5

Observables in gauge theories

Modern theories of fundamental interactions are gauge theories. The principle of local gauge invariance was introduced by H. Weyl for the electromagnetic interaction in analogy with general covariance in Einstein's theory of gravitation. An extension to non-Abelian gauge groups was given by Yang and Mills [YM54].

A crucial role in gauge theories is played by the phase factor which is associated with parallel transport in an external gauge field. The phase factors are observable in quantum theory, in contrast with the classical theory. For the electromagnetic field, this is known as the Aharonov–Bohm effect.

In this chapter we initially consider the matrix notation for the non-Abelian gauge fields and introduce proper non-Abelian phase factors. Then we discuss the relation between observables in classical and quantum theories.

5.1 Gauge invariance

The principle of local gauge invariance deals with the gauge transformation (g.t.) of a matter field ψ, which is given by

$$\psi(x) \;\stackrel{\text{g.t.}}{\longrightarrow}\; \psi'(x) \;=\; \Omega(x)\,\psi(x)\,. \tag{5.1}$$

Here $\Omega(x) \in G$ with G being a semisimple Lie group which is called the *gauge group* ($G = SU(3)$ for QCD). Equation (5.1) demonstrates that ψ belongs to the fundamental representation of G.

We restrict ourselves to a unitary gauge group when

$$\Omega^{-1}(x) \;=\; \Omega^{\dagger}(x)\,, \tag{5.2}$$

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while an extension to other Lie groups is obvious. Then we have

$$\psi^\dagger(x) \stackrel{\text{g.t.}}{\longrightarrow} \psi'^\dagger(x) = \psi^\dagger(x)\,\Omega^\dagger(x)\,. \tag{5.3}$$

In analogy with QCD, the gauge group $G = SU(N)$ is usually associated with *color*, while the proper index of ψ is called the color index.

The gauge transformation (5.1) of the matter field ψ can be compensated by a transformation of the non-Abelian gauge field $\mathcal{A}_\mu$ which belongs to the adjoint representation of G:

$$\mathcal{A}_\mu(x) \stackrel{\text{g.t.}}{\longrightarrow} \mathcal{A}'_\mu(x) = \Omega(x)\,\mathcal{A}_\mu(x)\,\Omega^\dagger(x) + \mathrm{i}\,\Omega(x)\,\partial_\mu\Omega^\dagger(x)\,. \tag{5.4}$$

We have introduced in Eq. (5.4) the Hermitian matrix $\mathcal{A}_\mu(x)$ with the elements

$$[\mathcal{A}_\mu(x)]^{ij} = g\sum_a A^a_\mu(x)\,[t^a]^{ij}\,. \tag{5.5}$$

Here $[t^a]^{ij}$ are the generators of G ($a = 1, \ldots, N^2-1$ for $SU(N)$) which are normalized such that*

$$\operatorname{tr} t^a t^b = \delta^{ab}, \tag{5.6}$$

where tr is the trace over the matrix indices i and j, while g is the gauge coupling constant.

Equation (5.5) can be inverted to give

$$A^a_\mu(x) = \frac{1}{g}\operatorname{tr}\mathcal{A}_\mu(x)\,t^a. \tag{5.7}$$

Substituting

$$\Omega(x) = \mathrm{e}^{\mathrm{i}\alpha(x)}, \tag{5.8}$$

we obtain for an infinitesimal α:

$$\delta\,\mathcal{A}_\mu(x) \stackrel{\text{g.t.}}{=} \nabla^{\text{adj}}_\mu\,\alpha(x)\,. \tag{5.9}$$

Here

$$\nabla^{\text{adj}}_\mu\,\alpha \equiv \partial_\mu\alpha - \mathrm{i}\,[\mathcal{A}_\mu, \alpha] \tag{5.10}$$

is the covariant derivative in the adjoint representation of G, while

$$\nabla^{\text{fun}}_\mu\,\psi \equiv \partial_\mu\psi - \mathrm{i}\,\mathcal{A}_\mu\psi \tag{5.11}$$

* Quite often another normalization of the generators with an extra factor of 1/2, $\operatorname{tr}\tilde{t}^a\tilde{t}^b = \frac{1}{2}\delta^{ab}$, is used for historical reasons, in particular, $\tilde{t}^a = \sigma^a/2$ for the $SU(2)$ group, where σ^a are the Pauli matrices. This results in the redefinition of the coupling constant, $\tilde{g}^2 = 2g^2$.

is that in the fundamental representation. It is evident that

$$\nabla_\mu^{\rm adj} B(x) \;=\; [\nabla_\mu^{\rm fun}, B(x)]\,, \tag{5.12}$$

where $B(x)$ is a matrix-valued function of x.

