

# STRUCTURE-REVERSIBILITY OF A TWO-DIMENSIONAL REFLECTING RANDOM WALK AND ITS APPLICATION TO QUEUEING NETWORK

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We consider a two-dimensional reflecting random walk on the non-negative integer quadrant. It is assumed that this reflecting random walk has skip-free transitions. We are concerned with its time-reversed process assuming that the stationary distribution exists. In general, the time-reversed process may not be a reflecting random walk. In this paper, we derive necessary and sufficient conditions for the time-reversed process also to be a reflecting random walk. These conditions are different from but closely related to the product form of the stationary distribution.

## 1. INTRODUCTION

We consider a two-dimensional reflecting random walk on the non-negative integer quadrant. We are interested in its stationary distribution for queueing applications. This stationary distribution is generally hard to analytically get, and recent research interests have been directed to its tail asymptotics. We now have good pictures for those tail asymptotics (e.g., see [8]), but many other characteristics like moments are not available. In this paper, we look at this problem up side down. Namely, we aim to find a class of the reflecting random walks whose stationary distributions are obtained in closed form.

For this, we use a time-reversed process for the reflecting random walk, and expect that the stationary distribution is analytically obtained when the time-reversed process is also a reflecting random walk. Kelly [4] pioneered to use this time-reversed idea for deriving the so-called product form solutions for various queueing network models. Here, the stationary distribution is said to be a product form solution if it is the product of marginal stationary distributions. This product form solution has been further studied (see, e.g., [2,4,10] and references therein). However, they are limited in use for applications.

While those traditional approaches use the Markov chains which are specialized to queueing models, we here use a different class of Markov chains. Namely, we take two-dimensional reflecting random walks for this class motivated by Miyazawa [9]. They may be

interpreted as queueing models, but our intention is to consider the time-reversed processes within a class of two-dimensional reflecting random walks. In this way, we study a class of the reflecting random walks which have tractable stationary distributions.

To make clear our arguments, we introduce the notion of structure-reversibility. A reflecting random walk is said to be structure-reversible if it has a stationary distribution and if its time-reversed process under this stationary distribution is a reflecting random walk, where its transition probabilities may not be identical with those of the original reflecting random walk. We then derive necessary and sufficient conditions for the reflecting random walk to be structure-reversible under appropriate assumptions. In this derivation, the stationary distribution is simultaneously obtained. This stationary distribution has product form in the interior of the quadrant but may not be of product form on the boundary.

It is notable that our stationary distribution is closely related to the one which was recently obtained by Latouche and Miyazawa [6]. They derived it by characterizing a class of the reflecting random walks whose stationary distribution has product form. In the interior of the quadrant, this two-dimensional distribution has geometric marginals whose rates are identical with those of ours. We have geometric interpretations to characterize such decay rates similarly to those of [6]. However, the stationary distribution under structure-reversibility is not necessary to have product form as we already remarked (see Corollary 3.1 and Remark 4.2 for further discussions). We also note that the terminology, “structure-reversibility”, is used for queueing networks with batch movements in Miyazawa [7]. Its spirit is the same as the one of the present paper, but the classes of models to be applied are different.

This paper is made up by five sections. We formally define our reflecting random walk and its time-reversed process in Section 2. In Section 3, we derive necessary and sufficient conditions of structure-reversibility assuming the existence of the stationary distribution. It is not easy to check those structure-reversibility conditions since they use the stationary distribution. In Section 4, we obtain the structure-reversibility conditions not using the stationary distribution, that is, only using modeling parameters. We finally in Section 5 discuss a class of queueing networks flexible at boundaries, and give examples to be structure-reversible.

## 2. TWO-DIMENSIONAL REFLECTING RANDOM WALK AND ITS TIME-REVERSED PROCESS

In this section, we briefly introduce a two-dimensional reflecting random walk and its time-reversed process. We use the following notations.

$\mathbb{Z}$  = the set of all integers,  $\mathbb{Z}_+ = \{i \in \mathbb{Z}; i \geq 0\}$ ,  $\mathbb{R}$  = the set of all real numbers.

Let  $\mathcal{S} \equiv \mathbb{Z}_+^2$  be a state space for the reflecting random walk. We partition it into following subsets.

$$\begin{aligned} \mathcal{S}_0 &= \{(0, 0)\}, & \mathcal{S}_1 &= \{(i, 0) \in \mathcal{S}; i \geq 1\}, \\ \mathcal{S}_2 &= \{(0, j) \in \mathcal{S}; j \geq 1\}, & \mathcal{S}_+ &= \{(i, j) \in \mathcal{S}; i, j \geq 1\}. \end{aligned}$$

Let  $\partial\mathcal{S} = \cup_{i=0}^2 \mathcal{S}_i$ . Clearly,  $\mathcal{S} = \mathcal{S}_+ \cup \partial\mathcal{S}$ . The subsets  $\mathcal{S}_+$  and  $\partial\mathcal{S}$  are called an interior and a boundary, and  $\mathcal{S}_i$  is called a boundary face for  $i = 0, 1, 2$ .

