general associative law $a(bc) = (ab)c$ as contrasted with the validity of the special associative laws $(aa)b = a(ab)$ and $a^{-1}(ab) = b$ is immediately apparent. In fact it is known that in a classical Desarguesian geometry the law $a(bc) = (ab)c$ follows directly from the result that in such geometries the standard construction of the point ab from the points a and b, leads to its unique determination as soon as points 0, 1, ∞ are assigned in the line: i.e., the point ab does not depend on the special position of points outside the line which are used in the construction. In contrast a failure of Desargues' theorem leads immediately to the fact

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that in general the position of the point ab in the line depends not only on the assignment of 0, 1 and ∞ but also on the assignment of two other special points outside the line. However, the axiom of the fourth harmonic point does lead to the fact that the points a^{-1} and a^2 are uniquely determined by a once 0, 1, ∞ are assigned and hence the laws: $a^{-1}(ab) = b$, $a^2b = a(ab)$.

It is possible to develop a multiplication in the line in which the product $a \circ b$ is uniquely determined by the "scale" 0, 1 ∞ in the line. The resulting algebra is a Jordan field. Unfortunately when such co-ordinates are extended to the whole plane the equations of straight lines do not take on simple linear forms which can be handled readily.

Finally, the collineation theory developed here can be applied to more general non-Desarguesian geometries. In the last section some of the results obtained are mentioned together with a discussion of possible directions in which the theory may be used and extended.

2. Axioms for plane projective geometries. The general projective plane geometry is characterized by the following set of axioms. There are two sets of elements called points and lines respectively and one relation called "incidence" such that for any point and any line the relationship of "incidence" either holds or does not hold. The relationship is subject to the following three axioms:

A(1). Two distinct points are incident on exactly one line;

A(2). Two distinct lines are incident on exactly one point;

A(3). There exist four distinct points no three of which are incident on the same line.

In what follows the usual geometric terminology will be used, i.e. points which are incident on the same line will be called collinear, lines which are incident on the same point will be called concurrent and if a point and line are incident the line will be said to pass through the point and the point will be said to be on the line.

From the axioms of incidence alone very little can be proved. The results of importance are:

(1) If one line contains exactly $n + 1$ points so does every other line and the total number of points is $n^2 + n + 1$.

(2) If $n = p^r$, where p is a prime and r a positive integer, there exists exactly one (apart from isomorphism) Desarguesian geometry with $n + 1$ points on each line.

(3) For $n = 2, 3, 4, 5, 7$ the only geometries are the classical ones; for $n = 6$ there is no geometry. In (5) Bruck and Ryser establish the non-existence of a projective plane for a certain class of n.

(4) There exist projective geometries for which Desargues' theorem fails; in particular finite non-Desarguesian geometries exist for $n = 9$.

At the later stage a new axiom will be introduced. Also in what follows some of the proofs used will assume that a line has at least five points. This latter assumption will enable us to avoid exceptional cases which in any case can easily be examined.

3. Projective transformations. In this section properties of projective transformations are discussed, the purpose being to contrast the properties of Desarguesian and non-Desarguesian plane projective geometries and also to build up a rudimentary theory of projective collineations in a non-Desarguesian plane. This theory will be of fundamental importance in the subsequent development.

Points A, B, C, D, . . . in a line m and points A', B', C', D', . . . on a line m' are said to be in perspective from a point O if AA', BB', CC', . . . all pass through O. If we think of this as a mapping $A \to A'$, $B \to B'$, $C \to C'$, etc., we will use the notation

$$A, B, C, D, \ldots \overset{O}{\underset{\wedge}{-}} A', B', C', D', \ldots$$

to describe the mapping. A transformation $A \to A'$, $B \to B'$, $C \to C'$, etc., from the points of a line m to those of a line m' will be called a projective transformation if the points A', B', C', . . . are obtained from A, B, C, . . . as a result of a finite sequence of perspectivities. In the classical theory projective transformations are studied with reference to three properties, viz., Desargues' theorem, Pappus' or Pascal's theorem, and the fundamental theorem of projective transformations in a line. Some of the more important properties are listed here:

(a) In any plane projective geometry (Desarguesian or not) there is always at least one projective transformation which carries any three collinear points into any three other collinear points.

(b) Desargues' theorem is equivalent to its converse.

(c) Desargues' theorem is valid if and only if the plane is embeddable in a projective three-space.

(d) In any plane projective geometry, Pappus' theorem implies Desargues' theorem.

(e) In a finite plane projective geometry Pappus' theorem is equivalent to Desargues' theorem.

(f) Pappus' theorem is equivalent to the fundamental theorem which states that there is exactly one projective transformation which maps a set of three collinear points onto any set of three collinear points.

Also to be discussed in this section are some of the consequences of the so called "little Desargues theorem" a statement which may or may not be valid in any specific plane projective geometry. Roughly speaking the "little Desargues theorem" states that Desargues' theorem holds for those pairs of triangles in which the centre of perspectivity is incident on the axis of perspectivity. A formal statement of the "little Desargues theorem" and its converse are given below.

LITTLE DESARGUES THEOREM. Let ABC (Figure 1) and $A'B'C'$ be two triangles such that AA', BB', CC' pass through O. Let AB meet $A'B'$ atC'', BC meet $B'C'$ at A'' and CA meet $C'A'$ at B''. If two of the points A'', B'', C'' are collinear with O then so is the third.

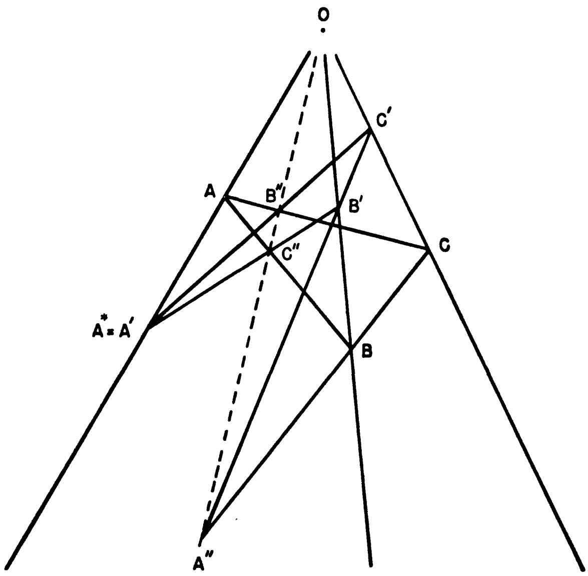

FIG. 1

Converse of the Little Desargues Theorem. Let ABC and $A'B'C'$ be two triangles such that AB meets $A'B'$ at C'', BC meets $B'C'$ at A'' and CA meets $C'A'$ at B'' and let A'', B'', C'' be collinear. If two of the lines AA', BB', CC' intersect at a point on A'', B'', C'' then the third line also passes through this point.

THEOREM 1. *In any plane projective geometry the "little Desargues theorem" is equivalent to its converse.*

Proof. As both parts of the equivalence are proved by the same means only the statement "the little Desargues' theorem implies its converse" is proved here. In Figure 1 using the above notation assume that A'', B'', C'' are collinear and that BB' meets CC' at O on $A''B''C''$. Let OA meet $C'B''$ at A^*. It is sufficient to show that $A^* = A'$. In the triangles $A\ B\ C$, $A^*\ B'\ C'$ the lines AA^*, BB', CC' are concurrent at O. Furthermore BC meets $B'C'$ at A'', AC meets A^*C' at B''. Since O, A'', B'' are collinear the little Desargues' theorem implies that A^*B' meets AB on the line $A''B''$. Hence A^*B' passes through C'' so that A^* is the point of intersection of $B'C''$ and $C'B''$. Hence $A^* = A'$.

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THEOREM 2. *If Desargues' theorem fails for a pair of triangles then there exists a line and a projective transformation in it, such that three points are fixed by the transformation and such that not all points in the line are fixed by the transformation.*

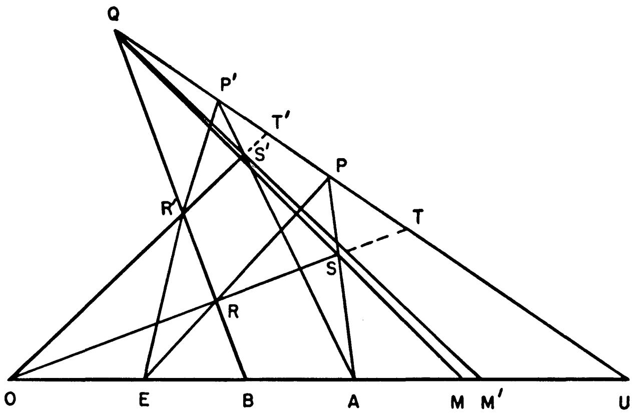

FIG. 2

Proof. The theorem actually follows from remarks (d) and (f) made previously but a direct proof is given here because the resulting diagram is used later in other connections. Furthermore, the proof will be given a second interpretation in what follows.

Since Desargues' theorem fails, its converse also fails. Let PRS and $P'R'S'$ be two triangles for which the converse of Desargues' theorem fails (Figure 2). Let PR meet $P'R'$ at E, PS meet $P'S'$ at A and RS meet $R'S'$ at O. Let A, O, E be collinear. Suppose PP' and RR' meet at Q. Since the converse of Desargues' theorem fails Q, S and S' are not collinear. Let QS meet OA at M and QS' meet OA at M'. Then $M \neq M'$. Let $QP'P$ meet OA at U and let $R'S'$ and RS meet PP' at T' and T respectively. Then

$$O, M, A, U \underset{\wedge}{\overset{S}{-}} T, Q, P, U, \underset{\wedge}{\overset{R}{-}} O, B, E, U \underset{\wedge}{\overset{R'}{-}} T', Q, P', U \underset{\wedge}{\overset{S'}{-}} O, M', A, U.$$

The resultant mapping sends $O \to O$, $A \to A$, $U \to U$, $M \to M' \neq M$.

Theorem 2 shows that failure of Desargues' theorem increases the number of projective transformations possible in a line. It will be seen subsequently that the effect of such failure can work in reverse in the case of transformations of the whole plane.

