WELL DISTRIBUTED SEQUENCES

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1. Introduction. In this note we discuss some properties of well distributed sequences. We take $0 \leqslant a < b \leqslant 1$ and let $I(x)$ denote the characteristic function of the interval $[a, b]$, so that

$$I(x) = \begin{cases} 1 \text{ if } x \in [a, b] \\ 0 \text{ otherwise.} \end{cases}$$

For convenience, we suppose that our sequences (s_n) satisfy $0 \leqslant s_n \leqslant 1$ for every positive integer n. A sequence (s_n) is said to be well distributed if

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=n+1}^{n+p} I(s_k) = b - a \tag{1}$$

holds uniformly in n, for every interval $[a, b]$. This may be regarded as a more stringent test of the regularity of distribution of a sequence (s_n) than the classical uniform distribution condition, where

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=1}^{p} I(s_k) = b - a \tag{2}$$

for every $[a, b]$. By a well-known theorem of Weyl **(1)**, the condition (2) may be expressed alternatively as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e(hs_k) = 0, \qquad h = 1, 2, \ldots,$$

where $e(t)$ denotes $e^{2\pi i t}$. A similar condition for well distributed sequences has been given by Petersen **(4)**. Thus, (s_n) is well distributed if, and only if,

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=n+1}^{n+p} e(hs_k) = 0, \qquad (h = 1, 2, \ldots) \tag{3}$$

uniformly in n, and this is the basis for our proof of Theorem 1. Throughout, we shall use $\{\theta\}$ to denote $\theta - [\theta]$, where $[\theta]$ is the largest integer $\leqslant \theta$.

2. THEOREM 1. *If (s_k) is well distributed and $s_k - t_k \to 0$ as $k \to \infty$, then (t_k) is well distributed.*

(With routine changes, the word "well" may be replaced both times by "uniformly".)

Proof. We will suppose that

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$$\frac{1}{p}\sum_{k=n+1}^{n+p} e(hs_k) \to 0, \qquad h = 1, 2, \ldots, \tag{4}$$

uniformly in n, as $p \to \infty$ and that $s_k - t_k \to 0$ as $k \to \infty$. Then, by (3), it suffices to prove that

$$\frac{1}{p}\sum_{k=n+1}^{n+p} e(ht_k) \to 0, \qquad h = 1, 2, \ldots, \tag{5}$$

uniformly in n, as $p \to \infty$.

Let $\epsilon > 0$. By our supposition that $s_k - t_k \to 0$ as $k \to \infty$, there is an $m_0 > 0$ such that

$$|e(h(t_m - s_m)) - 1| < \epsilon \quad \text{for all} \quad m > m_0, \qquad h = 1, 2, \ldots. \tag{6}$$

Here, m_0 may depend on h but is independent of n. Also, by our hypothesis concerning (4), there is a p_0 independent of n such that

$$\left|\frac{1}{p}\sum_{k=n+1}^{n+p} e(hs_k)\right| < \epsilon \quad \text{for all} \quad p > p_0, \qquad h = 1, 2, \ldots. \tag{7}$$

We apply these inequalities to the following identity:

$$\frac{1}{p}\sum_{k=n+1}^{n+p} e(ht_k) = \frac{1}{p}\sum_{k=n+1}^{n+p} e(hs_k) + \frac{1}{p}\sum_{k=n+1}^{n+p} e(hs_k)(e(h(t_k - s_k)) - 1) \tag{8}$$

and estimate the absolute values of the sums on the right. For the first, we simply use (7). For the second, it is convenient to consider two cases according as $n \geqslant m_0$ or $n < m_0$. If $n \geqslant m_0$, we use (6) and obtain the trivial estimate $p^{-1}(p\epsilon) = \epsilon$, valid for all integers $p \geqslant 1$. But if $n < m_0$, we express it in two parts:

$$\frac{1}{p}\sum_{k=n+1}^{n+p} = \frac{1}{p}\sum_{k=n+1}^{m_0} + \frac{1}{p}\sum_{k=m_0+1}^{n+p}.$$

Then, by applying (6) to the second term on the right, we get

$$\frac{1}{p}\left|\sum_{k=n+1}^{n+p}\right| < \frac{2m_0}{p} + \frac{1}{p}(p\epsilon), \tag{9}$$

since the summand is at most 2 in absolute value. Thus for $p > p_0' = 2m_0\epsilon^{-1}$, the terms on the right of (9) cannot exceed 2ϵ. Combining the two cases we see that, for all $p > \max(p_0, p_0')$,

$$\left|\frac{1}{p}\sum_{k=n+1}^{n+p} e(ht_k)\right| < \epsilon + 2\epsilon = 3\epsilon,$$

by (8). This completes the proof.