Let  $\mathbf{X}^{(+)} \equiv (X_1^{(+)}, X_2^{(+)})$  be a random vector taking value in  $\mathbb{U} \equiv \{-1, 0, 1\}^2$ , and for  $i = 0, 1, 2$ ,  $\mathbf{X}^{(i)} \equiv (X_1^{(i)}, X_2^{(i)})$  be a random vector taking values in  $\mathbb{U}$  such that  $\mathbf{X}^{(0)} \geq (0, 0)$ ,

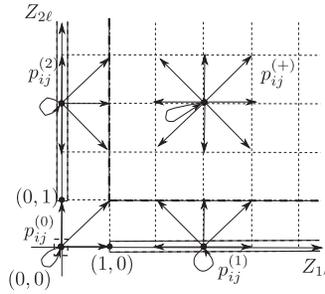


FIGURE 1. Transition diagram of  $\{Z_\ell\}$ .

$X_2^{(1)} \geq 0$  and  $X_1^{(2)} \geq 0$ . Define  $\{Z_\ell; \ell \in \mathbb{Z}_+\}$  as the Markov chain with state space  $\mathcal{S}$  and the following transition probabilities.

$$\mathbb{P}(Z_{\ell+1} = \mathbf{n}' | Z_\ell = \mathbf{n}) = \begin{cases} \mathbb{P}(\mathbf{X}^{(+)} = \mathbf{n}' - \mathbf{n}), & \mathbf{n} \in \mathcal{S}_+, \mathbf{n}' \in \mathcal{S}, \\ \mathbb{P}(\mathbf{X}^{(i)} = \mathbf{n}' - \mathbf{n}), & i = 0, 1, 2, \mathbf{n} \in \mathcal{S}_i, \mathbf{n}' \in \mathcal{S}. \end{cases} \tag{2.1}$$

By this definition,  $\{Z_\ell\}$  is skip-free, and has homogeneous transition probabilities in each boundary face  $\mathcal{S}_i$  and the interior  $\mathcal{S}_+$ . We refer to this Markov chain  $\{Z_\ell\}$  as a two-dimensional reflecting random walk. Throughout this paper, we assume that

(i)  $\{Z_\ell\}$  is irreducible.

The modeling primitives of this reflecting random walk are given by the four sets of the following probability distributions:

$$\begin{aligned} p_{ij}^{(0)} &= \mathbb{P}(\mathbf{X}^{(0)} = (i, j)), & i, j = 0, 1, \\ p_{ij}^{(1)} &= \mathbb{P}(\mathbf{X}^{(1)} = (i, j)), & i = 0, \pm 1, j = 0, 1, \\ p_{ij}^{(2)} &= \mathbb{P}(\mathbf{X}^{(2)} = (i, j)), & i = 0, 1, j = 0, \pm 1, \\ p_{ij}^{(+)} &= \mathbb{P}(\mathbf{X}^{(+)} = (i, j)), & i, j = 0, \pm 1. \end{aligned}$$

The transition diagram of  $\{Z_\ell\}$  is illustrated in Figure 1.

For the modeling parameter  $p_{ij}^{(+)}$ , we add the irreducibility assumption. Let  $\{Y_\ell \in \mathbb{Z}^2; \ell \in \mathbb{Z}_+\}$  be a two-dimensional random walk removing the boundary  $\partial\mathcal{S}$  of  $\{Z_\ell\}$ . We note that the distribution of increments for  $\{Y_\ell\}$  is identical with  $p_{ij}^{(+)}$ . For this random walk, we assume the following condition.

(ii)  $\{Y_\ell; \ell \in \mathbb{Z}_+\}$  is irreducible.

Of course, if  $\{Y_\ell\}$  is not irreducible, the irreducible condition (i) may be still satisfied because of the reflection at the boundary. We first note the following fact.

LEMMA 2.1: Under the condition (ii), at least one of  $p_{10}^{(+)}, p_{(-1)0}^{(+)}, p_{01}^{(+)}$  and  $p_{0(-1)}^{(+)}$  is positive.

PROOF: Suppose that our claim is not true, that is,  $p_{10}^{(+)} = p_{(-1)0}^{(+)} = p_{01}^{(+)} = p_{0(-1)}^{(+)} = 0$ . But then, for any  $n_1, n_2 \in \mathbb{Z}$ , if  $\mathbf{Y}_0 = (n_1, n_2)$ , then  $\mathbf{Y}_\ell$  does not arrive at  $(n_1 + 1, n_2)$  and  $(n_1, n_2 + 1)$  for any  $\ell \in \mathbb{Z}_+$ . This is a contradiction for irreducibility of  $\{Y_\ell\}$ . ■

Our problem is to find necessary and sufficient conditions for  $\{Z_\ell\}$  under which its time reversal is a reflecting random walk on  $\mathcal{S}$ . For this, we assume that

- (iii)  $\{Z_\ell\}$  has the stationary distribution.