The QCD action is given in the matrix notation as

$$S\left[\mathcal{A},\psi,\bar\psi\right] \;=\; \int {\rm d}^4x \left[\bar\psi\gamma_\mu\left(\partial_\mu - {\rm i}\,\mathcal{A}_\mu\right)\psi + m\bar\psi\psi + \frac{1}{4g^2}\,{\rm tr}\,\mathcal{F}_{\mu\nu}^2\right], \tag{5.13}$$

where

$$\mathcal{F}_{\mu\nu} \;=\; \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - {\rm i}\left[\mathcal{A}_\mu,\mathcal{A}_\nu\right] \tag{5.14}$$

is the (Hermitian) matrix of the non-Abelian field strength.

The action (5.13) is manifestly invariant under the local gauge transformation (5.1) and (5.4) since

$$\mathcal{F}_{\mu\nu}(x) \;\stackrel{\rm g.t.}{\longrightarrow}\; \Omega(x)\,\mathcal{F}_{\mu\nu}(x)\,\Omega^\dagger(x) \tag{5.15}$$

or

$$\delta\mathcal{F}_{\mu\nu}(x) \;\stackrel{\rm g.t.}{=}\; -{\rm i}\left[\mathcal{F}_{\mu\nu}(x),\alpha(x)\right] \tag{5.16}$$

for the infinitesimal gauge transformation.

For the Abelian group $G = U(1)$, the above formulas recover those of the previous part for QED where we have already used the calligraphic notation in Problem 3.6 on p. 61.

Problem 5.1 Rewrite classical equations of motion in the matrix notation.

Solution The non-Abelian Maxwell equation and the Bianchi identity are given, respectively, as

$$\nabla_\mu^{\rm adj}\,\mathcal{F}_{\mu\nu} \;=\; 0 \tag{5.17}$$

and

$$\nabla_\mu^{\rm adj}\,\widetilde{\mathcal{F}}_{\mu\nu} \;=\; 0\,, \tag{5.18}$$

where the dual field strength is defined by Eq. (3.49). Rewriting Eq. (5.14) as

$$\mathcal{F}_{\mu\nu} \;=\; {\rm i}\,[\nabla_\mu^{\rm fun},\nabla_\nu^{\rm fun}] \tag{5.19}$$

and using Eq. (5.12), we represent the Bianchi identity as

$$\epsilon_{\mu\nu\lambda\rho}[\nabla_\mu^{\rm fun},[\nabla_\nu^{\rm fun},\nabla_\lambda^{\rm fun}]] \;=\; 0 \tag{5.20}$$

which is obviously satisfied owing to the Jacobi identity.

We have thus proven the well-known fact that the Bianchi identity is satisfied explicitly in the second-order formalism, where $\mathcal{F}_{\mu\nu}$ is expressed via $\mathcal{A}_\mu$ by virtue of Eq. (5.14). In contrast, $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ are considered to be independent variables in the first-order formalism, where both equations (5.17) and (5.18) are essential. The concept of the first- and second-order formalisms comes from the theory of gravity.

5.2 Phase factors (definition)

In order to compare the phases of wave functions at distinct points, one needs a non-Abelian extension of the parallel transporter that was considered in Sect. 1.7. The proper extension of the Abelian formula (1.158) is written as

$$U[\Gamma_{yx}] \;=\; \boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{\Gamma_{yx}} \mathrm{d}z^\mu \mathcal{A}_\mu(z)}. \tag{5.21}$$

Although the matrices $\mathcal{A}_\mu(z)$ do not commute, the path-ordered exponential on the RHS of Eq. (5.21) is defined unambiguously by the general method of Sect. 1.3. This is obvious after rewriting the phase factor in an equivalent form

$$\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{\Gamma_{yx}} \mathrm{d}z^\mu \mathcal{A}_\mu(z)} \;=\; \boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_0^1 \mathrm{d}\sigma\, \dot{z}^\mu(\sigma)\, \mathcal{A}_\mu(z(\sigma))}. \tag{5.22}$$

Therefore, the path-ordered exponential in Eq. (5.21) can be understood as*

$$U[\Gamma_{yx}] \;=\; \prod_{t=0}^{\tau} \left[1 + \mathrm{i}\, \mathrm{d}t\, \dot{z}^\mu(t)\, \mathcal{A}_\mu(z(t))\right]. \tag{5.23}$$

We have already used this notation for the product on the RHS in Problem 1.9 on p. 22. Using Eq. (1.157), Eq. (5.23) can also be written as

$$U[\Gamma_{yx}] \;=\; \prod_{z\in\Gamma_{yx}} \left[1 + \mathrm{i}\, \mathrm{d}z^\mu \mathcal{A}_\mu(z)\right]. \tag{5.24}$$

If the contour Γ_{yx} is discretized as is shown in Fig. 1.3, then the non-Abelian phase factor is approximated by

$$U[\Gamma_{yx}] \;=\; \lim_{M\to\infty} \prod_{i=1}^{M} \left[1 + \mathrm{i}\,(z_i - z_{i-1})^\mu \mathcal{A}_\mu\left(\frac{z_i + z_{i-1}}{2}\right)\right], \tag{5.25}$$

which obviously reproduces (5.24) in the limit $\epsilon \to 0$.