At this point the notions of collineation and projective collineation are introduced. A point to point mapping of all the points of a plane is said to be a collineation if it is one-one, its inverse exists, if collinear points have as images collinear points and if collinear points are the images of collinear points. In the classical projective plane, where Desargues' theorem is valid a *projective* collineation is defined as follows. The plane Π is embedded in a

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three dimensional space and a perspectivity from the plane Π to a plane Π', distinct from Π, with centre 0 not on either plane is defined by mapping the point A in Π onto the point A' in Π' whenever A, A' and 0 are collinear. A *projective* collineation of the plane Π is a point-point mapping of the plane Π which is the result of a finite sequence of perspectivities. A classical result is that if the co-ordinate field has automorphisms distinct from the identity there exist collineations which are not projective. If one is to distinguish between projective and non-projective collineations in a non-Desargeusian geometry a different approach is necessary since it is not possible to embed a non-Desarguesian geometry in a projective three-space.

It is possible to define projective collineations in a non-Desarguesian plane by means of the notions of homology and elation and the definition now to be introduced has the property that when applied to Desarguesian geometries it yields the full projective collineation group. Let P be any point and l be any line. Let A and A' be any two points collinear with P but which are distinct from P and are not on l. In a Desarguesian plane there is always exactly one projective collineation which keeps all points on l fixed and all lines through P fixed and which maps $A \rightarrow A'$. If P is on l the mapping is called an elation and if P is not on l it is called a homology. Furthermore it is known that the homologies generate all projective collineations and that the elations generate a subgroup of collineations usually referred to as the unimodular subgroup. Before defining the projective collineation group for the non-Desarguesian case a few properties of homologies and elations valid for any projective plane will be developed.

THEOREM 3. *Let P be any point and l be any line. Let A and A' be two points not on l but collinear with P and distinct from P. There exists at most one collineation which keeps all points on l fixed, all lines through P fixed and which maps A into A'.*

Proof. The proof is independent of whether or not P is on l. In Figure 3 the case P not on l is shown. Let B be any point not on AA' and not on l. It is shown that the image B' of B is uniquely determined. Let AB meet l at M. Since all lines through P are fixed B' is on PB. Furthermore since $A \rightarrow A'$ and $B \rightarrow B'$ the line AB has as its image the line $A'B'$. Also the point M on AB is fixed and hence must lie on $A'B'$. Hence B' is determined as the point of intersection of PB and $A'M$. In the same way the fact that $B \rightarrow B'$, determines a unique image for any point on the line AA'.

THEOREM 4. *Using the notation of* Theorem 3, *a necessary and sufficient condition that the collineation described in* Theorem 3 *exists is that Desargues' theorem is valid for every pair of triangles ABC and $A'B'C'$ with centre of perspectivity P and axis of perspectivity l. More exactly, the condition is: if AA', BB', CC', pass through P and AB meets $A'B'$ at C'', AC meets $A'C'$ at B'' and if C'' and B'' are on l then BC meets $B'C'$ at A'' which is on l.*

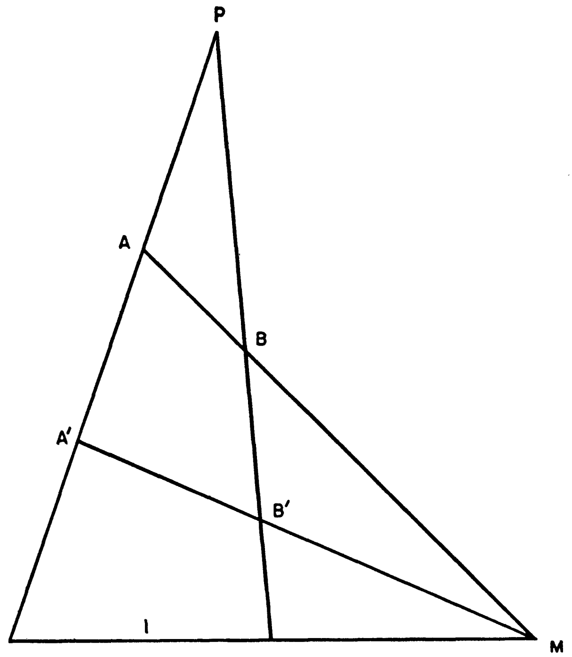

FIG. 3

Proof. Sufficiency; if Desargues' theorem holds for all such triangles then the image C' of C is the same whether obtained from $A \to A'$ or from $B \to B'$. The mapping is thus well defined and is obviously a collineation.

Necessity; Suppose the mapping is a collineation. Let ABC and $A'B'C'$ be any two triangles in perspective from P and with B'' and C'' on l. Then in the mapping $A \to A'$ implies $B \to B'$ and $C \to C'$. Let BC meet l at A''. Since BC maps into $B'C'$ and A'' is fixed it follows that $B'C'$ passes through A''.

In the collineations just described the line of fixed points is called the axis, the point of fixed lines the centre. The notation Elat $(P, l; A \to A')$ will be used to denote the elation with centre at P, axis at l and A' the image of A. Similarly the corresponding homology will be described as Hom $(P, l; A \to A')$.

In the projective plane if the collineation which maps A into A' and which keeps all points on l fixed and all lines through P fixed does not exist we will say the collineation is *obstructed;* otherwise the collineation will be said to be *unobstructed.* We define the projective collineation group as the set of all collineations of the plane generated by all the unobstructed homologies and elations. The subgroup generated by all unobstructed elations will be called the unimodular subgroup and any collineation which is representable as a product of elations will be termed a unimodular collineation. If no homology is obstructed the geometry will be said to admit a *full* projective collineation group, and if no elation is obstructed the geometry will be said to admit a

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full unimodular group. As corollaries to Theorem 4 we have the following two theorems.

THEOREM 5. *A necessary and sufficient condition for a full projective collineation group to exist is that the geometry be Desarguesian.*

THEOREM 6. *A necessary and sufficient condition for a full unimodular group to exist is that the little Desargues theorem is valid in the geometry.*

THEOREM 7. *If the little Desargues theorem is valid in a projective plane then any projective transformation between two lines of the plane can be embedded in a unimodular collineation of the plane.*

Proof. It is only necessary to show that any perspectivity between two lines can be embedded in an elation, since on representing a projective transformation as a product of perspectivities the resultant transformation is embedded in the collineation which results from multiplying the corresponding elations. Let l and m be two lines intersecting at the point A and suppose the points of l are mapped onto those of m by a perspectivity with P as centre. Let B on l be mapped into B' on m by this perspectivity. Then Elat $(P, PA; B \to B')$ embeds the given perspectivity.

It may be remarked here that Theorem 7 is not true necessarily for non-Desarguesian geometries for which the small Desargues' theorem is not valid. In fact it is possible to exhibit a non-Desarguesian plane and a projective transformation in a line which is not embeddable in any collineation of the plane.[1]

For geometries which satisfy the little Desargues' theorem, it is not generally true that there exists a projective collineation which maps any set of four points, no three of which are collinear into any other such set of four points. In this case the following weaker theorem is valid.

THEOREM 8. *Let* Π *be a projective plane for which the little Desargues' theorem is valid. Let* A, B, C, D *and* A', B', C', D' *be two sets of four distinct points such that* A, B, C *are collinear and* D *is not on* ABC *and* A', B', C' *are collinear and* D' *is not on* $A'B'C'$. *There exists a unimodular collineation which maps* $A \to A'$, $B \to B'$, $C \to C'$ *and* $D \to D'$.

Proof. By statement (a) of this section there is at least one projective transformation which maps $A \to A'$, $B \to B'$, $C \to C'$. Embed this transformation in a unimodular collineation U. Let U map $D \to D''$. If $D'' = D'$, the

[1]The geometry described (**15**, pp. 383–384) by Veblen and Wedderburn has this property. Using their notation the points of Figure 2 may be assigned coordinates as follows: $P(0, 1, 1)$, $S(2 + 2j, 1, 0)$, $R(0, j, 1)$, $P(1, 2, 1)$, $S'(j, j, 1)$, $R'(2, 1, 1)$ $Q(1 + j, 2 + j, 1)$, $E(0, 0, 1)$, $A(j + 1, 0, 1)$, $0(2 + j, 0, 1)$, $B(1 + 2j, 0, 1)$, $U(2, 0, 1)$, $M(1, 0, 1)$, $M'(2j, 0, 1)$. The projective transformation of Theorem 2 keeps the points $(2 + j, 0, 1)$, $(j + 1, 0, 1)$, $(2, 0, 1)$ fixed and maps $(1, 0, 1)$ into $(2j, 0, 1)$. It can be verified that in this geometry the projective mapping in the line cannot be extended to a collineation (projective or otherwise) of the plane.

collineation U has the required property. If $D'' \neq D'$, since D is not on ABC, D'' is not on $A'B'C'$. Let $D'D''$ meet A', B', C' at E. Let V be the collineation Elat $(E, A'B'; D'' \to D')$. The collineation UV then has the required property.

4. The axiom of the fourth harmonic point. In the subsequent development consideration will be restricted to those geometries which satisfy the following axiom which will be termed the axiom of the fourth harmonic point. Let A, B, C be any three points in line (Figure 4). Let M be any point not on AB. Let CEL be any line through C distinct from AB and not passing through M, the point L being on MA and E on MB. Let AE intersect BL at R and let MR intersect AB at D. Also, let K be point of intersection of CL and MR.