We denote it by  $\pi$ . Then, we can construct the stationary Markov chain  $\{Z_\ell; \ell \in \mathbb{Z}\}$  starting from  $-\infty$  with  $Z_0$  subject to  $\pi$ . We define the time-reversed process  $\{\tilde{Z}_\ell; \ell \in \mathbb{Z}\}$  by

$$\tilde{Z}_\ell = Z_{-\ell}, \quad \forall \ell \in \mathbb{Z}.$$

It is easy to see that  $\{\tilde{Z}_\ell\}$  is the Markov chain, and transition probability of  $\{\tilde{Z}_\ell\}$  is given by

$$\mathbb{P}(\tilde{Z}_{\ell+1} = \mathbf{n}' | \tilde{Z}_\ell = \mathbf{n}) = \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(Z_{\ell+1} = \mathbf{n} | Z_\ell = \mathbf{n}'), \quad \mathbf{n}, \mathbf{n}' \in \mathcal{S}. \tag{2.2}$$

The Markov chain  $\{\tilde{Z}_\ell\}$  is referred to as a time-reversed process under  $\pi$  (see, e.g., [1]).

*Remark 2.1:* In (2.2), we may not require that the  $\pi$  is the stationary distribution of  $\{Z_\ell\}$ . It is the stationary distribution if and only if

$$\sum_{\mathbf{n}' \in \mathcal{S}} \mathbb{P}(\tilde{Z}_{\ell+1} = \mathbf{n}' | \tilde{Z}_\ell = \mathbf{n}) = 1,$$

for all  $\mathbf{n} \in \mathcal{S}$  (see, e.g., [4,9]).

### 3. CHARACTERIZATION OF STRUCTURE-REVERSIBILITY

A Markov chain is said to be reversible if its time-reversed process is stochastically identical with the original Markov chain. However, this condition is too strong (see, e.g., [9] for queueing models). Instead of this reversibility, we consider weaker concept of reversibility for a reflecting random walk.

**DEFINITION 3.1:** *The reflecting random walk  $\{Z_\ell\}$  is said to be structure-reversible if it has a stationary distribution and if its time-reversed process  $\{\tilde{Z}_\ell\}$  is also a reflecting random walk.*

Thus, if  $\{Z_\ell\}$  has structure-reversibility, then the transition probabilities of  $\{\tilde{Z}_\ell\}$  are given by

$$\mathbb{P}(\tilde{Z}_{\ell+1} = \mathbf{n}' | \tilde{Z}_\ell = \mathbf{n}) = \mathbb{P}(\tilde{\mathbf{X}}^{(i)} = \mathbf{n}' - \mathbf{n}), \quad \ell \in \mathbb{Z}, i = 0, 1, 2, +, \mathbf{n} \in \mathcal{S}_i, \tag{3.1}$$

for some random vector  $\tilde{\mathbf{X}}^{(i)} = (\tilde{X}_1^{(i)}, \tilde{X}_2^{(i)}) \in \{0, 1, -1\}^2$ . From the reflection property of  $\{\tilde{Z}_\ell\}$ , it is required that  $X_1^{(i)} \geq 0$  for  $i = 0, 2$  and  $X_2^{(i)} \geq 0$  for  $i = 0, 1$ . We also note that the distributions of  $\mathbf{X}^{(i)}$  and  $\tilde{\mathbf{X}}^{(i)}$  may not be the same.

We are now ready to present conditions for structure-reversibility.

**THEOREM 3.1:** *For the reflecting random walk  $\{Z_\ell\}$ , assume the conditions (i), (ii) and (iii). Then  $\{Z_\ell\}$  is structure-reversible if and only if the following conditions hold.*

- (a1) *There exist  $c^{(1+)}, c^{(2+)} > 0$  such that  $c^{(1+)}p_{i1}^{(+)} = p_{i1}^{(1)}$  and  $c^{(2+)}p_{1i}^{(+)} = p_{1i}^{(2)}$  for any  $i = 0, \pm 1$ .*