Note that the non-Abelian phase factor (5.21) is, by construction, an element of the gauge group G itself, while $\mathcal{A}_\mu$ belongs to the Lie algebra of G.

* Sometimes the phase factor is defined using a similar formula but with the inverse order of multipliers. Our definition using Eq. (5.23) is exactly equivalent to Dyson's definition of the P-product (see the footnote on p. 3) which can be seen by choosing the contour Γ_{yx} to coincide with the temporal axis.

Problem 5.2 Write down an explicit expansion of the non-Abelian phase factor (5.21) in $\mathcal{A}_\mu$.

Solution Let us use the notation

$$\int_x^y \mathrm{d}z^\mu \cdots \equiv \int_{\Gamma_{yx}} \mathrm{d}z^\mu \cdots \tag{5.26}$$

for the integral along the contour Γ_{yx}. Then we have

$$\begin{aligned} &\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}z^\mu \mathcal{A}_\mu(z)} \\ &= \sum_{k=0}^{\infty} \mathrm{i}^n \int_x^y \mathrm{d}z_1^{\mu_1} \int_{z_1}^y \mathrm{d}z_2^{\mu_2} \cdots \int_{z_{k-1}}^y \mathrm{d}z_k^{\mu_k} \mathcal{A}_{\mu_k}(z_k)\cdots\mathcal{A}_{\mu_2}(z_2)\mathcal{A}_{\mu_1}(z_1)\,. \end{aligned} \tag{5.27}$$

The ordered integral in this formula can be rewritten in a more symmetric form as

$$\begin{aligned} &\int_0^\tau \mathrm{d}t_1 \int_{t_1}^\tau \mathrm{d}t_2 \ldots \int_{t_{k-1}}^\tau \mathrm{d}t_k\, \dot{z}^{\mu_1}(t_1)\dot{z}^{\mu_2}(t_2)\cdots\dot{z}^{\mu_k}(t_k) \\ &\qquad \times \mathcal{A}_{\mu_k}(z(t_k))\cdots\mathcal{A}_{\mu_2}(z(t_2))\mathcal{A}_{\mu_1}(z(t_1)) \\ &= \int_0^\tau \mathrm{d}t_1 \int_0^\tau \mathrm{d}t_2 \cdots \int_0^\tau \mathrm{d}t_k\, \theta(t_k, t_{k-1}, \ldots, t_2, t_1)\, \dot{z}^{\mu_1}(t_1)\dot{z}^{\mu_2}(t_2)\cdots\dot{z}^{\mu_k}(t_k) \\ &\qquad \times \mathcal{A}_{\mu_k}(z(t_k))\cdots\mathcal{A}_{\mu_2}(z(t_2))\mathcal{A}_{\mu_1}(z(t_1))\,, \end{aligned} \tag{5.28}$$

where

$$\theta(t_k, t_{k-1}, t_{k-2}, \cdots, t_2, t_1) = \theta(t_k - t_{k-1})\,\theta(t_{k-1} - t_{k-2})\cdots\theta(t_2 - t_1) \tag{5.29}$$

orders the points along the contour. We shall also denote this theta in a parametrization-independent form as

$$\theta(k, k-1, k-2, \ldots, 2, 1) \equiv \theta(t_k, t_{k-1}, t_{k-2}, \ldots, t_2, t_1)\,. \tag{5.30}$$

It satisfies the obvious identity

$$\begin{aligned} &\theta(k, k-1, k-2, \ldots, 2, 1) + \theta(k-1, k, k-2, \ldots, 2, 1) \\ &\qquad + (\text{other permutations of } k, \ldots, 1) \\ &= 1\,. \end{aligned} \tag{5.31}$$

For the Abelian case, when $\mathcal{A}_{\mu_i}(z_i)$ commute, Eq. (5.31) results in

$$\int_x^y \mathrm{d}z_1^{\mu_1} \int_{z_1}^y \mathrm{d}z_2^{\mu_2} \cdots \int_{z_{k-1}}^y \mathrm{d}z_k^{\mu_k} \mathcal{A}_{\mu_k}(z_k)\cdots\mathcal{A}_{\mu_2}(z_2)\mathcal{A}_{\mu_1}(z_1) = \frac{1}{k!}\left(\int_x^y \mathrm{d}z^\mu \mathcal{A}_\mu(z)\right)^k \tag{5.32}$$

so that the Abelian exponential of the contour integral is reproduced.

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Problem 5.3 Disentangle the non-Abelian phase factor using a path integral over Grassmann variables on a contour.