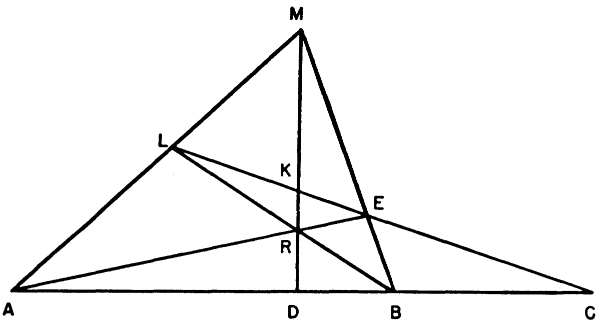

FIG. 4

Points A, B, C, D related by such a diagram will be said to satisfy the relation $H(\bar{A}, \bar{B}, C, D)$. The axiom of the fourth harmonic point may be stated as follows:

A(4). *If $H(\bar{A}, \bar{B}; C, D)$ then D is distinct from C and is uniquely determined by A, B and C.*

The axiom **A(4)** implies that D is independent of the choice of M and the choice of the line CEL. It is possible to weaken the axiom to the assumption that D is independent of the choice of the line CEL for fixed M, but it can readily be shown that the weakened axiom is equivalent to the stronger form.

Figure 4 will be referred to as the harmonic diagram. The points A, B, C, D may be described as follows: A and B are diagonal points of the quadrangle $MLRE$ and C and D are the points where the diagonals of $MLRE$ meet the line AB. Figure 4 is symmetric with respect to A and B and also with respect to C and D. Hence:

THEOREM 9. *$H(\bar{A}, \bar{B}; C, D)$ implies $H(\bar{B}, \bar{A}; C, D)$ and*
$H(\bar{A}, \bar{B}; C, D)$ implies $H(\bar{A}, \bar{B}; D, C)$.

THEOREM 10. *If* $H(\bar{A}, \bar{B}; C, D)$ *and* A, B, C, D *are in perspective with* L, E, C, K *then* $H(\bar{L}, \bar{E}; C, K)$.

Proof. If M is the centre of perspectivity, Figure 4 gives a construction of D from A, B, C. In Figure 4, $MABR$ is a quadrangle, L and E are its diagonal points and K and C are the points where the diagonals meet L and E. Hence, Figure 4 represents a construction for K such that $H(\bar{L}, \bar{E}; C, K)$.

Because of the symmetry of the harmonic construction between C and D a theorem similar to Theorem 10 holds if C is replaced by D.

THEOREM 11. *If* A, B, C, D *and* A', B', C', D' *are perspective and* $H(\bar{A}, \bar{B}; C, D)$ *then* $H(\bar{A}', \bar{B}'; C', D')$.

Proof. Let O be the centre of perspectivity and let CD' meet OA at A^* and OB at B^*. By theorem 10 $H(\bar{A}, \bar{B}; C, D)$ implies $H(\bar{A}^*, \bar{B}^*; C, D')$ which in turn implies $H(\bar{A}', \bar{B}'; C', D')$. As an obvious corollary, it follows that:

THEOREM 12. *If* A, B, C, D, *are four points in line related to* A', B', C', D' *by a projective transformation and* $H(\bar{A}, \bar{B}; C, D)$ *then* $H(\bar{A}', \bar{B}'; C', D')$.

THEOREM 13. $H(\bar{A}, \bar{B}; C, D)$ *implies* $H(\bar{C}, \bar{D}; A, B)$.

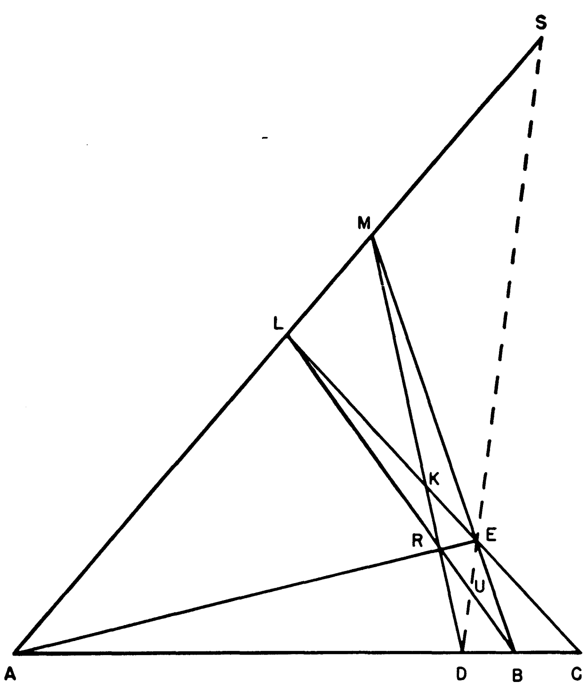

FIG. 5

Proof. The diagram of Figure 4 is extended to Figure 5 as follows: Obtain the point S as the intersection of AM and DE and the point U as the intersection of BL and DE.

$$A, B, C, D \overset{E}{\underset{\wedge}{-}} A, M, L, S \overset{R}{\underset{\wedge}{-}} E, D, U, S \overset{L}{\underset{\wedge}{-}} C, D, B, A.$$

Hence $H(\bar{A}, \bar{B}; C, D)$ *implies* $H(\bar{C}, \bar{D}; B, A)$ which in turn implies $H(\bar{C}, \bar{D}; A, B)$.

Theorem 13 allows us to remove the distinction between the pairs A, B and C, D of a harmonic tetrad. Hence it is unnecessary to indicate in the notation which points are capped. In what follows, $H(\bar{A}, \bar{B}; C, D)$ will be replaced by $H(A, B; C, D)$.

THEOREM 14. *Let A, B, C, D and A, B', C', D' be two sets of four points on distinct lines such that $H(A, B; C, D)$ and $H(A, B'; C'D')$. Then BB', CC', DD' are concurrent.*

Proof. Let BB' meet CC' at K and let KD meet AB' at D''. By Theorem 11, $H(A, B; C, D)$ implies $H(A, B'; C'D'')$. Also $H(A, B'; C', D')$ and $H(A, B'; C'D'')$ implies $D' = D''$ by axiom **A(4)**.

THEOREM 15. *Let P be a point and l be a line not through P. Let A be any point. If A is P or A is on l let $A' = A$. If A is distinct from P and is not on l let AP meet l in M and choose A' so that $H(A, A'; P, M)$. Then the mapping $A \to A'$ is a collineation.*

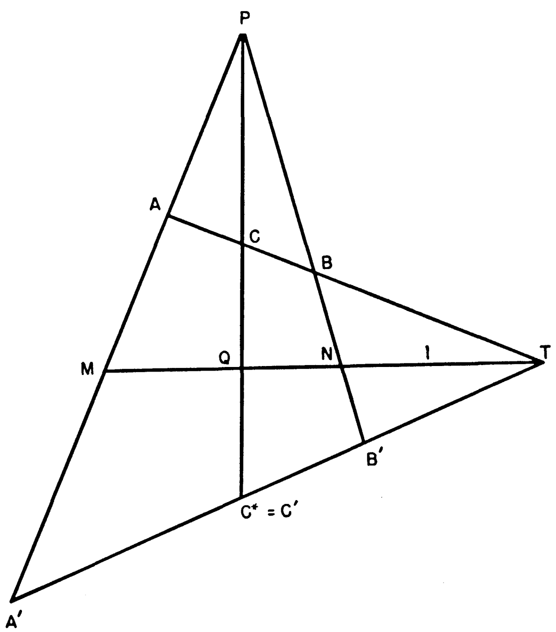

FIG. 6

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Proof (Figure 6). By definition and **A(4)** every point A has a unique image. Let B be any point not on l and not in line with AP. Let BP meet l at N. Since $H(P, M; A, A')$ and $H(P, N; B, B')$ the lines MN, AB and $A'B'$ are concurrent. Let T be point of concurrency. If C is on AB, let PC meet MN at Q and $A'B'$ at C^*. Since

$$P, M, A, A' \overset{T}{\underset{\wedge}{-}} P, Q, C, C^*$$

it follows that $H(P, Q; C, C^*)$. Hence $C^* = C'$ the image of C in the mapping. Hence collinear points map into collinear points.

This mapping is usually called a harmonic homology and is denoted here by Harm (P, l).

THEOREM 16. *A projective plane which satisfies the axiom of the fourth harmonic point admits a full unimodular group.*

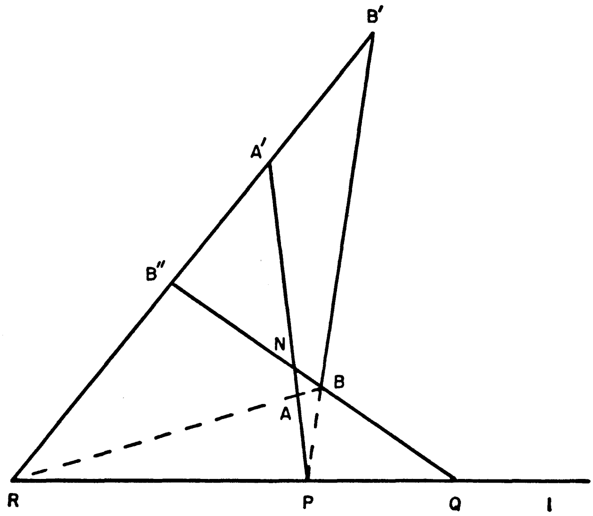

FIG. 7

Proof. Let P be any point on a line l (Figure 7). Let A and A' be in line with P. It is now shown that Elat $(P, l; A \to A')$, exists. Choose N so that $H(P, N; A, A')$. Then by Theorem 15, the product of Harm(N, l) and Harm(A', l) is a collineation. Now Harm(N, l) Harm(A', l) maps $A \to A'$ and keeps all points on l fixed. Let B be any point not on l and not on AA'. Let Harm(N, l) map $B \to B''$ and Harm(A', l) map $B'' \to B'$. Let BB'' meet l at Q and $B''B'$ meet l at R. Then $H(B, B''; N, Q)$ and $H(B', B''; A', R)$. By Theorem 14, BB', $A'N$, QK are concurrent. Hence BB' passes through P. This implies Harm(N, l) Harm(A', l) = Elat$(P, l; A \to A')$.

In **(6**, 5.28**)** Coxeter gives the above construction for the Desarguesian plane.

As a corollary to Theorems 6 and 16 the following result is obtained.

THEOREM 17. *The little Desargues' theorem is valid in any projective plane which satisfies the axiom of the fourth harmonic point.*

The converse of Theorem 17 is also true but a proof is not given here. A proof can readily be obtained from the observation that an elation with Q as centre and which maps $A \rightarrow B$ and $B \rightarrow C$ has the property that $H(Q, B; A, C)$.