- (a2) *There exist  $c^{(10)}, c^{(20)} \geq 0$  such that  $c^{(10)}p_{1j}^{(1)} = p_{1j}^{(0)}$  and  $c^{(20)}p_{j1}^{(2)} = p_{j1}^{(0)}$  for any  $j = 0, 1$ . If  $c^{(10)} = 0$  (resp.  $c^{(20)} = 0$ ), then  $p_{1j}^{(1)} = 0$  (resp.  $p_{j1}^{(2)} = 0$ ) for any  $j = 0, 1$ .*
- (a3) *If both  $c^{(10)} > 0$  and  $c^{(20)} > 0$ , then  $c^{(10)}c^{(1+)} = c^{(20)}c^{(2+)}$ .*
- (a4) *There exist  $\eta_1, \eta_2 \in (0, 1)$  such that*

$$\pi(n, 0) = \eta_1^{n-1}\pi(1, 0), \quad n \geq 1, \tag{3.2}$$

$$\pi(0, n) = \eta_2^{n-1}\pi(0, 1), \quad n \geq 1, \tag{3.3}$$

$$\pi(n_1, n_2) = \eta_1^{n_1-1}\eta_2^{n_2-1}\pi(1, 1), \quad n_1, n_2 \geq 1. \tag{3.4}$$

If  $c^{(10)} > 0$ , then

$$\pi(1, 0) = c^{(10)}\eta_1\pi(0, 0), \tag{3.5}$$

$$\pi(0, 1) = \frac{c^{(10)}c^{(1+)}}{c^{(2+)}}\eta_2\pi(0, 0), \tag{3.6}$$

$$\pi(1, 1) = c^{(10)}c^{(1+)}\eta_1\eta_2\pi(0, 0), \tag{3.7}$$

and if  $c^{(20)} > 0$ , then

$$\pi(1, 0) = \frac{c^{(20)}c^{(2+)}}{c^{(1+)}}\eta_1\pi(0, 0), \tag{3.8}$$

$$\pi(0, 1) = c^{(20)}\eta_2\pi(0, 0), \tag{3.9}$$

$$\pi(1, 1) = c^{(20)}c^{(2+)}\eta_1\eta_2\pi(0, 0). \tag{3.10}$$

*Remark 3.1:* Under the condition (a2), if  $c^{(10)} = c^{(20)} = 0$ , then  $\{\mathbf{Z}_\ell\}$  is not irreducible, that is, the condition (i) is not satisfied. Therefore, at least one of  $c^{(10)} > 0$  and  $c^{(20)} > 0$  holds, and we can obtain at least one of (3.5)–(3.7) and (3.8)–(3.10) since  $c^{(1+)}, c^{(2+)} > 0$  under the condition (a1).

*Remark 3.2:* From the condition (a3), if  $c^{(10)} > 0$  and  $c^{(20)} > 0$ , then (3.5)–(3.7) are identical with (3.8)–(3.10).

*Remark 3.3:* We can obtain the structure-reversibility condition even if the condition (ii) is not satisfied. Then, the condition (a4) is slightly changed. Such an example is given in Appendix A.

We now prove Theorem 3.1. We first verify that the conditions (a1)–(a4) are necessary for structure-reversibility.

**LEMMA 3.1:** *Under the conditions (i), (ii) and (iii), if  $\{\tilde{\mathbf{Z}}_\ell\}$  is a reflecting random walk, then the stationary distribution satisfies (3.4).*

**PROOF:** From Lemma 2.1, we divide our proof into the following three cases:

$$p_{i0}^{(+)} > 0 \text{ and } p_{0j}^{(+)} > 0 \text{ for some } i, j = \pm 1, \tag{3.11}$$

$$p_{i0}^{(+)} > 0 \text{ for some } i = \pm 1 \text{ and } p_{0j}^{(+)} = 0 \text{ for all } j = \pm 1, \tag{3.12}$$

$$p_{0i}^{(+)} > 0 \text{ for some } i = \pm 1 \text{ and } p_{j0}^{(+)} = 0 \text{ for all } j = \pm 1. \tag{3.13}$$

Cases (3.12) and (3.13) are symmetric, and therefore, we prove (3.4) for (3.11) and (3.12).

For the case (3.11), we only consider the case that  $p_{10}^{(+)} > 0$  and  $p_{01}^{(+)} > 0$  since the other cases are similarly proved. Consider the transition of the reversed process  $\{\tilde{Z}_\ell\}$  from  $\mathbf{n}$  to  $\mathbf{n}' = \mathbf{n} - \mathbf{e}_1$ , where  $\mathbf{e}_1 = (1, 0)$ . Then, it follows from (2.1) and (2.2) for all  $\mathbf{n} - \mathbf{e}_1 \in \mathcal{S}_+$  (also  $\mathbf{n} \in \mathcal{S}_+$ ) that

$$\mathbb{P}(\tilde{Z}_{\ell+1} = \mathbf{n} - \mathbf{e}_1 | \tilde{Z}_\ell = \mathbf{n}) = \frac{\pi(\mathbf{n} - \mathbf{e}_1)}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{X}^{(+)} = (1, 0)) = \frac{\pi(\mathbf{n} - \mathbf{e}_1)}{\pi(\mathbf{n})} p_{10}^{(+)} \tag{3.14}$$