Solution Let us define the average

$$\left\langle F[\psi,\bar\psi]\right\rangle_\psi = \frac{\displaystyle\int \mathcal{D}\bar\psi(t)\,\mathcal{D}\psi(t)\,\mathrm{e}^{-\int_0^\tau \mathrm{d}t\,\bar\psi(t)\dot\psi(t)-\bar\psi(0)\psi(0)}\,F[\psi,\bar\psi]}{\displaystyle\int \mathcal{D}\bar\psi(t)\,\mathcal{D}\psi(t)\,\mathrm{e}^{-\int_0^\tau \mathrm{d}t\,\bar\psi(t)\dot\psi(t)-\bar\psi(0)\psi(0)}}\,. \tag{5.33}$$

The path integral in this formula looks like those of Chapter 2 with $\bar\psi_i(t)$ and $\psi_j(t)$ being Grassmann variables which depend on the one-dimensional variable $t\in[0,\tau]$ that parametrizes a contour, and i and j are the color indices.

The simplest average, which describes propagation of the color indices along the contour, is

$$\left\langle \psi_i(t_2)\bar\psi_j(t_1)\right\rangle_\psi = \delta_{ij}\,\theta(t_2-t_1)\,, \qquad 0\le t_1,t_2\le\tau\,. \tag{5.34}$$

This can be easily checked, say, by deriving the Schwinger–Dyson equation

$$\frac{\partial}{\partial t_2}\left\langle \psi_i(t_2)\bar\psi_j(t_1)\right\rangle_\psi = \delta_{ij}\,\delta^{(1)}(t_2-t_1)\,, \qquad 0< t_1,t_2<\tau \tag{5.35}$$

as was done in Chapter 3. We now see that we need the Grassmann variables because the operator in the action in Eq. (5.33) is $\partial/\partial t$.

A special comment is needed concerning the term $\bar\psi(0)\psi(0)$ in the exponents in Eq. (5.33), the appearance of which in the disentangling procedure is clarified in [HJS77]. The need for this term can be seen from the discretized version of the exponent:

$$\int\limits_0^\tau \mathrm{d}t\,\bar\psi(t)\dot\psi(t)+\bar\psi(0)\psi(0) \;\to\; \sum_{n=1}^{M}\bar\psi(n\epsilon)\left[\psi(n\epsilon)-\psi(n\epsilon-\epsilon)\right]+\bar\psi(0)\psi(0)\,. \tag{5.36}$$

For this discretization we immediately obtain

$$\left\langle \psi_i(n\epsilon)\bar\psi_j(m\epsilon)\right\rangle_\psi = \begin{cases}\delta_{ij} & \text{for } n\ge m\,,\\ 0 & \text{for } n<m\,.\end{cases} \tag{5.37}$$

The term $\bar\psi(0)\psi(0)$ is needed to provide nonvanishing integrals over $\bar\psi(0)$ and $\psi(0)$. It can also be seen from the discretized version that the path integral in the denominator on the RHS of Eq. (5.33) is equal to unity.

The fermionic path-integral representation for the non-Abelian phase factor (see, for example, [GN80]) is given as

$$\left[\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_0^\tau \mathrm{d}t\,\dot z^\mu(t)\mathcal{A}_\mu(z(t))}\right]_{ij} = \left\langle \mathrm{e}^{\mathrm{i}\int_0^\tau \mathrm{d}t\,\dot z^\mu(t)\bar\psi(t)\mathcal{A}_\mu(z(t))\psi(t)}\psi_i(\tau)\bar\psi_j(0)\right\rangle_\psi\,. \tag{5.38}$$

There is no path-ordering sign on the RHS since the matrix indices of $\mathcal{A}_\mu$ are contacted by ψ and $\bar\psi$.

In order to prove Eq. (5.38), one expands the exponential in $\mathcal{A}_\mu$ and calculates the average using Eq. (5.34) and the rules of Wick's pairing, which yields

$$\frac{1}{k!}\left\langle \psi_i(\tau)\left[\int_0^\tau \mathrm{d}t\,\dot z^\mu(t)\,\bar\psi(t)\mathcal{A}_\mu(z(t))\psi(t)\right]^k \bar\psi_j(0)\right\rangle_\psi$$
$$= \int_0^\tau \mathrm{d}t_1\int_0^\tau \mathrm{d}t_2\cdots\int_0^\tau \mathrm{d}t_k\,\theta(\tau,t_k,\ldots,t_2,t_1,0)\,\dot z^{\mu_1}(t_1)\dot z^{\mu_2}(t_2)\cdots\dot z^{\mu_k}(t_k)$$
$$\times\left[\mathcal{A}_{\mu_k}(z(t_k))\cdots\mathcal{A}_{\mu_2}(z(t_2))\mathcal{A}_{\mu_1}(z(t_1))\right]_{ij}\,, \tag{5.39}$$

where $\theta(\tau,t_k,\ldots,t_2,t_1,0)$ is given by Eq. (5.29). It is crucial in the derivation of this formula that only connected terms contribute to the average (5.33). Equation (5.39) reproduces Eq. (5.27) from the previous Problem, which completes the proof of Eq. (5.38). Moreover, we can say that the path integral (5.33) is nothing but a nice representation of the thetas (5.29).

