TWO VERTEX-REGULAR POLYHEDRA

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1. Introduction. The definition of a regular polyhedron may be enunciated as follows:

(α) A polyhedron is said to be regular if its faces are equal regular polygons, and its vertex figures are equal regular polygons.

In a recent note[1] I gave three examples of uniform non-regular polyhedra, which I called *facially-regular*, using the definition:

(β) A polyhedron is said to be facially-regular if it is uniform and all its faces are equal.

It is evident that this definition may be replaced by one similar to that for regular polyhedra given above, such as[2]:

(γ) A polyhedron is said to be facially-regular if its faces are equal regular polygons, and its vertex figures are equal polygons.

A comparison of the definitions (α) and (γ) suggests that it would be of interest to investigate polyhedra which differ only from regular polyhedra in that the condition for regularity on the faces is dropped in (α), instead of the condition for regularity on the vertex figures (which gives (γ)). I call such polyhedra *vertex-regular*, using the definition:

(δ) A polyhedron is said to be vertex-regular if its faces are equal polygons and its vertex figures are equal regular polygons;

and in this note I describe two such polyhedra.

It will be noticed that if we define the regularity of a polyhedron in the more normal way:

(ϵ) A polyhedron is said to be regular if it possesses two particular symmetries; one which cyclically permutes the vertices of any face c, and one which cyclically permutes the faces that meet at a vertex C, C being a vertex of c;

then facially-regular polyhedra possess the first of these symmetries, and vertex-regular polyhedra possess the second.

For convenience I use the notation a^b to denote a vertex-regular polyhedron whose faces are a-gons, b of which meet at each vertex; but the symbol does not necessarily define the polyhedron uniquely.

The polyhedra which I describe are in this notation 12^3 and 9^3.

2. The polyhedron 12^3. From the ordinary space filling $[4^4]$, remove cubes

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[1] Can. J. Math., vol. 2 (1950), 326.

[2] For the sake of simplicity in this and the next definition I consider a polyhedron to be such that every face is accessible to any other face by paths crossing from one face to another by the edge common to both.

in such a manner that, for those remaining, each vertex of each member is a vertex of just one other member, and no two members have an edge in common.

Consider each cube as composed of 27 congruent component cubes. Squash the pile so that each cube retains its size and orientation, but corner components which originally touched come into coincidence (Fig. 1(a)).

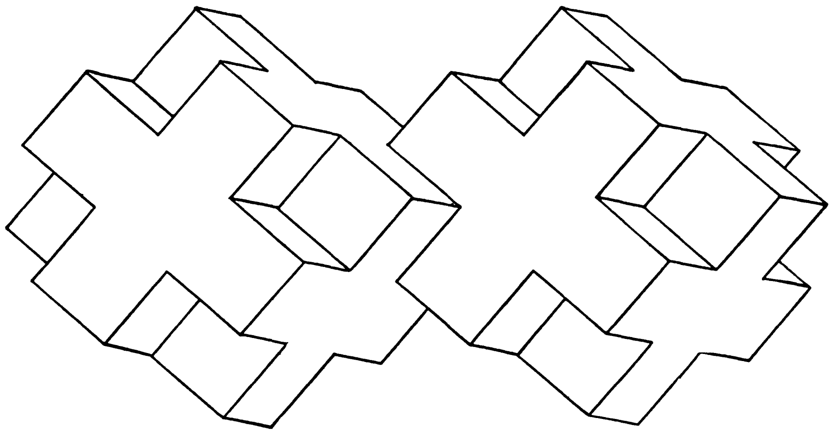

FIG. 1(a)

The surface of each cube not enclosed in the surface of any other cube now consists of six crosses (Fig. 1(b)), and the arrangement of all these crosses is such that each edge of each cross is in common with just one other cross. Further, three such crosses meet at each vertex, and their planes are mutually perpendicular, so that the vertex figure is an equilateral triangle. Hence the

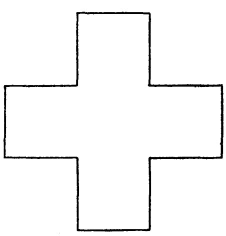

FIG. 1(b)

polyhedron formed by the aggregate of the crosses has for its faces eqqal polygons and for its vertex figures equilateral triangles: it is vertex-regular.

The vertices of this polyhedron can best be described by referring it to rectangular Cartesian coordinates. Classify the integers by their residues (mod 8), and label P those points having odd coordinates one of which is congruent to 1 or 7, and another of which is congruent to 3 or 5 (the third being congruent to any one of 1, 3, 5, 7). Then the points P are the vertices of a 12^3 having edge 2, vertex figures equilateral triangles, and faces equal to the "Greek cross" whose vertices are in order:

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$$\begin{array}{cccccc} (3, 1) & (1, 1) & (1, 3) & (-1, 3) & (-1, 1) & (-3, 1) \\ (-3, -1) & (-1, -1) & (-1, -3) & (1, -3) & (1, -1) & (3, -1). \end{array}$$

The coordinates of the centres of the faces of this 12^3 lie at the points having coordinates congruent (mod 8) to one of the sets 4, 4, 1; 4, 4, 7; 0, 0, 3; 0, 0, 5; in some order. As I showed in my previous note, these points are the vertices of a facially-regular polyhedron which I described as 3^{12}, so that the polyhedra 3^{12} and 12^3 are reciprocal in the sense that the vertices of a 3^{12} lie at the centres of the faces of a 12^3 (but not *vice versa*).

3. The polyhedron 9^3. For clarity I first give a short description of the facially-regular polyhedron 3^9. Consider a set of octahedra each coloured α on two opposite faces and β on the remainder, and a set of tetrahedra coloured γ on all faces. Join these polyhedra by their faces according to the rule $\alpha \leftrightarrow \gamma$; then the β faces are the faces of a 3^9. It will be seen that one could alternatively consider the 3^9, viewed as a solid body, as composed of suitably selected tetrahedra and octahedra from the space filling of octahedra and tetrahedra[3] [3^4]; and that the faces of 3^9 lie in planes parallel to any of its component tetrahedra according to the system shown in Fig. 2(a).

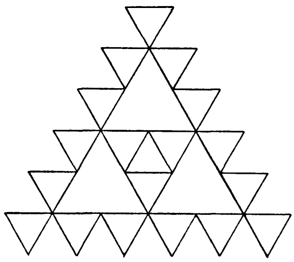

FIG. 2(a)

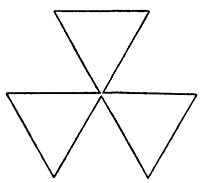

FIG. 2(b)

The faces of a 3^9 occur in sets of three. Regard each set of three as constituting an irregular enneagon, having as vertices the vertices of a regular hexagon and its centre (which is counted three times). Such an enneagon is shown slightly distorted in Fig. 2(b). The vertices, edges and face planes of the 3^9 remain unaltered, and as three such enneagons meet at each vertex, and each vertex has as its vertex figure an equilateral triangle, it follows that the polyhedron having the enneagons as its faces satisfies the definition of vertex-regularity. Hence there is a vertex-regular polyhedron 9^3, which occupies the same space as 3^9, and is in fact 3^9 thought of in another way.

The vertices of a 9^3 are also the centres of its faces, and again the vertices of a 3^9, so that the polyhedra 3^9 and 9^3 are reciprocal in the sense that the vertices of a 3^9 are the centres of the faces of a 9^3. The main point of interest is that the polyhedron 3^9 reciprocates in this sense into a polyhedron occupying the same space.

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[3]W. W. R. Ball, *Mathematical Recreations and Essays* (11th ed.), London, 1949, p. 147.

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