Since  $p_{10}^{(+)} > 0$  and  $\{\tilde{Z}_\ell\}$  is a reflecting random walk, the right-hand side must be a constant for all  $\mathbf{n}, \mathbf{n}' \in \mathcal{S}_+$ . Denote this constant by  $\eta_1^{-1}$ , that is,

$$\eta_1^{-1} = \frac{\pi(\mathbf{n} - \mathbf{e}_1)}{\pi(\mathbf{n})}, \quad \mathbf{n}, \mathbf{n} - \mathbf{e}_1 \in \mathcal{S}_+. \tag{3.15}$$

Similarly, we can show that  $\frac{\pi(\mathbf{n} - \mathbf{e}_2)}{\pi(\mathbf{n})}$  is a positive constant for all  $\mathbf{n}, \mathbf{n} - \mathbf{e}_2 \in \mathcal{S}_+$ , where  $\mathbf{e}_2 = (0, 1)$ , and denote it by  $\eta_2^{-1}$ . Thus, for  $\mathbf{n} = (n_1, n_2) \in \mathcal{S}_+$ , we have

$$\pi(\mathbf{n}) = \eta_1 \pi(\mathbf{n} - \mathbf{e}_1) = \eta_1^{n_1-1} \pi(1, n_2) = \eta_1^{n_1-1} \eta_2^{n_2-1} \pi(1, 1).$$

Obviously, from this equation, we have  $\eta_1, \eta_2 \in (0, 1)$  since  $\sum_{\mathbf{n} \in \mathcal{S}_+} \pi(\mathbf{n}) \leq 1$ .

We next assume the case (3.12). Since  $p_{i0}^{(+)} > 0$  for some  $i \in \{-1, 1\}$ , we also have (3.15) for the case (3.12). In what follows, we prove that  $\pi(\mathbf{n} - \mathbf{e}_2)/\pi(\mathbf{n})$  also must be constant for the case (3.12).

First assume both  $p_{10}^{(+)} > 0$  and  $p_{(-1)0}^{(+)} > 0$ . Under the condition (ii) and the case (3.12) with  $p_{10}^{(+)} > 0$  and  $p_{(-1)0}^{(+)} > 0$ , it is clear that the conditions  $p_{i(-1)}^{(+)} > 0$  and  $p_{j1}^{(+)} > 0$  hold for some  $i, j \in \{-1, 1\}$ , and we may assume  $p_{(-1)1}^{(+)} > 0$  since a proof is similar to the other cases. Then, it again follows from (2.1) and (2.2) for  $\mathbf{n} = (n_1, n_2) \in \mathcal{S}_+$  and  $\mathbf{n}' = \mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2$ , we have

$$\begin{aligned} \mathbb{P}(\tilde{Z}_{\ell+1} = \mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2 | \tilde{Z}_\ell = \mathbf{n}) &= \frac{\pi(\mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2)}{\pi(\mathbf{n})} p_{(-1)1}^{(+)} \\ &= \frac{\pi(\mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2)}{\pi(\mathbf{n})} \frac{\pi(\mathbf{n} + \mathbf{e}_1)}{\pi(\mathbf{n} + \mathbf{e}_1)} p_{(-1)1}^{(+)} \\ &= \frac{\pi(\mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2)}{\pi(\mathbf{n} + \mathbf{e}_1)} \eta_1 p_{(-1)1}^{(+)} \\ &= \frac{\pi(\mathbf{m} - \mathbf{e}_2)}{\pi(\mathbf{m})} \eta_1 p_{(-1)1}^{(+)}, \quad \mathbf{m}, \mathbf{m} - \mathbf{e}_2 \in \mathcal{S}_+, \end{aligned}$$

where  $\mathbf{m} \equiv \mathbf{n} + \mathbf{e}_1$  and the third equation is obtained by (3.15). Since  $p_{(-1)1}^{(+)} > 0$  and the left-hand side must be independent for  $\mathbf{m}$ , we have

$$\frac{\pi(\mathbf{m} - \mathbf{e}_2)}{\pi(\mathbf{m})} = \eta_2^{-1},$$

as long as  $\mathbf{m} \in \mathcal{S}_+$ . Thus, we have (3.4) for case (3.12) with  $p_{10}^{(+)} > 0$  and  $p_{(-1)0}^{(+)} > 0$ .

The rest of proof for (3.12) is the cases that  $p_{10}^{(+)} > 0$ ,  $p_{(-1)0}^{(+)} = 0$  and  $p_{10}^{(+)} = 0$ ,  $p_{(-1)0}^{(+)} > 0$ . These are also symmetric, so we only consider the case  $p_{10}^{(+)} > 0$  and  $p_{(-1)0}^{(+)} = 0$ .