Problem 5.4 Invert $(-\nabla^2+m^2)$ when ∇_μ is in the fundamental representation.

Solution The calculation is quite analogous to that of the Problem 1.13 on p. 29. We first use the path-integral representation of the inverse operator:

$$G(x,y;\mathcal{A})$$
$$\equiv \left\langle y\left|\frac{1}{-\nabla_\mu^{\rm fun}\nabla_\mu^{\rm fun}+m^2}\right|x\right\rangle$$
$$= \frac{1}{2}\int_0^\infty \mathrm{d}\tau\;\mathrm{e}^{-\frac{1}{2}\tau m^2}\int_{z_\mu(0)=x_\mu}\mathcal{D}z_\mu(t)\;\mathrm{e}^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\,\dot z_\mu^2(t)}\left\langle y\left|\boldsymbol{P}\,\mathrm{e}^{-\int_x^{z(\tau)}\mathrm{d}z^\mu\nabla_\mu^{\rm fun}}\right|x\right\rangle. \tag{5.40}$$

The integral over $z(\tau)$ – the final point of the trajectory – of the matrix element on the RHS equals

$$\int \mathrm{d}^d z(\tau)\left\langle y\left|\boldsymbol{P}\,\mathrm{e}^{-\int_x^{z(\tau)}\mathrm{d}z^\mu\nabla_\mu^{\rm fun}}\right|x\right\rangle = \boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}z^\mu\mathcal{A}_\mu(z)}. \tag{5.41}$$

Therefore, the result can be written as

$$G(x,y;\mathcal{A}) = {\sum_{\Gamma_{yx}}}'\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{\Gamma_{yx}}\mathrm{d}z^\mu\mathcal{A}_\mu(z)}, \tag{5.42}$$

where $\sum'$ is defined by Eq. (1.156).

Problem 5.5 Invert $(-\nabla^2+m^2)$ when ∇_μ is in the adjoint representation.

Solution Let us introduce

$$\nabla_\mu^{ab} = \partial_\mu\delta^{ab} - gf^{abc}A_\mu^c \tag{5.43}$$

and the Green function $G^{ab}(x,y;\mathcal{A})$ which obeys

$$\left(-\nabla_\mu^{ac}\nabla_\mu^{cb}+m^2\delta^{ab}\right)G^{bd}(x,y;\mathcal{A}) = \delta^{ad}\,\delta^{(d)}(x-y)\,. \tag{5.44}$$

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Then we obtain

$$G^{ab}(x,y;\mathcal{A}) \;=\; \sum_{\Gamma_{yx}}{}' \operatorname{tr} t^b\, U[\Gamma_{yx}]\, t^a\, U^\dagger[\Gamma_{yx}]\,, \tag{5.45}$$

where $U[\Gamma_{yx}]$ is given by Eq. (5.21).

Since matrices are rearranged in inverse order under Hermitian conjugation, one has*

$$U^\dagger[\Gamma_{yx}] \;=\; U[\Gamma_{xy}]\,. \tag{5.46}$$

In particular, the phase factors obey the *backtracking* condition

$$U[\Gamma_{yx}]\, U[\Gamma_{xy}] \;=\; 1\,. \tag{5.47}$$

We have chosen $\mathcal{A}_\mu$ in the discretized phase factor (5.25) at the center on the ith interval in order to satisfy Eq. (5.47) at finite ϵ.

Problem 5.6 Establish the relation between non-Abelian phase factors and the group of paths.

Solution The group of paths (or loops) is defined as follows. The elements of the group are the paths Γ_{yx}. The product of two elements Γ_{zx} and Γ_{yz} is the path Γ_{yx}, which is a composition of Γ_{zx} and Γ_{yz}. In other words, one first passes along the path Γ_{zx} and then the path Γ_{yz}. The product is denoted as

$$\Gamma_{yz}\,\Gamma_{zx} \;=\; \Gamma_{yx}\,. \tag{5.48}$$

The multiplication of paths is obviously associative but noncommutative. The inverse element is defined as

$$\Gamma_{yx}^{-1} \;=\; \Gamma_{xy}\,, \tag{5.49}$$

i.e. the path with opposite orientation.

It follows from definition (5.24) that

$$U[\Gamma_{yz}]\, U[\Gamma_{zx}] \;=\; U[\Gamma_{yz}\,\Gamma_{zx}]\,. \tag{5.50}$$

The backtracking condition (5.47) is then given by

$$U[\Gamma_{yx}\,\Gamma_{xy}] \;=\; 1\,. \tag{5.51}$$

In other words, the paths of opposite orientation cancel each other in the phase factors.

* The notation Γ_{yx} means that the contour is oriented from x to y, while Γ_{xy} denotes the opposite orientation from y to x. In the path-ordered product (5.24), these two contours result in opposite orders of multiplication for the matrices.