5. The addition of points in a line. In this section it will be shown that addition in a line can be defined in such a way that the points of the line, except for one point (the point at "infinity") form an abelian group with the further property that to each point a in a line there is a point $\frac{1}{2}a$ such that $\frac{1}{2}a + \frac{1}{2}a = a$.

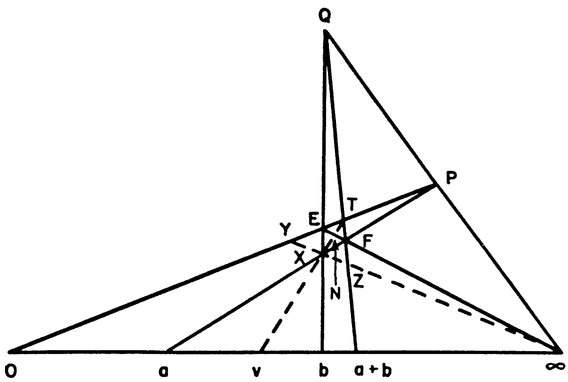

FIG. 8

Let l be any line and let 0 and ∞ be any two points on this line (Figure 8). Let P and Q be any two points in line with ∞ but not on l. Let a and b be any two points on l distinct from 0 and ∞. Join $0P$, aP, bQ and let E be the point of intersection of $0P$ and bQ. Join ∞E to meet aP at F and join QF to meet l at $a +$ b. The point $a + b$ is determined by a, b and the four points 0, ∞, P, Q. We refer to these latter four points as a scale and denote it by $\{0, \infty; P, Q\}$.

THEOREM 18. *The point $a + b$ is independent of the points P, Q used in the scale. In other words, addition in the line is completely determined by the points* 0, ∞.

Proof. In Figure 8, let X be the point of intersection of aP and bQ and T the point of intersection of $0P$ and FQ. Let TX meet EF at N and l at V. Let ∞X meet QF at Z and $0P$ at Y. From the quadrangle $QPFE$ it follows that $H(\infty, X; Y, Z)$. From

$$\infty, X, Y, Z \ \overset{T}{\underset{\wedge}{-}} \ \infty, V, 0, a + b$$

it follows that $H(\infty, V; 0, a + b)$. From

$$\infty, X, Y, Z \ \overset{T}{\underset{\wedge}{-}} \ \infty, N, E, F \overset{X}{\underset{\wedge}{-}} \infty, V, b, a$$

it follows that $H(\infty, V; b, a)$. From $H(\infty, V; b, a)$ it follows that a, b, ∞ uniquely determine V and from $H(\infty, V; 0, a + b)$ it follows that ∞, V and 0 uniquely determine $a + b$. Hence $a + b$ is uniquely determined by 0, ∞, a and b.

THEOREM 19. *For all a and b distinct from ∞ and 0, $a + b = b + a$.*

Proof. $a + b$ is determined by the harmonic relationships $H(\infty, V; b, a)$ and $H(\infty, V; 0, a + b)$. V is unchanged by the interchange of a and b. Hence $b + a$ is the same point as $a + b$.

The construction for a sum collapses when one of the points a or b is 0. We will define addition in this case by $a + 0 = a$ and $0 + b = b$, for all a, b distinct from ∞. We leave $a + \infty$ undefined.

THEOREM 20. *To each a distinct from ∞ there is a point $-a$ such that $a + (-a) = 0$.*

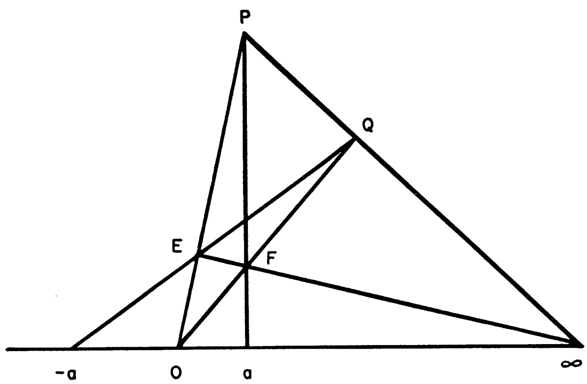

FIG. 9

Proof. If $a = 0$ choose $-a = 0$, otherwise choose $-a$ by the relationship $H(a, -a; 0, \infty)$ (Figure 9). Since 0, ∞ are diagonal points of the quadrangle $PQFE$ it is clear that $a + (-a) = 0$ from the construction by means of the scale $\{0, \infty; P, Q\}$.

THEOREM 21. *To each a distinct from ∞ there is a b such that $b + b = a$. We denote b by $\frac{1}{2}a$.*

Proof. If $a = 0$ choose $b = 0$. Otherwise choose b by the relationship $H(0, a; b, \infty)$. If a' is constructed from the relationship $b + b = a'$ using

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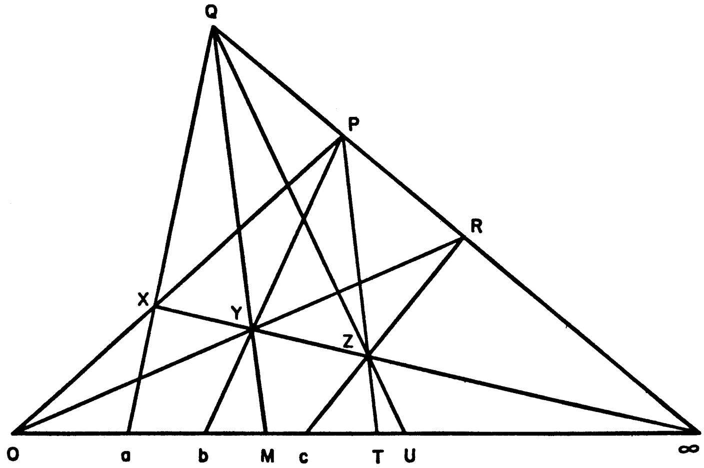
$\{0, \infty; P, Q\}$ it is clear that $H(0, a'; b, \infty)$. Hence $a' = a$. The proof also implies that b is uniquely determined by a.

THEOREM 22. *For all a, b, c distinct from ∞, $a + (b + c) = (a + b) + c$.*



$$M = a + b = b + a$$
$$T = c + b = b + c$$
$$U = a + (b + c) = (a + b) + c$$

FIG. 10

Proof. By definition the theorem is true if one or more of a, b, c are 0. Hence we assume all of a, b, c are distinct from 0. In the diagram only the case where a, b, c are distinct from each other is shown although the proof is valid in all cases. In Figure 10, let P and Q be any points in line with ∞ but not on $0\ \infty$. Let $0P$ meet aQ at X; $\infty\ X$ meet bP at Y; QY meet 0, ∞ at M; $0Y$ meet PQ at R; CR meet $\infty\ X$ at Z; ZP meet $0\ \infty$ at T and ZQ meet $0\ \infty$ at U.

Then $M = b + a = a + b$ from the scale $\{0, \infty; P, Q\}$

$T = c + b = b + c$ from the scale $\{0, \infty; R, P\}$

$U = c + (a + b) = (a + b) + c$ from the scale $\{0, \infty; R, Q\}$

$U = (b + c) + a = a + (b + c)$ from the scale $\{0, \infty; P, Q\}$

Hence $(a + b) + c = a + (b + c)$.

We note particularly that our proof employs only the fact that addition is uniquely defined by the points $0, \infty$. The breakdown of an analogous property for multiplication leads to a weakened associative law. Theroems 18 to 22 may be summarized as:

THEOREM 23. *Two distinct points 0, ∞ in a line l, determine an abelian group under addition for all the points of l distinct from ∞. Furthermore, to each a on l distinct from ∞ there is a $\frac{1}{2}a$, uniquely determined, such that $\frac{1}{2}a + \frac{1}{2}a = a$.*

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THEOREM 24. *Let l and l' be two lines. If there is a projective transformation from l to l' which maps* 0 *onto* $0'$ *and* ∞ *onto* ∞' *and if for each a on l the corresponding image on l' is denoted by a', then the mapping $a \to a'$ is an isomorphism of the corresponding additive groups, where* $0'$, ∞' *determines the addition in l'.*

Proof. By Theorems 7 and 17, the projective transformation between l and l' can be embedded in a collineation U. Let $\{0, \infty; P, Q\}$ be a scale for addition in l and let U map $0 \to 0'$, $\infty \to \infty'$, $P \to P'$, $P \to Q'$. Take $\{0', \infty'; P', Q'\}$ as a scale for addition in l'. The collineation U will then map the whole construction for a sum $a + b$ in l into the corresponding construction for $a' + b'$ in l'. Hence in the original projective transformation from l to l' the image of $a + b$ is $a' + b'$. Hence the mapping $a \to a'$ is an isomorphism of the additive groups.

THEOREM 25. *The additive group is uniquely determined apart from an isomorphism (i.e. is independent of the line used and the two points* 0, ∞, *chosen on it).*

Proof. This follows from the fact that there is always a projective transformation which carries two distinct points 0, ∞ into any other two distinct points $0'$, ∞'.

6. Some restricted Desargues' theorems. The axiom of the fourth harmonic point implies various variants of Desargues' theorem. It has already been shown that the little Desargues' theorem is valid. In this section some other cases in which Desargues' theorems are shown to be true.

THEOREM 26. *Desargues' theorem and its converse are true whenever the centre and axis of perspectivity are so related that the line joining corresponding vertices of both triangles meets the axis of perspectivity at the harmonic conjugate of the centre of perspectivity.*

Proof. The theorem is an immediate corollary of Theorems 15 and 4.