Repeatedly, from the condition (ii), we must have  $p_{(-1)1}^{(+)} > 0$  and  $p_{(-1)(-1)}^{(+)} > 0$  under the conditions  $p_{10}^{(+)} > 0$  and  $p_{(-1)0}^{(+)} = 0$ . Thus, using the same argument of the case for  $p_{10}^{(+)} > 0$  and  $p_{(-1)0}^{(+)} > 0$ , we obtain (3.4). ■

LEMMA 3.2: *Under the same assumptions of Theorem 3.1, if  $\{\tilde{Z}_\ell\}$  is a reflecting random walk, then we have (3.2) and (3.3).*

PROOF: We only obtain (3.2) since (3.3) is similarly proved. We separately consider the cases such that

$$\text{either } p_{10}^{(1)} > 0 \text{ or } p_{(-1)0}^{(1)} > 0, \tag{3.16}$$

$$\text{both } p_{10}^{(1)} = 0 \text{ and } p_{(-1)0}^{(1)} = 0. \tag{3.17}$$

For (3.16), to changing the proof of the case (3.11) in Lemma 3.1 from  $\mathcal{S}_+$  into  $\mathcal{S}_1$ , we have, for some constant  $\alpha_1 \in (0, 1)$ ,

$$\alpha_1^{-1} = \frac{\pi(\mathbf{n} - \mathbf{e}_1)}{\pi(\mathbf{n})}, \quad \mathbf{n}, \mathbf{n} - \mathbf{e}_1 \in \mathcal{S}_1. \tag{3.18}$$

From the conditions (ii), (3.15) and (3.18), we have  $p_{i(-1)}^{(+)} > 0$  for some  $i = 0, \pm 1$ , and therefore, for  $\mathbf{n} = (n_1, 0) \in \mathcal{S}_1$  and  $\mathbf{n}' = (n_1 - i, 1) \in \mathcal{S}_+$ ,

$$\mathbb{P}(\tilde{Z}_{\ell+1} = (n_1 - i, 1) | \tilde{Z}_\ell = (n_1, 0)) = \frac{\pi(n_1 - i, 1)}{\pi(n_1, 0)} p_{i(-1)}^{(+)} = \frac{\eta_1^{n_1-1-i} \pi(1, 1)}{\alpha_1^{n_1-1} \pi(1, 0)} p_{i(-1)}^{(+)}. \tag{3.19}$$

The left-hand side of this equation is independent of  $n_1$  since  $\{\tilde{Z}_\ell\}$  is a reflecting random walk, and therefore, we have  $\alpha_1 = \eta_1$ , which implies (3.2).

For (3.17), from the irreducible condition (i),  $p_{i1}^{(1)} > 0$  and  $p_{j(-1)}^{(+)} > 0$  for some  $i, j = 0, \pm 1$ . Then, from (2.1), (2.2) and (3.1), we have, for  $\mathbf{n} = (n_1 + j, 0) \in \mathcal{S}_1$  and  $\mathbf{n}' = (n_1, 1) \in \mathcal{S}_+$ ,

$$\mathbb{P}(\tilde{Z}_{\ell+1} = (n_1, 1) | \tilde{Z}_\ell = (n_1 + j, 0)) = \frac{\pi(n_1, 1)}{\pi(n_1 + j, 0)} p_{j(-1)}^{(+)}. \tag{3.20}$$

Since  $\{Z_\ell\}$  is structure-reversible and  $p_{j(-1)}^{(+)} > 0$ , the left-hand side of this equation is a positive constant which depends on  $j$ , and denote it by  $\tilde{p}_{(-j)1}^{(1)}$ . Similarly, for  $\mathbf{n} = (n_1 + i, 1) \in \mathcal{S}_+$  and  $\mathbf{n}' = (n_1, 0) \in \mathcal{S}_1$ ,

$$\begin{aligned} \mathbb{P}(\tilde{Z}_{\ell+1} = (n_1, 0) | \tilde{Z}_\ell = (n_1 + i, 1)) &= \frac{\pi(n_1, 0)}{\pi(n_1 + i, 1)} p_{i1}^{(1)} \\ &= \frac{\pi(n_1, 0)}{\pi(n_1 + i, 1)} \frac{\pi(n_1 + i + j, 0)}{\pi(n_1 + i + j, 0)} p_{i1}^{(1)} \\ &= \frac{\pi(n_1, 0)}{\pi(n_1 + i + j, 0)} \frac{p_{j(-1)}^{(+)}}{\tilde{p}_{(-j)1}^{(1)}} p_{i1}^{(1)}, \end{aligned}$$

where the third equality is given by (3.20). Since the left-hand side of this equation also does not depend on  $n_1$ , we directly obtain, for  $\eta_1 \in (0, 1)$  satisfying (3.15),

$$\frac{\pi(n_1, 0)}{\pi(n_1 + 1, 0)} = \eta_1^{-1}, \quad n_1 \geq 1, \tag{3.21}$$

if  $|i + j| = 1$  holds, and we have (3.2). On the other hand, from irreducible condition (i), if  $|i + j| \neq 1$ , then at least one of  $p_{10}^{(+)} > 0$  and  $p_{(-1)0}^{(+)} > 0$  holds, and we have, for  $k = 1$  or  $k = -1$ ,

$$\mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = (n_1, 1) | \tilde{\mathbf{Z}}_{\ell} = (n_1 + k, 1)) = \frac{\pi(n_1, 1)}{\pi(n_1 + k, 1)} p_{k0}^{(+)}$$