5.3 Phase factors (properties)

Under the gauge transformation (5.4) the non-Abelian phase factor (5.21) transforms as

$$U[\Gamma_{yx}] \quad \stackrel{\text{g.t.}}{\longrightarrow} \quad \Omega(y)\, U[\Gamma_{yx}]\, \Omega^{\dagger}(x)\,. \tag{5.52}$$

This formula stems from the fact that

$$\begin{aligned} [1+\mathrm{i}\,\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)] \quad &\stackrel{\text{g.t.}}{\longrightarrow} \quad \left[1+\mathrm{i}\,\mathrm{d}z^{\mu}\mathcal{A}'_{\mu}(z)\right] \\ &= \quad \Omega(z+\mathrm{d}z)\,[1+\mathrm{i}\,\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)]\,\Omega^{\dagger}(z) \end{aligned} \tag{5.53}$$

under the gauge transformation, which can be proven by substituting Eq. (5.4), so that $\Omega^{\dagger}(z)$ and $\Omega(z)$ cancel in the definition (5.24) at the intermediate point z.

One of the consequences of Eq. (5.52) is that $\psi(x)$, transported by the matrix $U[\Gamma_{yx}]$ to the point y, transforms under the gauge transformation as $\psi(y)$:

$$U[\Gamma_{yx}]\,\psi(x) \quad \stackrel{\text{g.t.}}{\sim} \quad \text{“}\psi(y)\text{”}, \tag{5.54}$$

and, analogously,

$$\bar{\psi}(y)\; U[\Gamma_{yx}] \quad \stackrel{\text{g.t.}}{\sim} \quad \text{“}\bar{\psi}(x)\text{”}. \tag{5.55}$$

Therefore, $U[\Gamma_{yx}]$ is, indeed, a parallel transporter.

It follows from these formulas that $\bar{\psi}(y)\,U[\Gamma_{yx}]\,\psi(x)$ is gauge invariant:

$$\bar{\psi}(y)\,U[\Gamma_{yx}]\,\psi(x) \quad \stackrel{\text{g.t.}}{\longrightarrow} \quad \bar{\psi}(y)\,U[\Gamma_{yx}]\,\psi(x)\,. \tag{5.56}$$

Another consequence of Eq. (5.52) is that the trace of the phase factor for a closed contour Γ is gauge invariant:

$$\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\oint_{\Gamma}\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)} \quad \stackrel{\text{g.t.}}{\longrightarrow} \quad \mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\oint_{\Gamma}\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)}. \tag{5.57}$$

These properties of the non-Abelian phase factor are quite similar to those of the Abelian one which was considered in Sect. 1.7.

Problem 5.7 Calculate $\partial U[\Gamma_{yx}]/\partial x_{\mu}$ and $\partial U[\Gamma_{yx}]/\partial y_{\mu}$.

Solution It is convenient to start from Eq. (5.25). Then only (z_1-x) in the last element of the product should be differentiated with respect to x or $(y-z_{M-1})$ in the first element of the product should be differentiated with respect to y. As $\epsilon\to 0$, we obtain

$$\left.\begin{aligned} \frac{\partial}{\partial x_{\mu}}\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)} \quad &= \quad -\mathrm{i}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)}\,\mathcal{A}_{\mu}(x)\,, \\ \frac{\partial}{\partial y_{\mu}}\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)} \quad &= \quad \mathrm{i}\,\mathcal{A}_{\mu}(y)\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)}. \end{aligned}\right\} \tag{5.58}$$

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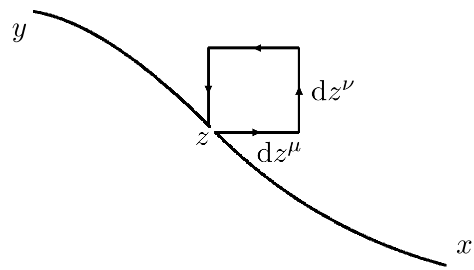

Fig. 5.1. The rectangular loop δC_{zz}, which is added to the contour Γ_{yx} at the intermediate point z in the (μ,ν)-plane.

These formulas are exactly the same as if one were to just differentiate the lower and upper limit in the path-ordered integral, bearing in mind the ordering of matrices.

One can rewrite Eq. (5.58) via the covariant derivatives as

$$\left.\begin{aligned} \nabla^{\rm fun}_\mu(y)\, U[\Gamma_{yx}] &= 0\,, \\ U[\Gamma_{yx}]\overleftarrow{\nabla}^{\rm fun}_\mu(x) &= 0\,. \end{aligned}\right\} \tag{5.59}$$

It is the property of the parallel transporter which is annihilated by the covariant derivative.

Problem 5.8 Prove that the sufficient and necessary condition for the phase factor to be independent on a local variation of the path is the vanishing of $\mathcal{F}_{\mu\nu}$.

