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CONTINUOUS IMAGES OF COMPACT SEMILATTICES

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ABSTRACT. Hyadic spaces are the continuous images of a hyperspace of a compact space. We prove that every non-isolated point in a hyadic space is the endpoint of some infinite cardinal subspace. We isolate a more general order-theoretic property of hyerspaces of compact spaces which is also enjoyed by compact semilattices from which the theorem follows.

1. **Introduction**. In topology an important method of generating, classifying and studying spaces is via continuous images of spaces which possess a restrictive but manageable structure of some kind. Hyadic spaces, i.e., continuous images of hyperspaces of compact spaces were introduced by E. van Douwen [2]. They include all dyadic spaces, all compact ordered spaces and even all normally supercompact spaces J. van Mill [6]. They omit all Stone-Cech compactifications βX where X is not pseudocompact. This latter fact was proven by E. van Douwen who also proved that every non-trivial G_δ in a hyadic space X contains the limit point of some non-trivial convergent sequence. Note that, in general, possessing a non-trivial convergent sequence is not a property that is preserved upon taking a continuous image, even for compact spaces.

Another theorem relating to convergent sequences is the following result of K. Hofmann et al. [4]: Every infinite compact semilattice contains a non-trivial convergent sequence.

The aim of our note is to strengthen both of the above results. Also, since the presence of an order compatible with the topology is the key element of both results, we achieve our improvements in a more general setting.

2. **Basic concepts**. All spaces considered in this note are assumed to be Hausdorff. A subspace A of a space X is called an ordinal subspace if A, with the subspace topology, is homeomorphic to some ordinal λ with the order topology. We say that p is the endpoint of an infinite cardinal subspace A of X if A is homeomorphic to an infinite cardinal λ and p is the unique complete accumulation point of A in X. In this case, we might also say that p is a λ-point of X.

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3. **Compact partial spaces**. Let X be a compact space and let $H(X)$ denote the hyperspace of all non-empty closed subsets of X endowed with the Vietoris topology. The Vietoris topology on $H(X)$ is compact and the union map $\cup : H(X) \times H(X) \to H(X)$ is a continuous surjection. Endowed with the union operation, $H(X)$ is an important example of a compact semilattice.

A compact semilattice S is a compact space endowed with a continuous, idempotent, commutative and associative binary operation $\bigvee$. Our main source for compact semilattice is G. Gierz et al. [3]. However, since we are motivated by the hyperspace, we will use the join $\bigvee$ as the operation and not the meet $\bigwedge$ as those authors do.

In order to reveal the essential ideas of our theorem we deal with partial orders. Let P be a set with a partial order $\leq$. If a subset A of P has a least upper bound (resp. greatest lower bound) then we denote it by $\sup A$ (resp. $\inf A$). If $p \in P$ then by $\downarrow p$ and $\uparrow p$ we mean the sets $\{q \in P : q \leq p\}$ and $\{q \in P : p \leq q\}$ respectively. An element p of P is called a compact element of P if whenever $p = \inf A$ then there exists a finite subset F of A with $p = \inf F$. If $A \subseteq P$ and $m \in A$ then we say that m is a maximal element of A if whenever $x \in A$ and $m \leq x$ then $m = x$. A conditionally complete partial order is one in which sup's (resp. inf's) exist for subsets bounded above (resp. bounded below).

We define a new notion. A compact partial space X is a compact space equipped with a conditionally complete partial order $\leq$ such that

(a) both $\downarrow x$ and $\uparrow x$ are closed in X for each $x \in X$.

(b) $\downarrow x$ is open in X for each compact element x of X.

Every compact semilattice S is a compact partial space under the ordering $x \leq y$ when $x \bigvee y = y$. The reader is referred to pages 272 and 280 of [3] for proofs of (a) and (b) in this case.