From Lemma 3.1, of the left-hand side of this equation is independent for  $n_1$ , and denote it by  $\tilde{p}_{(-k)0}^{(+)}$ . Thus, we obtain, from (3.20)

$$\begin{aligned} \mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = (n_1, 0) | \tilde{\mathbf{Z}}_{\ell} = (n_1 + i, 1)) &= \frac{\pi(n_1, 0)}{\pi(n_1 + i, 1)} p_{i1}^{(1)} \\ &= \frac{\pi(n_1, 0)}{\pi(n_1 + i, 1)} \frac{\pi(n_1 + i + k, 1)}{\pi(n_1 + i + k, 1)} \frac{\pi(n_1 + i + j + k, 0)}{\pi(n_1 + i + j + k, 0)} p_{i1}^{(1)} \\ &= \frac{\pi(n_1, 0)}{\pi(n_1 + i + j + k, 0)} \frac{p_{k0}^{(+)}}{\tilde{p}_{(-k)0}^{(+)}} \frac{p_{j(-1)}^{(+)}}{\tilde{p}_{(-j)1}^{(1)}} p_{i1}^{(1)}. \end{aligned}$$

From the irreducible condition (i) again, if  $|i + j| \neq 1$ , we must have  $|i + j + k| = 1$ , and therefore, we also have (3.21). We complete the proof since (3.21) implies (3.2). ■

LEMMA 3.3: *Under the assumptions in Theorem 3.1, if  $\{\tilde{\mathbf{Z}}_{\ell}\}$  is a reflecting random walk, then (a1), (a2) and (a3) hold, and the stationary distribution satisfies (3.5)–(3.10).*

The proof of Lemma 3.3 is deferred to Appendix B. We are now ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1: From Lemmas 3.1–3.3, we already prove the necessity of conditions. For sufficiency, suppose the conditions (a1)–(a4) hold. Then, from (2.1) and (2.2), we have, for  $i = 0, 1, 2, +, j, k = 0, \pm 1, \mathbf{n} \in \mathcal{S}_i$  and  $\mathbf{n}' = \mathbf{n} + (j, k) \in \mathcal{S}_i$ ,

$$\begin{aligned} \mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = \mathbf{n}' | \tilde{\mathbf{Z}}_{\ell} = \mathbf{n}) &= \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{X}^{(i)} = \mathbf{n} - \mathbf{n}') \\ &= \frac{\pi(\mathbf{n} + (j, k))}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{X}^{(i)} = (-j, -k)) \\ &= \eta_1^j \eta_2^k p_{(-j)(-k)}^{(i)}. \end{aligned} \tag{3.22}$$

Thus, the transition probabilities of  $\{\tilde{\mathbf{Z}}_{\ell}\}$  from  $\mathcal{S}_i$  to  $\mathcal{S}_i$  are homogeneous.

We next verify that the downward transitions from  $\mathcal{S}_+$  to  $\mathcal{S}_1$  and from  $\mathcal{S}_+$  to  $\mathcal{S}_+$  are homogeneous. To this end, let consider the transitions from  $\mathcal{S}_+$  to  $\mathcal{S}_1$ . For  $\mathbf{n} = (n_1, 1) \in \mathcal{S}_+$

and  $\mathbf{n}' = (n_1 + i, 0) \in \mathcal{S}_1$  and  $i = 0, \pm 1$ , we have, by the conditions (a1), (a2), (a3) and (a4)

$$\begin{aligned} \mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = (n_1 + i, 0) | \tilde{\mathbf{Z}}_{\ell} = (n_1, 1)) &= \frac{\pi(n_1 + i, 0)}{\pi(n_1, 1)} \mathbb{P}(\mathbf{X}^{(1)} = (-i, 1)) \\ &= \frac{\eta_1^i \pi(1, 0)}{\pi(1, 1)} p_{(-i)1}^{(1)} \\ &= \frac{1}{c^{(1+)}} \eta_1^i \eta_2^{-1} p_{(-i)1}^{(1)} \\ &= \eta_1^i \eta_2^{-1} p_{(-i)1}^{(+)}, \\ &= \mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = (n_1 + i, m - 1) | \tilde{\mathbf{Z}}_{\ell} = (n_1, m)), \quad m \geq 2. \end{aligned}$$

