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A STOCHASTIC PROOF OF AN EXTENSION OF A THEOREM OF RADO

by BERNT ØKSENDAL

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1. Introduction

The purpose of this article is to illustrate how the theorem of Lévy about conformal invariance of Brownian motion can be used to obtain information about boundary behaviour and removable singularity sets of analytic functions. In particular, we prove a Frostman–Nevanlinna–Tsuji type result about the size of the set of asymptotic values of an analytic function at a subset of the boundary of its domain of definition (Theorem 1). Then this is used to prove the following extension of the classical Radó theorem: If ϕ is analytic in $B\backslash K$, where B is the unit ball of $\mathbb{C}^n$ and K is a relatively closed subset of B, and the cluster set $\mathscr{C}$ of ϕ at K has zero harmonic measure w.r.t. $\phi(B\backslash K)\backslash\mathscr{C}\neq\emptyset$, then ϕ extends to a meromorphic function in B (Theorem 2).

2. Brownian motion and boundary behaviour of analytic functions

Let $B_t(\omega)$, $0\leqq t<\infty$, $\omega\in\Omega$, denote Brownian motion in $\mathbb{C}^n=\mathbb{R}^{2n}$. Let P^x denote the probability law of $B_t(\omega)$ starting at the point $x\in\mathbb{C}^n$. We may regard P^x as a measure on the space Ω of continuous paths $\omega:[0,\infty)\to\mathbb{R}^{2n}$.

An n-dimensional version of the theorem of P. Lévy about conformal invariance of Brownian motion can be stated as follows:

Let $U\subset\mathbb{C}^n$ be open, $x\in U$ and $\phi: U\to\mathbb{C}$ analytic.

Let

$$\tau=\tau_U(\omega)=\inf\{t>0; B_t\notin U\}$$

be the first exit time from U of $B_t(\omega)$, and define

$$\sigma(t)=\int_0^t|\phi'(B_s)|^2\,ds \quad \text{for} \quad 0\leqq t\leqq\tau.$$

Let $\hat{B}_t(\hat{\omega}); 0\leqq t<\infty$, $\hat{\omega}\in\hat{\Omega}$ with probability law $\hat{P}^y$, be Brownian motion in $\mathbb{C}=\mathbb{R}^2$ and define $W_t(\cdot):\Omega\times\hat{\Omega}\to\mathbb{R}^2$ by

$$W_t(\omega,\hat{\omega})=\begin{cases}\phi(B_{\sigma^{-1}(t)}(\omega)); t<\sigma(\tau)\\ \phi^*(\omega)+\hat{B}_{t-\sigma(\tau)}(\hat{\omega}); t\geqq\sigma(\tau),\end{cases} \tag{2.1}$$

where the limit

$$\phi^*(\omega) = \lim_{u \uparrow \tau} \phi(B_u(\omega)) \quad \text{exists a.e. on} \quad \{\omega; \sigma(\tau) < \infty\}. \tag{2.2}$$

Then $W_t(\cdot)$, with the probability law $P^x \times \hat{P}^0$ on $\Omega \times \hat{\Omega}$, induces a measure on the space of paths in $\mathbb{C}$ which is precisely that of 2-dimensional Brownian motion starting at $y = \phi(x)$.

A proof of an extended Lévy theorem can be found in [**2**]. The existence of the limit (2.2) is not proved explicitly in [**2**]. A proof in a more general setting, which also includes a proof of (2.2), can be found in [**5**].

The following result can be regarded as a partial extension of a theorem of Frostman–Nevanlinna–Tsuji (see [**12**], Th. VIII. 44, p. 339). If $V \subset \mathbb{C}^n$ is open, we call a function $\phi: V \to \mathbb{C} \cup \{\infty\}$ *meromorphic* in V if each $x \in V$ has a neighbourhood $V_x \subset V$ such that either ϕ or $1/\phi$ is analytic on V_x.

Let $f: V \to \mathbb{C}$ be continuous and $E \subset \bar{V}$, the closure of V. We say that $y \in \mathbb{C} \cup \{\infty\}$ is an *asymptotic value* of f at E if there exists a curve γ in V terminating at some point $z \in E$ such that

$$y = \lim_{\substack{x \to z \\ x \in \gamma}} f(x)$$

The set of all asymptotic values of f at E is denoted by $A_f(E)$, the *asymptotic set* of f at E.

Theorem 1. (Extended Frostman–Nevanlinna–Tsuji theorem) *Let $V \subset \mathbb{C}^n$ be connected, open, $K \subset \partial V$ (the boundary of V) compact. Suppose ϕ is meromorphic in V, let $A = A_\phi(K)$ be the asymptotic set of ϕ at K. If $\phi(V) \backslash A \neq \emptyset$ and A has harmonic measure* 0 *w.r.t. $\phi(V) \backslash A$, then K has harmonic measure* 0 *w.r.t. V. In particular, if* $\operatorname{cap}(A) = 0$, *then K has harmonic measure* 0 *w.r.t. V.*

Here cap denotes the logarithmic capacity.

Proof. Put $K_1 = K \cup \phi^{-1}(\infty)$ and let $U = V \backslash K_1$. Then ϕ is analytic in U and we apply the Lévy theorem above.

Let $G = \{\omega; \sigma(\tau) < \infty\}$, $H = \{\omega; B_{\tau(\omega)}(\omega) \in K_1\}$. For a.a. $\omega \notin G$ the path of $\phi(B_{\sigma^{-1}(t)})$; $0 \leqq t < \sigma(\tau) = \infty$ is dense in $\mathbb{C}$. So, if the limit $b = B_{\tau(\omega)}(\omega)$ exists, we have $A_\phi(\{b\}) = \mathbb{C}$ and therefore $b \notin K_1$ for a.a. $\omega \notin G$. Therefore $P^x(H \backslash G) = 0$.