Solution Let us add to Γ_{yx} at the point $z \in \Gamma_{yx}$ an infinitesimal loop δC_{zz} that lies in the (μ,ν)-plane and encloses the area $\delta\sigma_{\mu\nu}(z)$. Then the variation of the phase factor is

$$\delta U[\Gamma_{yx}] \;\equiv\; U[\Gamma_{yz}\,\delta C_{zz}\,\Gamma_{zx}] - U[\Gamma_{yx}] \;=\; \mathrm{i}\, U[\Gamma_{yz}]\,\mathcal{F}_{\mu\nu}(z)\,U[\Gamma_{zx}]\,\delta\sigma_{\mu\nu}(z)\,. \tag{5.60}$$

We can rewrite Eq. (5.60) as

$$\delta U[\Gamma_{yx}] \;=\; \mathrm{i}\,\boldsymbol{P}\,U[\Gamma_{yx}]\,\mathcal{F}_{\mu\nu}(z)\,\delta\sigma_{\mu\nu}(z) \tag{5.61}$$

since the P-product will automatically put $\mathcal{F}_{\mu\nu}(z)$ at the point z on the contour Γ_{yx}.

A convenient way to prove Eq. (5.60) is to choose δC_{zz} to be a rectangle which is constructed from the vectors $\mathrm{d}z^\mu$ and $\mathrm{d}z^\nu$, as depicted in Fig. 5.1. Using the representation (5.41), we see that the phase factor acquires the extra factor

$$[1+\mathrm{d}z^\nu\nabla_\nu]\,[1+\mathrm{d}z^\mu\nabla_\mu]\,[1-\mathrm{d}z^\nu\nabla_\nu]\,[1-\mathrm{d}z^\mu\nabla_\mu] \;=\; 1-\mathrm{d}z^\mu\mathrm{d}z^\nu\,[\nabla_\mu,\nabla_\nu] \tag{5.62}$$

at the proper order in the path-ordered product. Then Eq. (5.19) results in Eq. (5.61). Alternatively, one can prove Eq. (5.61) using the discretized formula (5.25).

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Problem 5.9 Derive a non-Abelian version of the Stokes theorem.

Solution The ordered contour integral can be represented as the double-ordered surface integral [Are80, Bra80]

$$\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\oint_{C_{xx}}\mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)} \;=\; \boldsymbol{P}_{\sigma}\,\boldsymbol{P}_{\tau}\,\mathrm{e}^{\mathrm{i}\int_{S}\mathrm{d}\sigma^{\mu\nu}\,\text{“}\mathcal{F}_{\mu\nu}(x)\text{”}}, \tag{5.63}$$

where τ and σ parametrize the surface S (spanned by C but arbitrary otherwise), the element of which is given by

$$\mathrm{d}\sigma^{\mu\nu} \;=\; \mathrm{d}\tau\,\mathrm{d}\sigma\left(\frac{\partial z_{\mu}}{\partial\tau}\frac{\partial z_{\nu}}{\partial\sigma} - \frac{\partial z_{\mu}}{\partial\sigma}\frac{\partial z_{\nu}}{\partial\tau}\right). \tag{5.64}$$

“$\mathcal{F}_{\mu\nu}(x)$” in Eq. (5.63) means that $\mathcal{F}_{\mu\nu}(z(\tau,\sigma))$ is parallel-transported to the initial point x.

Remark on an analogy with differential geometry

The formulas of the type of Eq. (5.60) are well-known in differential geometry where parallel transport around a small closed contour determines the curvature. Therefore, $\mathcal{F}_{\mu\nu}$ in Yang–Mills theory is the proper curvature in an internal color space while $\mathcal{A}_{\mu}$ is the connection.

A historical remark

An analog of the phase factors was first introduced by Weyl [Wey19] in his attempt to describe the gravitational and the electromagnetic interaction of an electron on an equal footing. What he did is associated in modern language with the scale rather than the gauge transformation, i.e. the vector-potential was not multiplied by i as in Eq. (1.158). This explains the term “gauge invariance” – gauging literally means fixing a scale. The factor of i was inserted by London [Lon27] after the creation of quantum mechanics and the recognition of the fact that the electromagnetic interaction corresponds to the freedom of choice of the phase of a wave function and not to a scale transformation. However, the terminology has remained.

5.4 Aharonov–Bohm effect

The simplest example of a gauge field is the electromagnetic field, for which transverse components describe photons. Otherwise, the longitudinal components of the vector-potential, which are changeable under the gauge transformation, are related to gauging the phase of a wave function, i.e. permit one to compare its values at different space-time points when an electron is placed in an external electromagnetic field.