PROPOSITION 3.1. *Let X be a compact partial space with ordering $\leq$.*

(i) *If A is an up-directed (resp. down-directed) subset of X then $\sup A$ (resp. $\inf A$) exists and moreover, as a net under $\leq$ (resp. under $\geq$), A topologically converges to $\sup A$ (resp. $\inf A$).*

(ii) *If F is closed in X and $x \in F$ then there exists a maximal element m of F with $x \leq m$.*

(iii) *If x is not a compact element of X then there exists an infinite regular cardinal λ and a decreasing λ-sequence $(x_\alpha)_{\alpha<\lambda}$ with $x = \inf_{\alpha<\lambda} x_\alpha$.*

(iv) *If U is open in X and m is a maximal element of U then $\downarrow m$ is open in X.*

PROOF. (i) This is implied by compactness of X, Hausdorffness of X and condition (a) of being a partial space.

(ii) Use Zorn's Lemma to get a maximal chain M in F above x. Since M, as a net under $\leq$, topologically converges to $\sup M$ and F is closed we get $\sup M \in F$.

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(iii) If x is not a compact element of X then let A be a subset of X of least cardinality λ such that $x = \inf A$ but $x \neq \inf F$ for any finite subset F of A. Enumerate A as $\{a_\alpha : \alpha < \lambda\}$. Put $x_\alpha = \inf_{\beta<\alpha} a_\beta$ for each $\alpha < \lambda$. Then $x = \inf_{\alpha<\lambda} x_\alpha$ and λ is infinite and regular.

(iv) If U is open in X and m is a maximal element of U then since m is not in the closure of $X - U$, from (iii) and (i) we see that m must be a compact element of X. Hence, by condition (b) of being a partial space, we deduce that $\downarrow m$ is open in X. □

THEOREM 3.2. *Every non-isolated point in a continuous image of a compact partial space is the endpoint of some infinite cardinal subspace.*

PROOF. Assume that Y is a compact partial space with ordering $\leq$, $\mathcal{B}$ is the collection of all clopen subsets of Y, φ is a continuous map of Y onto X and that p is a non-isolated point of X. Put $P = \varphi^{-1}(p)$ and choose an $a \in \mathcal{B}d_Y P$. We consider three cases.

CASE 1. *There exists $B \in \mathcal{B}$ with $a \in B$ such that for each $b \in B - P$ either $a \notin \downarrow b$ or $\downarrow b \notin \mathcal{B}$.*

Put $M = \{m : m$ is a maximal element of $B - P\}$. M might be the empty set, but, since $B - P$ is open in Y, we have that for each $m \in M$, $\downarrow m \in \mathcal{B}$.

If $B - P \subseteq \bigcup_{m\in M} \downarrow m$ then, our case 1 assumption implies that M must be infinite. We claim that p is an ω-point of X. To see this, note that if K is a compact subspace of $B - P$ then $K \subseteq \bigcup_{m\in M} \downarrow m$ and so there exists a finite subset F of M such that $K \subseteq \bigcup_{m\in F} \downarrow m$. Since M consists of pairwise incomparable elements we deduce that K can contain at most finitely many members of M. Hence, every open neighbourhood of P contains all but finitely many members of M.

If $B - P \nsubseteq \bigcup_{m\in M} \downarrow m$ then there exists $r \in B - P$ such that for every $m \in M$ we have $r \nleq m$. Using the facts that r is in some fibre $B \cap \varphi^{-1}(x)$, that every element of any fibre $B \cap \varphi^{-1}(y)$ can be extended to a maximal element of that fibre $B \cap \varphi^{-1}(y)$ and that no element of $B - P$ which is greater than r is a maximal element of $B - P$ we can build, for some limit ordinal λ, an increasing λ-sequence $(r_\alpha)_{\alpha<\lambda}$ in $B - P$ such that distinct α and β have $\varphi(r_\alpha) \neq \varphi(r_\beta)$, limit ordinals α have $r_\alpha = \sup_{\beta<\alpha} r_\beta$ and finally, $\sup_{\alpha<\lambda} r_\alpha \in P$. Thus, p is the endpoint of the limit ordinal subspace $\{\varphi(r_\alpha) : \alpha < \lambda\}$ of X.