THEOREM 27. *Desargues' theorem and its converse is true for any pair of triangles TPV and $T^*P^*V^*$ if T^*V^* meets TV at a point on the line PP^*.*

Proof. Assume that the triangles TPV and $T^*P^*V^*$ are in perspective from Q and suppose that TV meets T^*V^* at ∞ (Figure 11), on the line PP^*. T^*P^* meets PT at 0. Let Q, V, V^* meet 0∞ at R and let P^*V^* and PV meet 0∞ at a^* and a respectively. From the scale $\{0, \infty; P, Q\}$ it follows that $R = a + b$ and from $\{0, \infty; P^*, Q\}$ it follows that $R = a^* + b$. Hence $a^* + b = a + b$ or $a = a^*$. Conversely, suppose TPV and $T^*P^*V^*$ are such that TV meets T^*V^* at ∞ on the line PP^*. Suppose also that T^*P meets TP at 0 and PV meets P^*V^* at a and that 0, a, ∞ are collinear. Let TT^* and PP^* intersect at Q and let QV^* and QV meet 0∞ at R and R^* respectively. Let QTT^* meet

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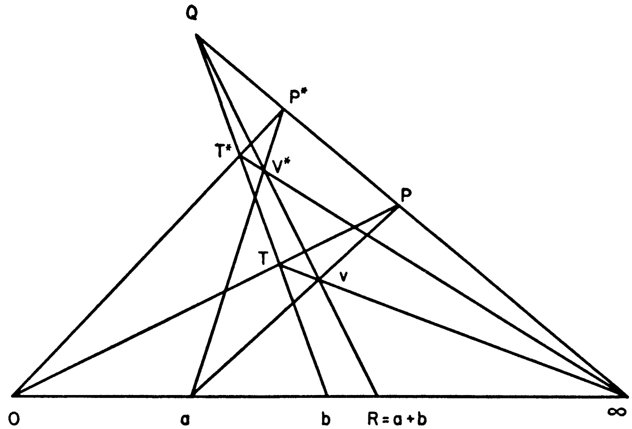

FIG. 11

0∞ at b. Then from the scale $\{0, \infty; P, Q\}$, $R = a + b$ and from the scale $\{0, \infty; P^*, Q\}$, $R^* = a + b$. Hence $R = R^*$.

THEOREM 28. *Let ABC and $A'B'C'$ be two triangles such that C' is on the line AB. Desargues' theorem and its converse are both true for such a pair of triangles.*

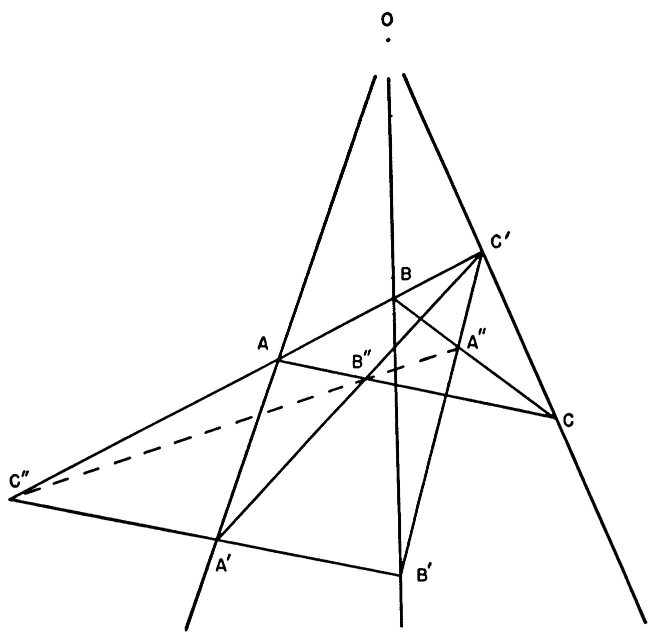

FIG. 12

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Proof. In Figure 12, let A'', B'', C'' be the points of intersection of BC, $B'C'$; CA, $C'A'$ and AB, $A'B'$ respectively. Theorem 27 applied to triangles $AA'B''$; $BB'A''$ is then equivalent to Theorem 28 for triangles ABC and $A'B'C'$.

It may be remarked here that Theorems 27, 28 and 17 are all actually equivalent to the axiom of the fourth harmonic point. The same may be said for Theorem 26 after a slight reformulation. In what follows these theorems could all be avoided by arguments involving collineations similar to those used in §5. However, in many cases these theorems lead to more direct proofs and will be used in the next section.

7. Multiplication in the line. There are several equivalent ways of defining multiplication in a Desarguesian plane but these may lead to inequivalent definitions for the non-Desarguesian case. For our purposes the following definition is taken. Let l be a line (Figure 13) and let 0, 1, ∞ be any three distinct points on it. Let P and Q be two distinct points in line with ∞ but not on l. The system $\{0, 1, \infty; P, Q\}$ is said to be a scale for multiplication in the line l, products being defined as follows. If a and b are two points on l distinct from 0 and ∞ let $1P$ meet bQ at R, $0R$ meet aP at S and let QS meet the line l at a point c. The point c is called the product ab. Products $a\infty$ or ∞a are not defined. The products $0a$ and $a0$ are both defined to be 0. Also for all $a(\neq \infty)$ it follows immediately that $1a = a1 = a$. It is a result of Desarguesian geometry that ab is determined by the points 0, 1, ∞ and is independent of the points P and Q. This result is no longer valid in the non-Desarguesian case as is shown by the following theorem.

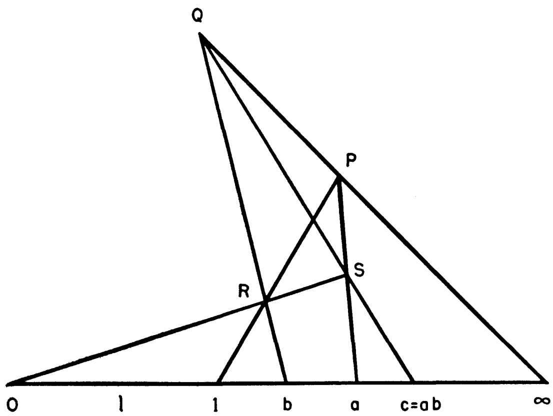

FIG. 13

THEOREM 29. *If the converse of Desargues' theorem fails for two triangles P, R, S and P', R', S' there is a line l and six distinct points on it viz.* 0, 1, ∞, *a, b, M, M' such that in the scale* $\{0, 1, \infty; P, Q\}$, *$M = ab$ while in the scale*

$\{0, 1, \infty; P'Q\}$, $M' = ab$. The point Q is the point of intersection of PP' and RR'.

Proof. We refer to Figure 2 whose construction is given in Theorem 2. If the points O, E, B, A, M, M', U of Figure 2 be relabelled $0, 1, b, a, M, M'$, respectively, the point M is actually constructed as the product ab in the scale $\{0, 1, \infty; P, Q\}$ and M' is the product ab in the scale $\{0, 1, \infty; P', Q\}$.

It follows from Theorem 29, and the theorems valid in a Desarguesian geometry, that Desargues' theorem is equivalent to the statement that multiplication as defined above is uniquely determined by the choice of $0, 1, \infty$ in a line. Hilbert has shown that Desargues' theorem implies associative multiplication, and it is also easy to give a direct proof that if $0, 1, \infty$ uniquely determine multiplication in a line then multiplication is associative (**2**, p. 79). In the subsequent development it is shown that in special cases the axiom of the fourth harmonic point determines a product dependent only on $0, 1, \infty$ and that in these cases the associative law is valid.

THEOREM 30. *To each point a distinct from* 0 *and* ∞ *on a line l there is a point a^{-1} such that $aa^{-1} = a^{-1}a = 1$. Furthermore the point a^{-1} is uniquely determined by the choice of* $0, 1, \infty$ *on the line l.*

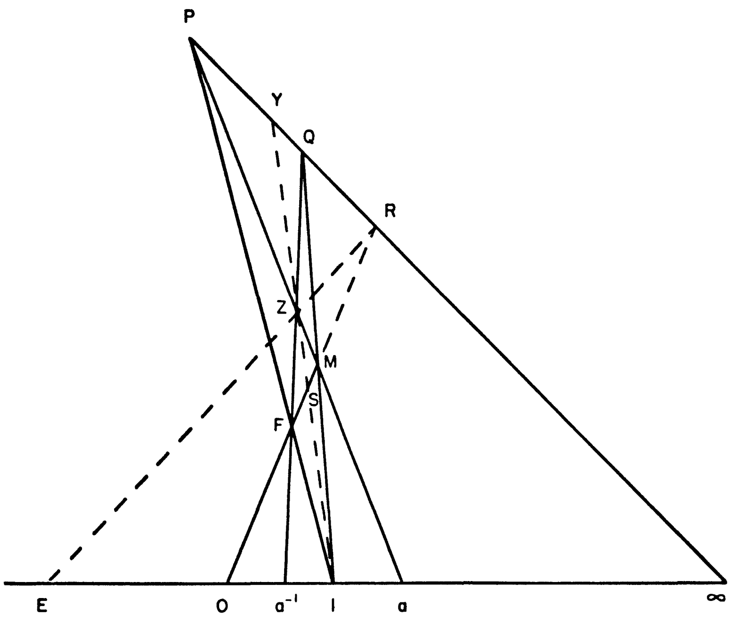

FIG. 14

Proof. Let $\{0, 1, \infty; P, Q\}$ be a scale for multiplication in l (Figure 14). Let aP meet $1Q$ at M, $M0$ meet $1P$ at F, QF meet 0∞ at a^{-1}. By construction,

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it follows that $aa^{-1} = 1$ in the scale $\{0, 1, \infty; P, Q\}$. To show a^{-1} is independent of P and Q the construction is continued as follows: Let QF meet aP at Z; $M0$ meet $Q\infty$ at R, $Z1$ meet MF and $Q\infty$ at S and Y, respectively; Let ZR meet 0∞ at E. From the quadrangle $F, 1, M, Z$ it follows that $H(P, Q; R, Y)$. Also

$$P, Q, R, Y \overset{Z}{\overline{\wedge}} a, a^{-1}, E, 1$$

so that $H(a, a^{-1}; 1, E)$. Also

$$1, E, 0, \infty \overset{R}{\overline{\wedge}} 1, Z, S, Y \overset{P}{\overline{\wedge}} F, M, S, R \overset{Z}{\overline{\wedge}} a^{-1}, a, 1, E.$$

Hence, $H(1, E; 0, \infty)$. From the relation $H(1, E; 0, \infty)$ it follows that E is uniquely determined by $0, 1, \infty$ and from $H(a^{-1}, a; 1, E)$ it follows that a^{-1} is uniquely determined by $a, 1, E$. Hence a^{-1} is determined uniquely by $0, 1, \infty, a$. Furthermore since the harmonic relationships which determine a^{-1} from a are symmetric with respect to a and a^{-1} it follows that $a^{-1}a = 1$.