If $A = A_\phi(K)$ has harmonic measure 0 w.r.t. $\phi(V) \backslash A$, then A has harmonic measure 0 w.r.t. $\phi(U) \backslash A$. Let $(\hat{B}_t, \hat{\Omega}, \hat{P})$ denote Brownian motion in $\mathbb{C}$, $\hat{\tau}$ the first time from $\phi(U) \backslash A$ of $\hat{B}_t$. Then

$$\hat{P}[\exists t \leqq \hat{\tau} \quad \text{with } \hat{B}_t \in A] = 0$$

Since $\sigma(\tau) \leqq \hat{\tau}$ a.s. $P \times \hat{P}^0$, we deduce that

$$P[\exists t \leqq \sigma(\tau) < \infty \quad \text{with} \quad \phi(B_{\sigma^{-1}(t)}) \in A] = 0.$$

This implies that

$$P[\sigma(\tau)<\infty, B_\tau \in K_1]=0.$$

i.e.

$$P[H\cap G]=0.$$

Therefore

$$P^x(H)=0 \quad \text{for all} \quad x\in U,$$

i.e. K_1 has harmonic measure 0 w.r.t. $U=V\backslash K_1$.

Thus, since $\phi^{-1}(\infty)\subset V$ is never hit by Brownian motion in $\mathbb{C}^n$ a.s., we conclude that K has harmonic measure 0 w.r.t. V.

Remark. For results similar to Theorem 1 in a more general context (harmonic morphisms) but with cluster sets instead of asymptotic sets, see [**6**] and (**8**].

3. An extension of a theorem of Radó

A classical theorem of T. Radó from 1924 [**10**] states that if ϕ is continuous on $\Delta=\{z\in C; |z|<1\}$ and analytic on $\Delta\backslash\phi^{-1}(0)$, then ϕ is analytic on Δ. Subsequently several extensions of this result have been found. See [**1**], [**3**], [**7**], [**9**], [**11**] and—for a more abstract version—[**4**]. Using Theorem 1 we now prove a version which contains essentially the extensions in [**1**], [**3**], [**7**] and [**11**].

If $V\subset\mathbb{C}^n$ is open and $f: V\to\mathbb{C}$ is continuous we let $\mathscr{C}=\mathrm{Cl}_f(E)$ denote the set of cluster values of f at the set $E\subset\bar{V}$.

Theorem 2. (Extended Radó theorem) *Let B be the open unit ball in $\mathbb{C}^n$, K a relatively closed subset of B. Let $\phi: B\backslash K\to\mathbb{C}$ be analytic. Put $\mathscr{C}=\mathrm{Cl}_\phi(K)$ and $A=A_\phi(K)$.*

(i) *Suppose $\phi(B\backslash K)\backslash\mathscr{C}\neq\varnothing$ and $\mathscr{C}$ has harmonic measure 0 w.r.t. $\phi(B\backslash K)\backslash\mathscr{C}$. Then ϕ extends to a meromorphic function on B.*

(ii) *Suppose ϕ is bounded, $\phi(B\backslash K)\backslash A\neq\varnothing$ and A has harmonic measure 0 w.r.t. $\phi(B\backslash K)\backslash A$. Then ϕ extends to an analytic function on B.*

Remark. The conditions on $\mathscr{C}$ and A in (i) and (ii) are satisfied if, for example, $\mathrm{cap}(\mathscr{C})=0$ and $\mathrm{cap}(A)=0$, respectively.

Proof. (i) Choose a component W of $B\backslash K$ such that $\phi(W)\backslash\mathscr{C}\neq\varnothing$. By Theorem 1, K has harmonic measure 0 w.r.t. W. So Brownian motion starting from a point $x\in W$ does not hit K before it hits ∂B. This implies that $C_{2n}(K)=0$ and that $W=B\backslash K$. Here C_{2n} is the capacity associated to the kernel $|z|^{-2n+2}$ for $n>1$ and $\log(1/|z|)$ for $n=1$ (i.e. $C_2=\mathrm{cap}$). Choose a point $a\in K$. Since $\mathrm{Cl}_\phi(\{a\})\neq\mathbb{C}$ there exists $b\in\mathbb{C}$ and a neighbourhood V of a such that $\psi(z)=1/(\phi(z)-b)$ is bounded in $V\cap W$. But then $K\cap V$ is a removable singularity set for $\psi(z)$, so that $\psi(z)$ extends analytically across V. Hence ϕ extends to a meromorphic function in V, and since $a \in K$ was arbitrary, the proof of (i) is complete. The first part of this argument proves (ii).

Remark. Both Theorem 1 and Theorem 2(i) remain valid if we only assume that ϕ is meromorphic in the weaker sense that ϕ is locally (in V or $B\backslash K$) of the form f/g, where f and g are analytic. This is because the set of singularities of such functions has C_{2n}-capacity zero (as can be seen, for example, from Theorem 1). In Theorem 2(i) the conclusion is still that ϕ, locally at K, is meromorphic in the stronger sense defined above. We conjecture that Theorem 2(i) also holds if we replace the cluster set $\mathscr{C}$ by the asymptotic set A.

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Department of Mathematics
University of Edinburgh
The King's Buildings
Edinburgh EH9 3JZ
Scotland

Present address:
Mathematical Institute
University of Oslo
Blindern, Oslo 3
Norway