CASE 2. *There exists $B \in \mathcal{B}$ with $a \in B$ such that for each $b \in B \cap P$* either *$a \notin \downarrow b$* or *$\downarrow b \notin \mathcal{B}$.*

Since $B \cap P$ is closed in Y and $a \in B \cap P$ there exists a maximal element m in $B \cap P$ such that $a \leq m$. Our case 2 assumption implies that $\downarrow m$ is not open in Y. Hence m is not a compact element of Y and so there exists an infinite regular cardinal λ and a decreasing λ-sequence $(m_\alpha)_{\alpha<\lambda}$ such that $\inf_{\alpha<\lambda} m_\alpha = m$. Since φ may not be one to one on this λ-sequence one more step is required. Inductively, for $\beta < \lambda$, we choose $\beta' < \lambda$ such that (a) if β is a limit ordinal then $\beta' = \sup_{\gamma<\beta} \gamma'$ and (b) for each $\gamma < \beta$ and for each $\delta \geq \beta'$ we have $\varphi(m_\delta) \neq \varphi(m_{\gamma'})$. This is easily done since λ is

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regular, $(m_\alpha)_{\alpha<\lambda}$ topologically converges to m and each $\varphi^{-1}(\varphi(m_{\gamma'}))$ is a closed set which does not contain m. Now we have p the endpoint of the cardinal subspace $\{\varphi(m_{\beta'}) : \beta < \lambda\}$.

CASE 3. *For each $B \in \mathcal{B}$ with $a \in B$ there exists $b \in B - P$ and there exists $c \in B \cap P$ such that $a \in \downarrow b \cap \downarrow c$* and $\{\downarrow b, \downarrow c\} \subseteq \mathcal{B}$.

Inductively we construct $B_n \in \mathcal{B}$ and points b_n such that

(a) $b_n \in B_n - P$ for n even and $b_n \in B_n \cap P$ for n odd and

(b) $b_{n+1} < b_n$. Once completed, we have $(b_{2k})_{k<\omega}$ converging to $\inf_{n \text{ odd}} b_n \in P$.

Thus, p is an ω-point of X.

To do the induction, put $B_0 = Y$ and choose $b_0 \in B - P$ so that $a \in \downarrow b_0$ and $\downarrow b_0 \in \mathcal{B}$. In order to get B_{n+1} and b_{n+1} from B_n and b_n just apply the case 3 assumption to $B = \downarrow b_n$ to get b_{n+1} (in $B - P$ if $n + 1$ is even and in $B \cap P$ if $n + 1$ is odd) and put $B_{n+1} = \downarrow b_{n+1}$. □

As an example of a new compact space which we now know is not hyadic consider the first countable, locally compact, countably compact, but noncompact space X from A. Ostazewski [7] which does not contain the ordinal space ω_1. The one point compactification of X cannot be hyadic since the point at infinity is not a λ-point for any infinite cardinal λ.

We particularly want to mention the analogue of van Douwen's result on βX.

COROLLARY 3.3. *If βX is a continuous image of a compact partial space then X is pseudocompact. In particular, βN is not a continuous image of a compact partial space.*

PROOF. If X is not pseudocompact then βX maps onto βR. If βX was a continuous image of a compact partial space then βR would also be one. However, it follows from standard facts about βR that no point of $\beta R - R$ can be endpoint of an infinite cardinal subspace of βR. □

4. **Conclusion**. The circle with the Euclidean topology is not a compact semilattice, G. Aumann [1]; however it is easy to show that it is a compact partial space. We do not know whether the continuous images of these two classes of spaces coincide.

Are all compact semilattices hyadic? It is known that if X is a compact semilattice then the mapping $\varphi : H(X) \twoheadrightarrow X$ defined by $\varphi(F) = \sup F$ is continuous precisely when X is a "continuous lattice"; we refer the reader to page 285 of [3]. Every compact 0-dimensional semilattice is a "continuous lattice" but there are compact semilattices for which the above φ is not continuous, J. Lawson [5].

Lastly, we mention an interesting global open question for hyadic spaces (or for continuous images of compact semilattices). Is a hyadic space Fréchet-Urysohn (i.e., when the closure of any subset coincides with the sequential closure of that subset) precisely when it does not contain a copy of the ordinal space $\omega_1 + 1$?

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