It may be noted that the point E is identical with the point -1 and that $1^{-1} = 1$, and $(-1)^{-1} = -1$.

A simpler proof of Theorem 30 could be given using a restricted Desargues' theorem but such a proof would not exhibit the harmonic relationships connecting a and a^{-1}.

THEOREM 31. *For all a distinct from 0 and ∞ and for all b distinct from ∞, $a^{-1}(ab) = b$ and $(ba)a^{-1} = b$.*

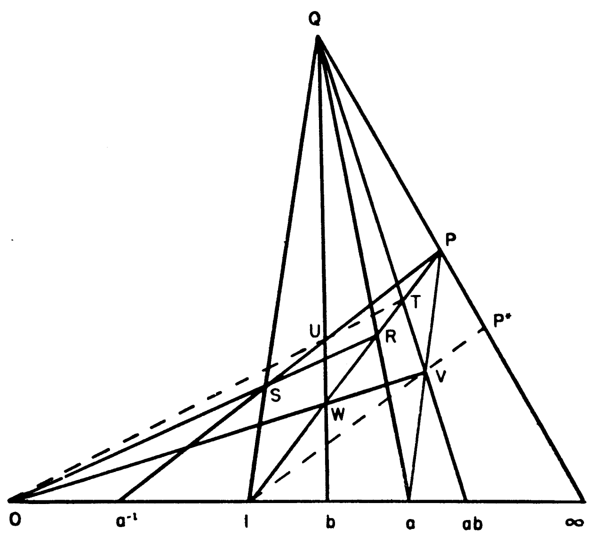

FIG. 15

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Proof. The theorem is obvious if $b = 0$. Assume $b \neq 0$. Only the relationship $a^{-1}(ab) = b$ is proved here as the proof of the other case is along the same lines. In Figure 15, let $\{0, 1, \infty; P, Q\}$ be a scale for multiplication. Let $1P$ meet bQ at W, $0W$ meet aP at V and QV meet 0∞ at ab. Let aQ meet $1P$ at R and $0R$ meet $1Q$ at S and PS meet 0∞ at a^{-1}. Let $a^{-1}W$ meet PQ at P^*. Let U be the intersection of bQ and $a^{-1}P$. The theorem will be proved provided it can be shown that 0, U, T are collinear. It is first shown that 1, V, P^* are collinear. This follows from the equation $aa^{-1} = 1$, using the scale $\{0, 1, \infty; PP^*\}$ and Theorem 30. Consider now the triangles $Pa^{-1}1$ and QWV. PQ, $a^{-1}W$ and $1V$ all pass through P^*. Furthermore W is on the line $P1$. By the restricted Desargues' theorem 28 it follows that 0, U, T are collinear. Hence from the scale $\{0, 1, \infty; P, Q\}$ $a^{-1}(ab) = b$.

THEOREM 32. *For all points a distinct from* ∞ *on the line* l a^2 *is determined uniquely by the points* $0, 1, \infty$.

Proof. Let $\{0, 1, \infty; P, Q\}$ be a scale for multiplication in the line. The theorem is proved in three stages: (1), b^2 is independent of position of P on ∞Q; (2), b^2 is independent of position of Q on ∞P, and (3), b^2 is independent of which line through ∞ is used.

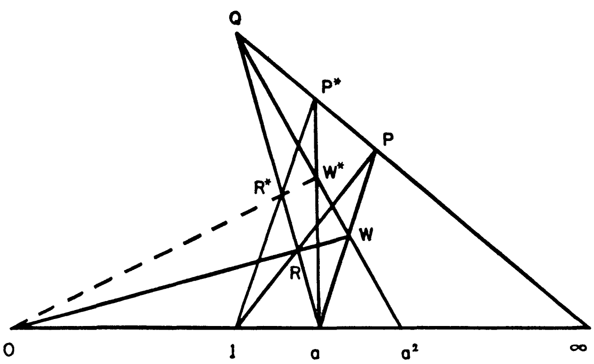

FIG. 16

Proof of (1). In Figure 16, let $\{0, 1, \infty; P, Q\}$ be a scale and let P^* be any other point on ∞, P, Q. Let $1P$ meet aQ at R, $0R$ meet aP at W, QW meet 0∞ at a^2. Let $1P^*$ meet aQ at R^* and Qa^2 meet aP^* at W^*. To show a^2 in scale $\{0, 1, \infty; P^*, Q\}$ is the same point as a^2 in scale $\{0, 1, \infty; P, Q\}$ it is sufficient to show that $0R^*W^*$ are collinear. In triangles PWR; $P^*W^*R^*$, PP^*, WW^*, RR^* all pass through Q. Furthermore PW meets P^*W^* at a which is on RR^*. By Theorem 27, 0, R^*, W^* are collinear.

Proof of (2). In Figure 17, let $\{0, 1, \infty; P, Q\}$ be a scale and let Q^* be any other point on ∞PQ. Let $1P$ meet aQ at R, $0R$ meet aP at W, QW meet 0∞ at a^2. Let aQ^* meet $1P$ at R^*, $0R^*$ meet aP at W^*. From triangles QWR;

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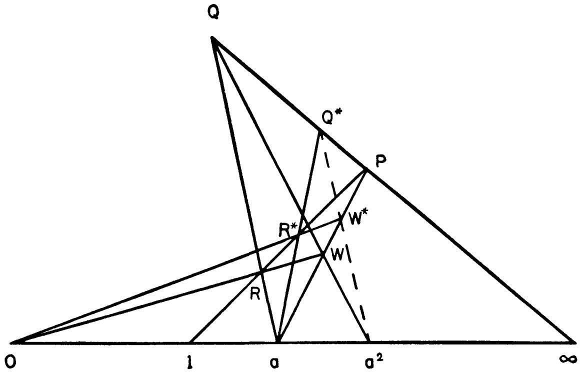

FIG. 17

$Q^*\ W^*\ R^*$; QQ^*, WW^*, and RR^* are concurrent at P. Furthermore QR meets Q^*R^* at a on WW^*. Hence by Theorem 27, Q^*, W^*, and a^2 are collinear, so that a^2 is the same for the scales $\{0, 1, \infty; P, Q\}$ and $\{0, 1, \infty; P, Q^*\}$.

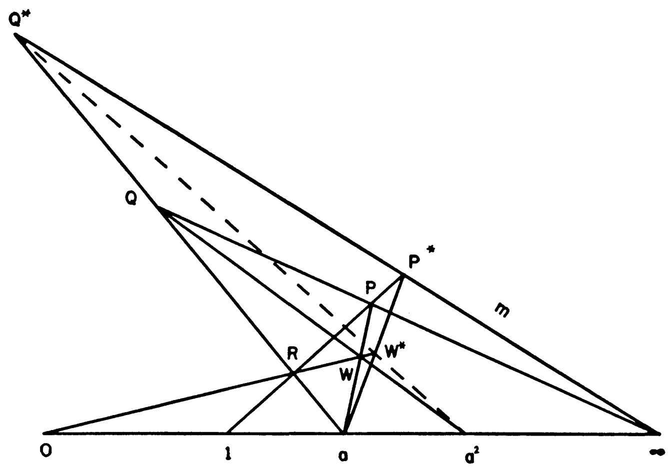

FIG. 18

Proof of (3). In Figure 18, let $\{0, 1, \infty; P, Q\}$ be a scale and let m be any line through ∞ distinct from $0, 1, \infty$ and from ∞, P, Q. Let $1P$ meet m at P^* and aQ meet m at Q^*. Let $1P$ meet aQ at R, $0R$ meet aP at W and QW meet 0∞ at a^2. Let aP^* meet $0R$ at W^*. In triangles WPQ; $W^*P^*Q^*$, WW^*, PP^*, QQ^* are concurrent at R. Also PW meets P^*W^* at a on Q^*Q. Hence by Theorem 27, Q^*, W^*, a^2 are collinear. Hence a^2 is the same for the scales $\{0, 1, \infty; P, Q\}$ and $\{0, 1, \infty; P^*, Q^*\}$.

It is clear that one can go from scale $\{0, 1, \infty; P, Q\}$ to scale $\{0, 1, \infty; A, B\}$ by a series of transformations of types (1), (2), (3).

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Remarks on Theorem 32. An alternative proof of Theorem 32 which does not employ the restricted Desargues' theorem can be given. This proof is dependent on the fact that a^2 is determined from a and $-a$ by the relationship $H(a, -a; 1, a^2)$. Furthermore the uniqueness of a^{-1} and of a^2 are not independent facts algebraically. In fact, the equation

$$a^2 = \cfrac{1}{\cfrac{1}{a} - \cfrac{1}{a + \cfrac{1}{1 - \cfrac{1}{a}}}}$$

yields an independent algebraic proof of the uniqueness of a^2. In a problem in the American Mathematical Monthly **(11)** the author has shown how to express a^2b (or aba the case of non-commutative multiplication) in terms of a and b using only addition, subtraction and reciprocation. The equivalent of this identity was used by Hua in **(9)** to obtain properties of division rings.