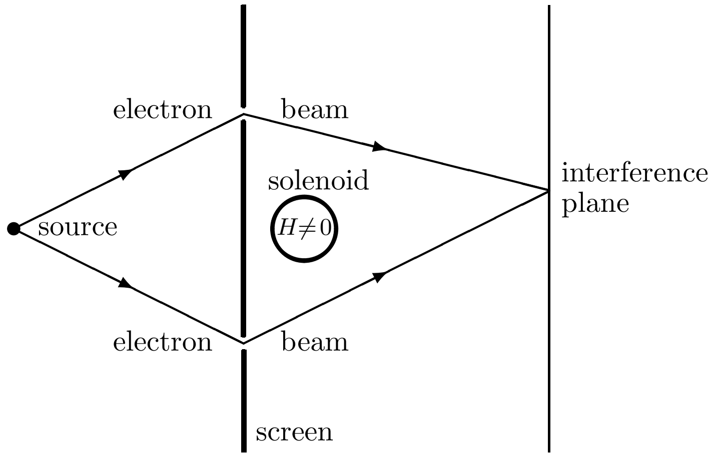

Fig. 5.2. Principal scheme of the experiment that demonstrates the Aharonov–Bohm effect. Electrons do not pass inside the solenoid where the magnetic field is concentrated. Nevertheless, a phase difference arises between the electron beams passing through the two slits. The interference picture changes with the value of the electric current.

As is well-known in quantum mechanics, the wave-function phase itself is unobservable. Only the phase differences are observable, for example via interference phenomena. For the electron in an electromagnetic field, the current (gauged) value of the phase of the wave function ψ at the point y is related, as is discussed in Sect. 1.7, to its value at some reference point x by the parallel transport which is given by Eq. (1.163). Therefore, the phase difference depends on the value of the phase factor for a given path Γ_{yx} along which the parallel transport is performed.

It is essential that the phase factors are observable in quantum theory, in contrast to classical theory. This is seen in the Aharonov–Bohm effect. The corresponding experiment is depicted schematically in Fig. 5.2.

It allows one to measure the phase difference between electrons passing through the two slits and, therefore, going across opposite sides of the solenoid. The fine point is that the magnetic field is nonvanishing only inside the solenoid where electrons do not penetrate. Hence the electrons pass throughout the region of space where the magnetic field strength vanishes! Nevertheless, the vector potential A_μ itself does not vanish which results in observable consequences.

The probability amplitude for an electron to propagate from a source at the point x to the point y in the interference plane is given by the

Minkowski-space analog of Eq. (1.155):

$$\Psi(x,y) = \sum_{\Gamma^+_{yx}}{}' \mathrm{e}^{\mathrm{i}e\int_{\Gamma^+_{yx}} \mathrm{d}z^\mu A_\mu(z)} + \sum_{\Gamma^-_{yx}}{}' \mathrm{e}^{\mathrm{i}e\int_{\Gamma^-_{yx}} \mathrm{d}z^\mu A_\mu(z)}, \tag{5.65}$$

where the contour Γ^+_{yx} passes through the upper slit, while the contour Γ^-_{yx} passes through the lower one.

The intensity of the interference pattern is given by $|\Psi(x,y)|^2$ which contains, in particular, the term proportional to (the real part of)

$$\mathrm{e}^{\mathrm{i}e\int_{\Gamma^+_{yx}} \mathrm{d}z^\mu A_\mu(z)}\, \mathrm{e}^{-\mathrm{i}e\int_{\Gamma^-_{yx}} \mathrm{d}z^\mu A_\mu(z)} = \mathrm{e}^{\mathrm{i}e\oint_\Gamma \mathrm{d}z^\mu A_\mu(z)}, \tag{5.66}$$

where the closed contour Γ is composed from Γ^+_{yx} and Γ^-_{xy}. This is nothing but the phase factor associated with a parallel transport along the closed contour Γ.

For the given process this phase factor does not depend on the shape of Γ^+_{yx} and Γ^-_{yx}. Applying the Stokes theorem, one obtains

$$\mathrm{e}^{\mathrm{i}e\oint_\Gamma \mathrm{d}z^\mu A_\mu} = \mathrm{e}^{\mathrm{i}e\int \mathrm{d}\sigma^{\mu\nu} F_{\mu\nu}} = \mathrm{e}^{\mathrm{i}eHS}, \tag{5.67}$$

where HS is the magnetic flux through the solenoid. Therefore, the interference picture changes when H changes.*

Remark on quantum vs. classical observables

A moral from the Aharonov–Bohm experiment is that the phase factors are observable in quantum theory while in classical theory only the electric and magnetic field strengths are observable. The vector potential plays, in classical theory, only an auxiliary role in determining the field strength.

For the non-Abelian gauge group $G = SU(N)$, a quark can alter its color under the parallel transport so the non-Abelian phase factor (5.21) is a unitary $N \times N$ matrix. A non-Abelian analog of the quantity, which is measurable in the Aharonov–Bohm experiment, is the trace of the matrix of the parallel transport along a closed path. It is gauge invariant according to Eq. (5.57).

It looks promising to reformulate gauge theories entirely in terms of these observable quantities. How this can be achieved will be explained in Part 3.

* A detailed computation of the interference picture for the Aharonov–Bohm experiment is contained, for example, in the review by Kobe [Kob79].