ON KADISON'S CONDITION FOR EXTREME POINTS OF THE UNIT BALL IN A B^*-ALGEBRA

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(Received 4th July 1968)

1. Introduction

Let B be a complex Banach algebra with an identity 1 and an involution $x \to x^*$. Kadison (1) has shown that, if B is a B^*-algebra, the set of extreme points of its unit ball coincides with the set $\mathfrak{E}$ of elements x of B for which

$$(1-x^*x)B(1-xx^*) = (0). \tag{1}$$

This elegant result is very useful for B^*-algebra theory; see (1) and (3). In this paper we examine the set $\mathfrak{E}$ for algebras B which are not necessarily B^*-algebras. It is shown that the spectral radius of each $x \in \mathfrak{E}$ is at least one. In Section 4 we consider the set $\mathfrak{E}$ for the special case where B is the algebra of all bounded linear operators on the infinite-dimensional Hilbert space H. Here $\mathfrak{E}$ is the set of semi-unitary elements T ($TT^* = 1$ or $T^*T = 1$). For such T we show that there exists a complex number b, $|b| = 1$, such that $b-T$ is not a semi-Fredholm operator on H. (For this notion see Section 4 or (2)). This then says that b lies in the essential spectrum of T when we use the rather restrictive definition of essential spectrum due to Kato (2, p. 243).

2. Algebraic considerations

We begin with some pure ring theory. Let A be a ring with identity 1 and an involution $x \to x^*$. Let $\mathfrak{E}(A)$ denote the set $x \in A$ for which

$$(1-x^*x)A(1-xx^*) = (0).$$

For $x \in \mathfrak{E}(A)$ we have $1 = xx^* \circ x^*x = x^*x \circ xx^*$ where we use the familiar notation (4) that $u \circ v = u+v-uv$.

Proposition 1. *Let* $x \in \mathfrak{E}(A)$. *Then* $x^n \in \mathfrak{E}(A)$ *for* $n = 1, 2, \ldots$.

Proof. Let $x \in \mathfrak{E}(A)$. The following computations use ideas of Miles (3, p. 631).

First we show that

$$(1-(x^*)^n x^n)A(1-xx^*) = (0) \tag{2}$$

for $n = 1, 2, \ldots$. By hypothesis, this is valid for $n = 1$ and we suppose it is true for the integer n. Note that

$$1-(x^*)^{n+1}x^{n+1} = 1-(x^*)^n x^n + (x^*)^n(1-x^*x)x^n. \tag{3}$$

[1] This research was supported by the National Science Foundation Grant NSF-GP-7516. The author is grateful to the referee for his helpful comments.

Set $w = (1-(x^*)^{n+1}x^{n+1})y(1-xx^*)$. Using (3), we write $w = a+b$ where

$$a = (1-(x^*)^n x^n)y(1-xx^*) = 0,$$

$$b = (x^*)^n(1-x^*x)x^n y(1-xx^*) = 0$$

since $x \in \mathfrak{E}(A)$. Likewise the relation

$$1-x^{n+1}(x^*)^{n+1} = 1-x^n(x^*)^n + x^n(1-xx^*)(x^*)^n \tag{4}$$

leads to the conclusion that

$$(1-x^*x)A(1-x^n(x^*)^n) = (0) \tag{5}$$

for $n = 1, 2, \ldots$.

Now we show that $x^n \in \mathfrak{E}(A)$ by induction. Suppose this holds for exponents $k = 1, \ldots, n$. Set

$$w = (1-(x^*)^{n+1}x^{n+1})y(1-x^{n+1}(x^*)^{n+1}).$$

Using (3) and (4), we rewrite w as the sum of four terms each of which must be zero by (2) and (5) and the induction hypothesis. This establishes the desired result. In particular, $x \in \mathfrak{E}(A)$ is never a nilpotent element of A.

For further results we assume that the involution is proper ($x^*x = 0$ implies $x = 0$). One then readily verifies that the four statements (a) x^*x is an idempotent, (b) xx^* is an idempotent, (c) $x = xx^*x$, (d) $x^* = x^*xx^*$ are equivalent. We then call x a *partial isometry*. Arguments of Miles (3, p. 630) show that any $x \in \mathfrak{E}(A)$ is a partial isometry. These results can fail if the involution is not proper. For an example let A be the ring of all numbers of the form $a+bi$, $i^2 = -1$, under the usual operations, where a and b lie in the ring of integers modulo 16. For $x = a+bi$, set $x^* = a-bi$. Then $x = 2+i$ lies in $\mathfrak{E}(A)$ but xx^* is not an idempotent.