THEOREM 33. *For all a, b distinct from ∞, $a^2b = a(ab)$ and $ba^2 = (ba)a$.*

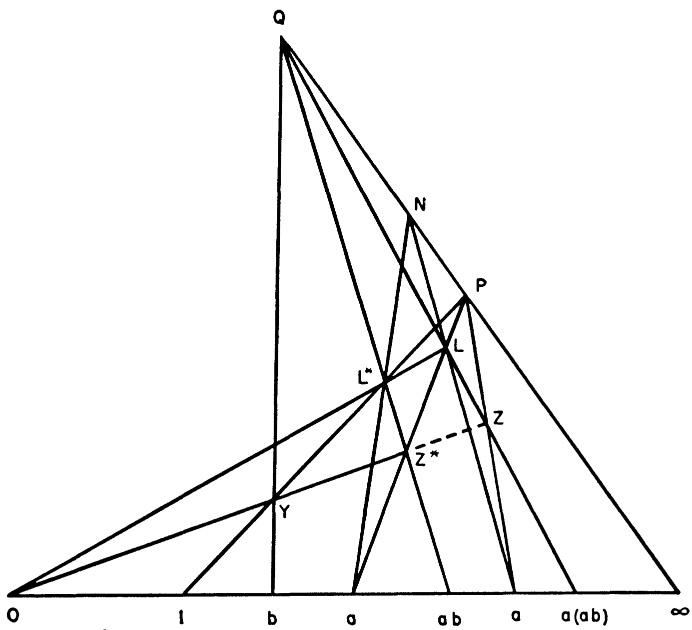

FIG. 19

Proof. In Figure 19, let $\{0, 1, \infty; P, Q\}$ be a scale and let $1P$ meet bQ at Y, $0Y$ meet aP at Z^*, QZ^* meet 0∞ at ab. Let $1P$ meet abQ at L^*, $0L^*$ meet aP at L, QL meet 0∞ at $a(ab)$. Let aL^* meet P, Q at N. The point a^2

is constructed from $\{0, 1, \infty; P, N\}$ by joining NL to meet 0∞ at a^2. Let a^2P meet QL at Z. The equation $a^2b = a(ab)$ is valid provided 0, Z^*, Z are collinear. To show this, consider triangles ZLa^2 and Z^*L^*a. ZL meets Z^*L^* at Q, Za^2 meets Z^*a at P and La^2 meets L^*a at N. Furthermore, P, N and Q are collinear and L is on the line Z^*a. By Theorem 28 Z, Z^* and 0 are collinear.

THEOREM 34. *For all a, b, c distinct from ∞, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.*

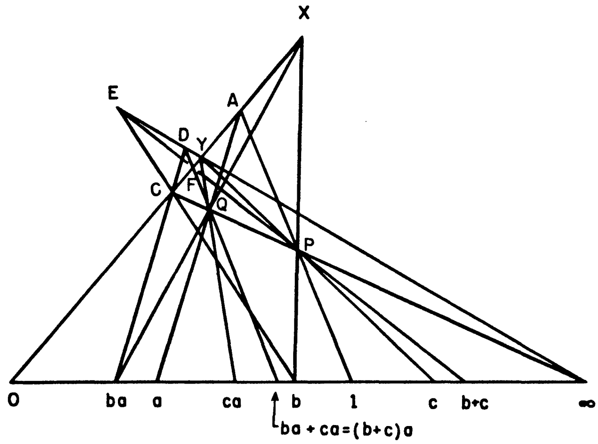

FIG. 20

Proof. Both statements are proved along the same lines so only the second is proved here. The theorem is obvious if any one of a, b, or c is 0. Hence assume a, b, c are all distinct from 0. In Figure 20, let $\{0, 1, \infty; P, Q\}$ be a scale for multiplication in the line 0, 1, ∞. Let $1P$ meet aQ at A, $0A$ meet bP at X, QX meet 0∞ at ba. Let cP meet $0A$ at Y, YQ meet 0∞ at ca. Let $0A$ meet PQ at C; bC meet ∞Y at E; ba, C meet ∞Y at D; DQ meet 0∞ at $ba + ca$. Let EP meet 0∞ at $b + c$; $(b + c)$, P meet $0A$ at F; FQ meets 0∞ at $(b + c)a$. All points on the line 0∞ have been properly labelled with respect to addition and multiplication. The equation $(b + c)a = ba + ca$ will be valid if F is on the line QD. In triangles PQX; EDC, PQ meets ED at ∞. PX meets EC at b, and QX meets DC at ba. Furthermore ∞, b and ba are collinear and C is on PQ. By Theorem 28, F, Q, D are collinear.

The main results of this section and the last may be summarized in the statement:

THEOREM 35. *Under addition and multiplication the points on a line which are distinct from ∞ form an alternative division ring, for any fixed scale $\{0, 1, \infty; P, Q\}$.*

In her paper **(12)** Moufang has shown that starting with an alternative division ring of characteristic distinct from 2 one can set up a non-Desarguesian plane projective geometry which satisfies the axiom of the fourth harmonic point.

In the next few theorems some properties of multiplication are developed which are interesting in themselves but which are not needed for the main development.

THEOREM 36. *Let* $\{0, 1, \infty; P, Q\}$ *be a scale for both addition and multiplication in a line* l (i.e. *if* $\{0, 1, \infty; P, Q\}$ *is a multiplicative scale the set* $\{0, \infty; P, Q\}$ *is taken for the additive scale*). *Let* m *be any other line in the plane, and let* $0'$, $1'$, ∞' *be three arbitrary points of* m. *There is a scale for addition and multiplication in* m *such that the algebra of points in* l *is isomorphic to that of the points in* m.

Proof. Take any projective transformation which maps $0 \to 0'$, $1 \to 1'$, $\infty \to \infty'$ and embed this in a unimodular collineation of the plane. Let P' and Q' be the images of P and Q in this collineation and take $\{0', 1', \infty'; P', Q'\}$ as a scale for the line m. Let the image of a on l be a' on m. The mapping $a \to a'$ is the required isomorphism. This follows from the fact that any construction for a sum or product using the scale $\{0, 1, \infty; P, Q\}$ is mapped by the collineation into the corresponding construction using the scale $\{0', 1', \infty'; P', Q'\}$.

Since multiplication in the line as developed here is not uniquely determined by the scale points in the line, a natural question which one may ask is: can multiplication be defined in such a way that products are uniquely determined by the scale points 0, 1, ∞ in the line. An affirmative answer is given in the next theorem.

THEOREM 37. *The point* $a \circ b$ *defined as* $a \circ b = \frac{1}{2}(ab + ba)$ *is uniquely determined by the points* 0, 1, ∞ *in the scale, and the points of the line distinct from* ∞ *form a Jordan field under the operations of* $+$ *and* $\circ$.

Proof.
$$a \circ b = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$
and the uniqueness follows from Theorems 18, 20, 21, and 32. The Jordan associative law $a^2 \circ (b \circ a) = (a^2 \circ b) \circ a$ follows from a direct computation using the definition of $a \circ b$.

Geometrically the relationship between ab and $a \circ b$ is given by the harmonic relationship $H(ab, ba; a \circ b, \infty)$.

In spite of the intrinsic nature of Jordan multiplication, i.e., its independence of the position of the points P and Q, it is not of much use since when coordinates are introduced into the whole plane it does not lead to linear expressions for the equations of straight lines.

THEOREM 38. *If* $ab = ba$ *the point* ab *is uniquely determined by* 0, 1, ∞. *Furthermore if* $ab = ba$ *for all* a *and* b *in any line the same is true for any other line and the geometry is Desarguesian.*

Proof. If $ab = ba$ then $ab = a \circ b$, so that by Theorem 37, ab is uniquely determined by $0, 1, \infty$. By Theorem 36, if $ab = ba$ for all ab in any line the same relationship holds for any other line. By a previous remark the uniqueness of ab for all a and b is equivalent to Desargues' theorem. As an immediate corollary the following theorem is true.

THEOREM 39. *Any alternative division ring of characteristic different from* 2 *in which multiplication is commutative is a field.*

Remark. The interest here is that the proof is almost entirely geometric. The only algebraic relationship used was

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = \frac{ab+ba}{2}$$

and even this may be dispensed with. It is the author's belief that a completely geometric proof of the fundamental theorem of alternative division rings, namely; that every non-associative alternative division ring is a Cayley division algebra over its centre is not an unreasonable expectation.

8. Co-ordinates in the plane. At this point we introduce non-homogeneous co-ordinates to all the points of the plane except those on one line—the line at infinity. The procedure is straightforward except that some care must be exercised in order to arrange that addition and multiplication in the various lines are consistent. The reason for this, of course, is that the points $0, 1, \infty$ are not sufficient to determine uniquely the multiplication in a line. However, Theorem 36 will be used as a basis for connecting the algebras of the various lines.

Let O, U, V, L (Figure 21) be four points in the plane no three of which are collinear. Let UL meet OV at T, VL meet OU at M, OL meet UV at W and MT meet UV at P. We are not interested in assigning co-ordinates to the points on the line UV although we will label some of these points. To every other point, a pair of numbers (x, y) will be assigned. First co-ordinates are assigned to O, M, U, W, V, T, L as follows: $O(0, 0)$; $M(1, 0)$; $U(\infty, 0)$; $W(\infty, \infty)$; $V(0, \infty)$; $T(0, 1)$; $L(1, 1)$. Second, every point on the line O, L, W will be assigned co-ordinates of the form (a, a) and our rules of addition and multiplication will apply to the first co-ordinate. On O, L, W take the scale$\{(0, 0), (1, 1), (\infty, \infty); U, V\}$ to determine addition and multiplication of the first co-ordinates. Third, every point on the line O, M, U will be assigned a co-ordinate $(k, 0)$ the point $(k, 0)$ being defined as the intersection of the line joing V to (k, k) and the line OU. In the same way points on the line OV will be assigned co-ordinates $(0, m)$ where $(0, m)$ is the intersection of the lines OV and $U(m, m)$. Let X be any other point of the plane and let UX meet OV at the point $(0, y)$ and VX meet the line OU at the point $(x, 0)$. Assign to X the co-ordinates (x, y).

