MINIMAL COHESIVE BASIC SETS

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1. Introduction

A *basic set* (formerly *basic sequence*) $\mathscr{B}$ is a set of pairs (a, b) of positive integers satisfying

(1) if $(a, b) \in \mathscr{B}$, then $(b, a) \in \mathscr{B}$,

(2) $(a, bc) \in \mathscr{B}$ if and only if $(a, b) \in \mathscr{B}$ and $(a, c) \in \mathscr{B}$,

(3) $(1, k) \in \mathscr{B}$, $k = 1, 2, \ldots$.

Some familiar examples of basic sets are

$$\mathscr{S} = \bigcup_{k=1}^{\infty} S_k, \text{ where } S_k = \{(1, k), (k, 1)\},$$

$$\mathscr{M} = \{(a, b) |\ a \text{ and } b \text{ are relatively prime positive integers}\},$$

$$\mathscr{L} = \{(a, b) |\ a \text{ and } b \text{ are any positive integers}\}.$$

If Φ is any set of pairs of positive integers, the basic set *generated* by Φ is the intersection of all basic sets which contain Φ. If $\mathscr{B}$ is generated by Φ, we write

$$\mathscr{B} = \Gamma[\Phi].$$

A pair (p, q) is called a *primitive pair* if both p and q are primes.

A basic set $\mathscr{B}$ is *cohesive* if, for each positive integer k, there is an integer $a = a(k) > 1$ such that $(k, a) \in \mathscr{B}$. $\mathscr{M}$ and $\mathscr{L}$ are cohesive, as is the basic set

$$\mathscr{B}[p^*] = \Gamma[\bigcup_{q \in P} (p^*, q)]$$

generated by the primitive pairs $\bigcup_{q \in P} (p^*, q)$, where p^* is any fixed prime and P is the set of all primes. $\mathscr{B}$ is *minimally cohesive* provided

(1) $\mathscr{B}$ is cohesive,

(2) if $\mathscr{B}' \subset \mathscr{B}$ but $\mathscr{B}' \neq \mathscr{B}$ then $\mathscr{B}'$ is not cohesive.

The function-theoretic and combinatorial properties of arbitrary basic sets were discussed in (**1**) and (**2**), and those of cohesive basic sets in (**3**). We confine ourselves in this note to a further investigation of the combinatorial

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properties of cohesive basic sets. In particular, we give a complete determination of the collection of all minimal cohesive basic sets and show that there are no minimal cohesive basic sets contained in $\mathscr{M}$.

Our principal result is the following

Theorem. *A basic set $\mathscr{B}$ is minimally cohesive if and only if $\mathscr{B} = \mathscr{B}[p^*]$ for some prime p^*.*

2. Proof of the Theorem

We will use several lemmas leading to our main result.

Lemma 1. *Suppose that $\mathscr{B}$ is cohesive and that $\Phi_{\mathscr{B}}$ is the set of all primitive pairs in $\mathscr{B}$. Then $\mathscr{B}$ is minimally cohesive if and only if*

$$\mathscr{B}(p, q) = \Gamma[\Phi_{\mathscr{B}} - \{(p, q), (q, p)\}]$$

is not cohesive for every primitive pair (p, q) in $\mathscr{B}$.

Lemma 2. *If $\mathscr{B}$ is cohesive and $\mathscr{B} \subset \mathscr{M}$, then for every positive integer k there are infinitely many primes r such that $(k, r) \in \mathscr{B}$.*

Lemma 3. *There are no minimal cohesive basic sets in $\mathscr{M}$.*

Proof. Let $\mathscr{B}$ be any cohesive basic set contained in $\mathscr{M}$, and let (p_0, q_0) be any primitive pair in $\mathscr{B}$. Choose any integer $k > 1$. By Lemma 2, there is a prime r different from p_0 and q_0 for which $(k, r) \in \mathscr{B}$. For each prime divisor p of k, $(p, r) \in \mathscr{B}$ and also $(p, r) \in \mathscr{B}(p_0, q_0)$. Hence $(k, r) \in \mathscr{B}(p_0, q_0)$. It follows that $\mathscr{B}(p_0, q_0)$ is cohesive and so, by Lemma 1, $\mathscr{B}$ is not minimally cohesive. That proves Lemma 3.