Proposition 2. *Suppose that the involution in A is proper. If $x^n \in \mathfrak{E}(A)$ for some integer n and x is a partial isometry, then $x \in \mathfrak{E}(A)$.*

Proof. Suppose that $x^n \in \mathfrak{E}(A)$ and x is a partial isometry. For each $y \in A$ we can, using (1) with x^n instead of x, obtain an expression for y as

$$y = (x^*)^n x^n y + yx^n(x^*)^n - (x^*)^n x^n yx^n(x^*)^n. \tag{6}$$

We use (6) in $(1-x^*x)y(1-xx^*)$ and the facts $x = xx^*x$, $x^* = x^*xx^*$ to see that $(1-x^*x)y(1-xx^*) = 0$.

We use the following language customary in the theory of von Neumann algebras. A *projection* is a self-adjoint idempotent. A projection p is called *abelian* if pAp is an abelian ring and is called *minimal* if pA is a minimal right ideal.

Proposition 3. *Suppose that the involution in A is proper. Let $x \in A$ be a partial isometry. Then x^*x is an abelian (minimal) projection if and only if xx^* is an abelian (minimal) projection.*

Proof. Let $p = x^*x$, $q = xx^*$. Suppose that pAp is commutative and let $y, z \in A$. Then

$$px^*yxpx^*zxp = px^*zxpx^*yxp. \tag{7}$$

But $px^* = x^*xx^* = x^*$ and $xpx^* = q$. Multiplying (7) on the left by x and on the right by x^* shows that $qyqzq = qzqyq$ or q is an abelian projection.

Suppose that p is a minimal projection. By algebraic theory, either $xpA = (0)$ or xpA is a minimal right ideal. But $xpx^* = q \neq 0$. Then

$$(0) \neq qA \subset xpA.$$

Proposition 4. *Suppose that the involution in A is proper. Let $x \in \mathfrak{E}(A)$. Suppose that x^*x is an abelian (minimal) projection. Then A is a commutative ring (division ring).*

Proof. Let $p = x^*x$, $q = xx^*$. Suppose that p is an abelian projection. Then

$$pqpxp = pxpqp. \tag{8}$$

But $xp = x = qx$. Moreover, since $x \in \mathfrak{E}(A)$, we have, as noted earlier, $p \circ q = q \circ p = 1$. In particular, $pq = qp$. Then

$$pqpxp = qpxp = qpx = pqx = px.$$

On the other hand, $pxpqp = pxpq = pxq$. Then (8) yields $px(1-q) = 0$. We combine this with $(1-p)x(1-q) = 0$ to obtain $x(1-q) = 0$. This gives $p(1-q) = x^*x(1-q) = 0$. Since $p+q-pq = 1$ we get $q = 1$. But, by Proposition 3, q is an abelian projection. Hence A is abelian.

Suppose now that p is a minimal projection. First we show that $pq = 0$ is impossible. For suppose otherwise. Then $p+q = 1$ and $x(p+q) = x$. Consequently, $xq = 0 = x^2(x^*)^2$. Since the involution is proper, $x^2 = 0$. This contradicts Proposition 1. This implies that pq is a non-zero projection. Since $pqA = pA$, we may invoke a lemma of Rickart (**4**, p. 261) to see that $p = pq$. Via Proposition 3 we also get $q = qp$. Then $p = q = 1$. Since 1 is a minimal projection, A is a division ring.

3. The set $\mathfrak{E}$ for a Banach algebra B

For $x \in B$, a Banach algebra, we use the notation of (**4**), $\nu(x) = \lim \| x^n \|^{1/n}$ where $\nu(x)$ is also the spectral radius of x. We assume that B has an identity 1 and an involution $x \rightarrow x^*$ but do not suppose that the involution is proper.