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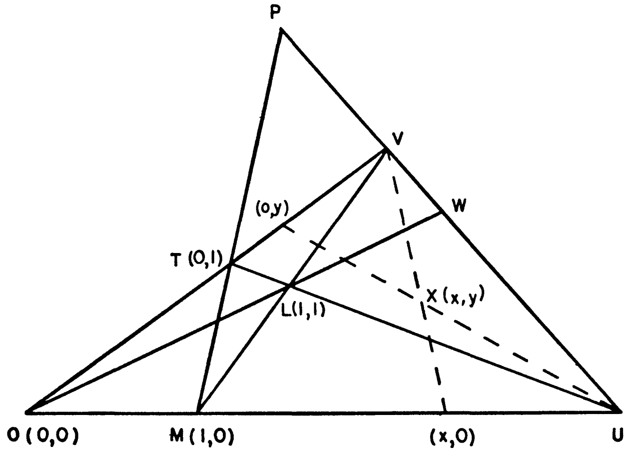

FIG. 21

The transformation Elat $\{V, OV; L \to M\}$ maps the point (b, b) on OW into $(b, 0)$ on OU. Furthermore it is easily seen that this elation maps $U \to P$ and $V \to V$. Hence by Theorem 36, using the scale $\{(0, 0), (1, 0), (\infty, 0); P, V\}$ addition and multiplication in OU is consistent with addition and multiplication in OW where addition and multiplication in OU is applied to the first co-ordinate. In the same way Elat $\{U, OU; L \to T\}$ maps (b, b) on OW into $(0, b)$ on OV. Also $V \to P$ and $U \to U$. Hence, using the scale $\{(0, 0), (0, 1), (0, \infty); U, P\}$ addition and multiplication in OV is consistent with that in OW.

THEOREM 40. *Every line through* $(0, 0)$ *has an equation of the form* $x - y\beta = 0$.

Proof. The lines OV and OU (using the notation of Figure 21) have equations $x = 0$, and $y = 0$ respectively. The line OW has equation $x - y = 0$.

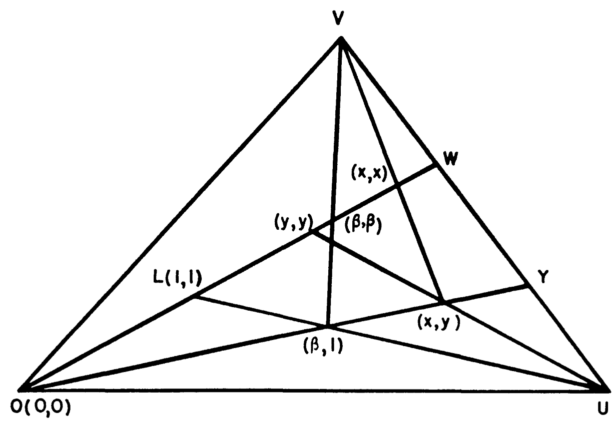

FIG. 22

Let OY be any other line through O (Figure 22). Let OY meet UL at the point $(\beta, 1)$ and let (x, y) be any other point on OY. Let U, (x, y) meet OW at (y, y); V, (x, y) meet OW at (x, x); V, $(\beta, 1)$ meet OW at (β, β). From the scale of multiplication $\{(0, 0), (1, 1), (\infty, \infty); U, V\}$ in the line OW and using just the first co-ordinates, Figure 22 shows that the product $y\beta$ is the point x. Hence $x - y\beta = 0$.

THEOREM 41. *Any line which does not pass through* $(0, 0)$ *and which is distinct from* UV *has an equation of the form* $x - y\beta - \alpha = 0$ *or* $y - \gamma = 0$.

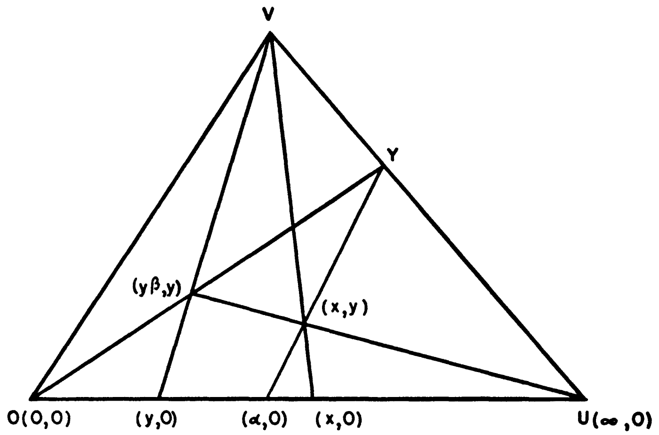

FIG. 23

Proof. In Figure 23, using the same notation as in Figure 21, if the line passes through U and meets OV at $(0, \gamma)$ its equation is $y - \gamma = 0$. If it passes through V and meets OU at $(\alpha, 0)$ its equation is $x - \alpha = 0$. Now suppose the line passes through $(\alpha, 0)$ and let it meet UV at Y. Join OY. By Theorem 40, OY has an equation $x - y\beta = 0$. Let (x, y) be any point on the given line. Let U, (x, y) meet OY at $(y\beta, y)$ and let V, $(y\beta, y)$ meet OU at $(y\beta, 0)$. In OU addition is defined for the first co-ordinates and using the scale $\{(0, 0), (\infty, 0); Y, V\}$ Figure 23 represents a construction of $x = \alpha + y\beta$. Hence $x - y\beta - \alpha = 0$.

It has now been shown that every line in the plane distinct from UV has an equation of one of the forms $x - y\beta - \alpha = 0$, $y - \gamma = 0$.

9. Concluding observations. It has been the point of view of this paper to relate the theorem of Desargues to the notion of a projective collineation, the basic theorems being 3, 4, 5, 6, 7, 8 where failure of Desargues' theorem is related to non-existence of certain collineations. In the case of geometries satisfying the axiom of the fourth harmonic point the success of the method is due mainly to the fact that a full unimodular group exists.

For more general non-Desarguesian geometries this approach can be used to give more or less precise information concerning the way in which Desargues' theorem breaks down. This information could be conceivably used to classify non-Desarguesian geometries. We illustrate what is intended by an example.

Let F be a finite "near-field", i.e. a finite set of elements which satisfy all the axioms of a field except the right distributive law and the commutative law of multiplication. Such near-fields exist and their complete determination has been carried out by Zassenhaus in **(16)**. Following Veblen and Weddenburn **(15)** a projective plane geometry can be constructed from F as follows. A point is any one of the following three types of triplets; $(1, 0, 0)$, $(a, 1, 0)$ or $(b, c, 1)$ where a, b, c are in F. Actually any triplet (a, b, c) (except $(0, 0, 0)$) may be regarded as a point provided one identifies (a, b, c) with $(\rho a, \rho b, \rho c)$ where $\rho \neq 0$. Note however that (a, b, c) is *not* identified with $(a\rho, b\rho, c\rho)$. A line is defined as any set of points satisfying an equation of one of[2] $x + ya + zb = 0$, $y + zc = 0$, $z = 0$.

Such points and lines do form a plane projective geometry. It is easily verified that for such a geometry the following transformations are collineations:

$$\begin{aligned} \rho x' &= \phi(x) + \phi(y)a + \phi(z)b. \\ \rho y' &= \phi(y)c. \\ \rho z' &= \phi(z)d. \end{aligned}$$

and

$$\begin{aligned} \rho x' &= \phi(x) + \phi(y)a + \phi(z)b. \\ \rho y' &= \phi(z)c. \\ \rho z' &= \phi(y)d. \end{aligned}$$

where a and b are arbitrary elements in F; ρ, c, d are any elements of F distinct from 0, and the mapping $k \to \phi(k)$ is an automorphism of F.

Among such collineations the transformation

$$\begin{aligned} \rho x' &= x + yA + zB. \\ \rho y' &= y. \\ \rho z' &= z. \end{aligned}$$

[2]It is important to note that the equation $xa + yb + zc = 0$ is not in general the equation of a line. The footnote on page 383 of the Veblen-Weddenburn paper **(14)** erroneously assumed that the equation

$$x(1 + j) + y(1 + j) + z = 0$$

represented a line. The statement to which their footnote referred is nevertheless correct.

for varying A and B in F represents all elations with centre (1, 0, 0) and arbitrary axis through (1, 0, 0). In other words no elation with centre (1, 0, 0) is obstructed. By Theorem 4 this can be translated into a statement concerning Desargues' theorem as follows:

THEOREM 42. *In any Veblen-Weddenburn geometry based on a near-field the little Desargues' theorem holds for all pairs of triangles in perspective from the point* (1, 0, 0).

The author has shown (in some work not yet published) that for all such geometries based on near-fields whether finite or infinite the point (1, 0, 0) is the only point with this property. Hence for such geometries every collineation keeps fixed the point (1, 0, 0). A question of interest is whether there are any other non-Desarguesian geometries with a point so specialized.

Another direction in which the investigation may be carried out is suggested by the following consideration. In Desarguesian geometry it is a fundamental property that for any two sets of four points no three of which are collinear there is a projective collineation which carries the first set into the second. It follows that if two systems of co-ordinates are set up based on these two tetrads the co-ordinates of any point in the plane based on the first tetrad can be expressed in terms of the co-ordinates based on the second tetrad by making use of the equations of the collineation. In the case where the Desarguesian property is weakened to the axiom of the fourth harmonic point this situation is no longer valid. Instead the most we can call on is Theorem 8. Nevertheless, given four points no three of which are collinear co-ordinates in the plane can be set up. If for two such tetrads it happens that a projective collineation exists mapping the first set onto the second it would be possible to relate both systems of co-ordinates once the equations of the collineation were determined. On the other hand if the two tetrads are not connected by a projective collineation the most that can be said is that the co-ordinates come from the same alternative field. There would be no way in which the co-ordinates with respect to one tetrad could be related to the co-ordinates of the second. We could then define two tetrads as conjugate if they are joined by a projective collineation. It is the author's conviction that a study of the manner in which the geometry breaks down into conjugate classes of tetrads would lead to a geometric proof of the fundamental theorem of alternative division rings. At present the results are too meagre to give any real information. Of course this notion of conjugate tetrad could be applied to any non-Desarguesian geometry. However, in the general case co-ordinates chosen from two distinct non-conjugate tetrads need not even belong to isomorphic algebraic systems.

Pickert **(14)** has published a book containing most of the known results concerning non-Desarguesian planes.

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