For a basic set $\mathscr{B}$ and a positive integer k, let $C_{\mathscr{B}}(k)$ denote the set of prime companions of k in $\mathscr{B}$; that is,

$$C_{\mathscr{B}}(k) = \{p \mid p \in P, (p, k) \in \mathscr{B}\}.$$

Note that $\mathscr{B}$ is cohesive if and only if $C_{\mathscr{B}}(k)$ is never empty for any k.

We are now ready for the proof of the main theorem.

Suppose p^* is a fixed (but arbitrary) prime and let $\mathscr{B} = \mathscr{B}[p^*]$. $\mathscr{B}$ is clearly cohesive since $(k, p^*) \in \mathscr{B}$ for every positive integer k. Moreover, $\mathscr{B}(p^*, q)$ is not cohesive for any prime q. For if $q \neq p^*$, then $C_{\mathscr{B}(p^*, q)}(q)$ is empty, and if $q = p^*$, then $C_{\mathscr{B}(p^*, p^*)}(p^* q')$ is empty, where q' is any prime different from p^*. Therefore $\mathscr{B}[p^*]$ is minimally cohesive.

Conversely, suppose that $\mathscr{B}$ is any minimally cohesive basic set. It is sufficient to show that the primitive pairs (p^*, q) are in $\mathscr{B}$ for some fixed prime p^* and every prime q in P.

By Lemma 3, $\mathscr{B} \not\subset \mathscr{M}$. Set

$$\mathscr{B}_1 = \mathscr{B} \cap \mathscr{M}.$$

$\mathscr{B}_1$ is a basic subset of $\mathscr{M}$, and we assert that $\mathscr{B}_1$ is not cohesive. For if (p, q) is any primitive pair in $\mathscr{B}_1$, then also $(p, q) \in \mathscr{B}$, and Lemma 1 and the minimal cohesiveness of $\mathscr{B}$ imply that $\mathscr{B}(p, q)$ is not cohesive. But $\mathscr{B}_1(p, q) \subset \mathscr{B}(p, q)$, and so $\mathscr{B}_1(p, q)$ is not cohesive. Therefore if $\mathscr{B}_1$ were cohesive, then by Lemma 1 it would be a minimally cohesive basic subset of $\mathscr{M}$, contrary to Lemma 3.

Since $\mathscr{B}_1$ is not cohesive, there is an integer $k_0 > 1$ such that, for every integer $a > 1$, $(k_0, a) \notin \mathscr{B}_1$. Now if p is any prime in $C_{\mathscr{B}}(k_0)$, then p must divide k_0, for otherwise p and k_0 would be relatively prime, so $(p, k_0) \in \mathscr{B} \cap \mathscr{M} = \mathscr{B}_1$, contrary to the choice of k_0. In particular $C_{\mathscr{B}}(k_0)$ is finite.

Enumerate all the primes: $q_1, q_2, \ldots$, and set

$$R_l = C_{\mathscr{B}}(k_0) \cap C_{\mathscr{B}}(q_1) \cap C_{\mathscr{B}}(q_2) \cap \ldots \cap C_{\mathscr{B}}(q_l),$$

for $l = 1, 2, \ldots$. We assert that $R_l \neq \varnothing$ $(l = 1, 2, \ldots)$. Since $\mathscr{B}$ is cohesive, there is an integer $a > 1$ for which

$$(a, k_0 q_1 q_2 \ldots q_l) \in \mathscr{B}.$$

It follows that $(a, k_0) \in \mathscr{B}$, so each prime divisor of a is in $C_{\mathscr{B}}(k_0)$. Moreover, $(a, q_i) \in \mathscr{B}$ $(i = 1, \ldots, l)$, so each prime divisor of a is also in $C_{\mathscr{B}}(q_i)$ $(i = 1, \ldots, l)$. Hence $R_l \neq \varnothing$, as claimed.

It follows from the preceding that $\{R_l\}_{l=1}^{\infty}$ is a nested, decreasing sequence of non-empty, finite sets of primes. Therefore

$$R = \bigcap_{l=1}^{\infty} R_l \neq \varnothing.$$

If p^* is any prime in R, then $(p^*, q) \in \mathscr{B}$ for every prime q in P.

That completes the proof of the theorem.

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