Theorem 1. *For each $x \in \mathfrak{E}$ we have $\nu(x) \geqq 1$. If B is a B^*-algebra, then $\nu(x) = 1$ for $x \in \mathfrak{E}$.*

Proof. Let $x \in \mathfrak{E}$. Proposition 1 gives $x^n \in \mathfrak{E}$, $n = 1, 2, 3, \ldots$. This implies that

$$(1-(x^n)^*x^n)(1-x^n(x^n)^*) = 0.$$

Hence, for $n = 1, 2, 3, \ldots$, 1 belongs to the spectrum of $(x^n)^*x^n$ or $x^n(x^n)^*$, leading to the conclusion that

$$\begin{aligned} 1 &\leqq \nu((x^n)^*x^n) = \nu(x^n(x^n)^*) \\ &\leqq \|(x^n)^*x^n\| \leqq \|(x^*)^n\| \, \|x^n\|, \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

Therefore $1 \leqq \nu(x^*)\nu(x) = \nu(x)^2$.

https://doi.org/10.1017/S0013091500012761 Published online by Cambridge University Press

If B is a B^*-algebra, Proposition 1 and the cited result of Kadison (1) make $\| x^n \| = 1$, for $x \in \mathfrak{E}$, $n = 1, 2, 3, \ldots$. Hence $v(x) = 1$.

We show, by example, that one can have a Banach algebra B where $v(x)$ is as large as desired for a suitable $x \in \mathfrak{E}$. Let n be a positive integer, and let Y be the subset of the real line, $Y = [0, 1] \cup \{2, 3\}$ with the usual topology. Let $C(Y)$ be the Banach algebra of all complex continuous functions on Y with the sup norm. We define an involution $x \to x^\#$ on Y by the rule that $x^\#(t) = \overline{x(t)}$ if $t \in [0, 1]$, $x^\#(2) = \overline{x(3)}$ and $x^\#(3) = \overline{x(2)}$. One sees that the function $x(t) = 1$, $t \in [0, 1]$, $x(2) = n$, $x(3) = n^{-1}$ lies in $\mathfrak{E}$ and $v(x) = n$.

Corollary 1. *Let K be a proper two-sided closed *-ideal of B. Then*

$$\text{dist}\,(\mathfrak{E}, K) \geqq 1.$$

Proof. Let π be the natural homomorphism of B onto B/K and let $x \in \mathfrak{E}$. Clearly $\pi(x) \in \mathfrak{E}(B/K)$ and, by Theorem 1, $\text{dist}\,(x, K) = \| \pi(x) \| \geqq v(\pi(x)) \geqq 1$.

Corollary 2. *Let B_1 be a B^*-algebra with an identity and T be an algebraic *-homomorphism of B_1 onto a dense subset of B. Then $v(T(x)) = 1$ and $\| T(x) \| \geqq \| x \|$ for each $x \in \mathfrak{E}(B_1)$.*

Proof. In this situation, $T(x) \in \mathfrak{E}(B)$. Then, by Theorem 1,

$$v(x) \geqq v(T(x)) \geqq 1 = v(x) = \| x \|.$$

Since we also have $\| T(x) \| \geqq v(T(x))$, the desired relations follow.

4. The set $\mathfrak{E}$ for operator algebras

First we consider the algebra $B(X)$ of all bounded linear operators on a complex Banach space X and the closed two-sided ideal $K(X)$ of compact operators. Let $R(T)$ denote the range of $T \in B(X)$. We define $\text{nul}\,(T)$ as the dimension of $T^{-1}(0)$ and $\text{def}\,(T)$ as the dimension of $X/R(T)$ (these are called ∞ if they are not finite). As usual (2) T is called *semi-Fredholm* if $R(T)$ is closed and either $\text{nul}\,(T) < \infty$ or $\text{def}\,(T) < \infty$. If $R(T)$ is closed and both $\text{nul}\,(T) < \infty$, $\text{def}\,(T) < \infty$, T is said to be a *Fredholm operator.* For a Fredholm operator we take as its *index*, $\text{ind}\,(T) = \text{nul}\,(T) - \text{def}\,(T)$.

Let σ denote the natural homomorphism of $B(X)$ onto $B(X)/K(X)$. Following (2), p. 242, we let $\Delta = \Delta(T)$ denote the semi-Fredholm region for $T \in B(X)$. This is the set of complex numbers a for which $a - T$ is a semi-Fredholm operator. Also Δ_F denotes the subset consisting of all a for which $a - T$ is a Fredholm operator. Then Δ_F can also be described as the a for which $\sigma(a - T)$ has a two-sided inverse in $B(X)/K(X)$; see (5), p. 617. We are also concerned with the essential spectrum $\Sigma_e = \Sigma_e(T)$ in the sense of (2), p. 243, which is the complement of $\Delta(T)$. That $\Delta_F \cup \Sigma_e$ does not in general exhaust the complex plane adds interest to Theorem 2. We use the notation

$$r = r(T) = \lim \| (\sigma(T))^n \|^{1/n}.$$

Theorem 2. *Let $T \in B(X)$ where X is infinite-dimensional. Then each complex number a, $| a | = r$, lies in $\Delta_F \cup \Sigma_e$. At least one such a lies in Σ_e.*