3. THEOREM 2. *If (s_k) is a countable everywhere dense sequence in the interval* $(0, 1)$, *then (s_k) can be enumerated in such a way as to be well distributed.*

Proof. It is known that $\{k\theta\}$, where θ is irrational, is well distributed. Since (s_k) is everywhere dense, we can select a subsequence (s_k') so that

$$|s_k' - \{k\theta\}| < \frac{1}{k}.$$

The terms of (s_k') may exhaust those of (s_k) in which case our statement follows from the previous theorem. But if this is not the case, we omit the terms s_{ν^3}' and this gives us a countable set of spaces to fill anew and we fill them with the set made up from those s_k not used and the s_k' omitted. This change will not affect any interval since if $r_k = 0$ for $k \neq \nu^3$ and $r_{\nu^3} = 1$, then

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=n+1}^{n+p} r_k = 0$$

uniformly in n (see Lorentz **(2)**). Hence in either case we have a well distributed sequence.

THEOREM 3. *For every irrational θ, there exists a sequence (n_k) such that*

$$\frac{n_k}{n_{k-1}} > \lambda > 1$$

and $\{n_k\theta\}$ is well distributed.

Proof. Since $\{n\theta\}$ is well distributed and everywhere dense we can choose a sequence (n_k) with $n_k/n_{k-1} > \lambda > 1$ such that $|\{n_k\theta\} - \{k\theta\}| < 1/k$. Theorem 1 then gives the result immediately.

4. Any real number θ can be represented uniquely in the form

$$c_0 + \sum_{i=2}^{\infty} \frac{c_i}{a_1 a_2 \ldots a_i}$$

where a_i and c_i are integers with $a_i > 1$ for all i and $0 \leqslant c_i \leqslant a_i - 1$, see **(3)**.

THEOREM 4. *If $a_i \geqslant a_{i-1} + 1$ for every i,*

$$\left\{\left(\prod_{i=1}^{k} a_i\right)\theta\right\}, \qquad k = 1, 2, \ldots,$$

is well distributed if, and only if,

$$\left(\frac{c_k}{a_k}\right)$$

is well distributed.

Proof. We have

$$\left\{\left(\prod_{i=1}^{k} a_i\right)\theta\right\} = \frac{c_{k+1}}{a_{k+1}} + \frac{c_{k+2}}{a_{k+1}a_{k+2}} + \ldots = \frac{c_{k+1}}{a_{k+1}} + R_{k+1}.$$

Since

$$\lim_{k\to\infty} R_{k+1} = 0,$$

our statement follows from Theorem 1.

For every such sequence (a_i), we can evidently construct a θ such that

$$\left\{\left(\prod_{i=1}^{k} a_i\right)\theta\right\}$$

is well distributed by choosing c_k so that

$$\lim_{k\to\infty}\left|\frac{c_k}{a_k} - \{k\sqrt{2}\}\right| = 0.$$

Similar remarks apply to uniform distribution. We have $\{n!\theta\}$ uniformly distributed for almost all θ (**5**, Satz 21). Hence, if

$$\theta = \sum_{n=1}^{\infty} \frac{a_n}{n!},$$

(a_n/n) is uniformly distributed for almost all θ.

In the special case when $a_i = r$ for all i, we have our numbers expressed to the base r. For any r, a number θ is said to be a *normal* number if and only if the sequence $\{\theta\}$, $\{r\theta\}$, $\{r^2\theta\}, \ldots$, is uniformly distributed. By a theorem of Hardy-Littlewood (**1**, Ch. IX, §28) it is known that almost all θ are normal. For a result in the opposite direction, we have

THEOREM 5. *If p, q are positive integers, the sequence*

$$\left\{\left(\frac{p}{q}\right)^k\theta\right\}, \qquad k = 1, 2, \ldots$$

is not well distributed for any θ.

Proof. We may suppose that the sequence is uniformly distributed since otherwise, there is nothing to prove. Then, given N however large, we can find an $m = m(N)$ such that

$$\left\{\frac{p^m}{q^m}\theta\right\} < \frac{\pi}{4p^N q^N}.$$

Then

$$\sum_{k=m+1}^{m+N} e\left(q^N \frac{p^k}{q^k}\theta\right) = \sum_{k=1}^{N} e\left(q^N \frac{p^k}{q^k}\frac{p^m}{q^m}\theta\right) = \sum_{k=1}^{N} e\left(q^{N-k}p^k\left\{\frac{p^m}{q^m}\theta\right\}\right)$$

where

$$0 < p^k q^{N-k}\left\{\frac{p^m}{q^m}\theta\right\} < \frac{\pi p^k q^{N-k}}{4p^N q^N} < \frac{\pi}{4} \qquad \text{for all } k \leqslant N.$$

Hence

$$\left|\sum_{k=m+1}^{m+N} e\left(q^N \frac{p^k}{q^k}\theta\right)\right| > \sum_{k=1}^{N} \cos\frac{\pi}{4} = N/\sqrt{2},$$

and the result follows from our criterion (3).

References

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