Proof. Let $|a| = r$ and suppose that $a \notin \Delta_F$. First we show that $a-T$ cannot have a closed range together with nul $(a-T)<\infty$. For suppose otherwise. As $a \notin \Delta_F$, def $(a-T) = \infty$. Let $x_1, \ldots, x_n$ be a (finite) basis for the null space of $a-T$. There exist $y_1, \ldots, y_n$ in X, linearly independent modulo $R(a-T)$. Choose $x_j^* \in X^*, j = 1, \ldots, n$, such that $x_j^*(x_k) = \delta_{jk}, j, k = 1, \ldots, n$ and set

$$V(x) = \sum_{j=1}^{n} x_j^*(x)y_j.$$

Then $V(x_j) = y_j, j = 1, \ldots, n$ and $V \in K(x)$.

We claim that $a-T-V$ is one-to-one. For suppose $V(z) = (a-T)(z)$. Then $(a-T)(z)$ is a linear combination of $y_1, \ldots, y_n$ so that $(a-T)(z) = 0 = V(z)$. Then z can be written as $z = b_1x_1 + \ldots + b_nx_n$ and $V(z) = b_1y_1 + \ldots + b_ny_n = 0$. Thus each $b_j = 0$ and $z = 0$.

It is clear that $R(a-T-V) \subset R(a-T) \oplus R(V)$. To see the reverse set inequality, suppose that $u = (a-T)(x)$ and $v = V(y)$. Let

$$w = \sum_{i=1}^{n} x_i^*(y)x_i, \quad z = x - \sum_{i=1}^{n} x_i^*(x)x_i.$$

Then an easy computation shows that

$$(a-T-V)(z-w) = u+v.$$

Hence $R(a-T-V)$ is closed, $R(a-T-V) \neq X$. By (5), p. 618, there exists $\varepsilon>0$ such that, for $|\lambda-a|<\varepsilon$, $\lambda-T-V$ is a one-to-one bicontinuous linear mapping of X onto a proper closed subspace of X. This is the case for a special choice of a complex number b, $|b|>|a|$, $|b-a|<\varepsilon$. Since $|b|>r$, $b-T$ is invertible in $B(X)/K(X)$ and $b-T-V$ is a Fredholm operator. In view of (5), Lemma 2.4, ind $(b-T-V) = 0$. But nul $(b-T-V) = 0$ so that def $(b-T-V) = 0$. But then $R(b-T-V) = w$, which is a contradiction.

We show next that $a-T$ cannot be semi-Fredholm with nul $(a-T) = \infty$ and def $(a-T)<\infty$. For suppose otherwise. Then $a-T^*$ is a semi-Fredholm operator on the Banach space X^*, nul $(a-T^*)<\infty$, def $(a-T^*) = \infty$, whereas $\lambda-T^*$ is a Fredholm operator if $|\lambda|>|a|$. The above reasoning will again lead to a contradiction.

Finally not all complex numbers a, $|a| = r$ can be in Δ_F. For suppose otherwise. First consider the case $r = 0$. Then the spectrum of $\sigma(T)$ in

$$B(X)/K(X)$$

would be void. For the case $r>0$, we use the fact that Δ_F is open. Note that $\lambda \in \Delta_F$ if $|\lambda|>r$. Then there exists $s<r$ such that $\lambda \in \Delta_F$ if $|\lambda|>s$. But then $\sigma(\lambda-T)$ is invertible in $B(X)/K(X)$ for all $|\lambda|>s$. This makes $r<s$, which is a contradiction.

Corollary 3. *Let H be an infinite-dimensional Hilbert space and T be a semi-unitary element of $B(H)$. Then there exists a complex number b, $|b| = 1$, which lies in $\Sigma_e(T)$.*

https://doi.org/10.1017/S0013091500012761 Published online by Cambridge University Press

Proof. Consider again the natural homomorphism σ of $B(H)$ onto

$$B(H)/K(H).$$

Then $\sigma(T)$ is a semi-unitary element of the quotient algebra and $r(T) = 1$ in the notation of Theorem 2. By that result, the desired conclusion follows.

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