where the last equation is obtained by (3.22). Note that  $(n_1, m), (n_1 + i, m - 1) \in \mathcal{S}_+$  since we assume  $n_1, n_1 + i \geq 1$ . Thus, the downward transitions for the direction of  $n_1$ -axis in the interior are homogeneous. Using a similar argument, we can prove that another downward transitions are also homogeneous. That is,  $\{\tilde{\mathbf{Z}}_{\ell}\}$  has the homogeneous transitions in each subset  $\mathcal{S}_i$ . Hence,  $\{\tilde{\mathbf{Z}}_{\ell}\}$  is a reflecting random walk if the conditions (a1)–(a4) hold. We complete the proof. ■

We discuss about the relationship between structure-reversibility and product form stationary distribution.

**COROLLARY 3.1:** *Suppose that the reflecting random walk  $\{\mathbf{Z}_{\ell}\}$  is structure-reversible. Then the stationary distribution of  $\{\mathbf{Z}_{\ell}\}$  has a product form solution if and only if the following conditions are satisfied.*

$$c^{(1+)} = c^{(20)} \text{ if } c^{(20)} > 0, \tag{3.23}$$

$$c^{(2+)} = c^{(10)} \text{ if } c^{(10)} > 0. \tag{3.24}$$

The proof of this corollary is deferred to Appendix C.

#### 4. GEOMETRIC CONDITIONS FOR STRUCTURE-REVERSIBILITY

We characterize the structure-reversibility in Theorem 3.1. However, the condition (a4) may not be easily checked because it uses the stationary distribution. Thus, we replace it by conditions only using modeling primitives. To this end, we introduce the following notations.

$$\begin{aligned} \gamma_0(\mathbf{z}) &= p_{00}^{(0)} + \frac{c^{(0)}}{c^{(1+)}} p_{(-1)0}^{(1)} z_1^{-1} + \frac{c^{(0)}}{c^{(2+)}} p_{0(-1)}^{(2)} z_2^{-1} + c^{(0)} p_{(-1)(-1)}^{(+)} z_1^{-1} z_2^{-1}, \\ \gamma_1(\mathbf{z}) &= p_{00}^{(1)} + p_{10}^{(1)} z_1 + p_{(-1)0}^{(1)} z_1^{-1} + c^{(1+)} \left( p_{1(-1)}^{(+)} z_1 z_2^{-1} + p_{0(-1)}^{(+)} z_2^{-1} + p_{(-1)(-1)}^{(+)} z_1^{-1} z_2^{-1} \right), \\ \gamma_2(\mathbf{z}) &= p_{00}^{(2)} + p_{01}^{(2)} z_2 + p_{0(-1)}^{(2)} z_2^{-1} + c^{(2+)} \left( p_{(-1)1}^{(+)} z_1^{-1} z_2 + p_{(-1)0}^{(+)} z_1^{-1} + p_{(-1)(-1)}^{(+)} z_1^{-1} z_2^{-1} \right), \\ \gamma_+(\mathbf{z}) &= \sum_{i=-1}^1 \sum_{j=-1}^1 p_{ij}^{(+)} z_1^i z_2^j, \end{aligned}$$

for  $\mathbf{z} \in \mathbb{R}^2$  such that  $\mathbf{z} > (0, 0)$ , where  $c^{(k+)}$  is non-negative constants defined in Theorem 3.1 for  $k = 1, 2$ , and

$$c^{(0)} \equiv \max(c^{(10)} c^{(1+)}, c^{(20)} c^{(2+)}).$$

Note that  $\gamma_+$  is the generating function of  $\mathbf{X}^{(+)}$ . Furthermore, if the condition (a3) and  $c^{(10)} > 0$  hold, then we have

$$c^{(0)} = c^{(10)}c^{(1+)}, \quad \frac{c^{(0)}}{c^{(1+)}} = \frac{c^{(10)}c^{(1+)}}{c^{(1+)}} = c^{(10)}, \tag{4.1}$$

and if  $c^{(20)} > 0$ ,

$$c^{(0)} = c^{(20)}c^{(2+)}, \quad \frac{c^{(0)}}{c^{(2+)}} = \frac{c^{(20)}c^{(2+)}}{c^{(2+)}} = c^{(20)}. \tag{4.2}$$

In addition, if  $c^{(10)} > 0$  and  $c^{(20)} > 0$ , then we have  $c^{(0)} = c^{(10)}c^{(1+)} = c^{(20)}c^{(2+)}$ . We prepare the next lemma, which will be proved at the end of this section.

LEMMA 4.1: *Suppose that conditions (i), (ii) and (a1)–(a3) hold. Then, the condition (iii) and (a4) are equivalent to*

$$(a5) \text{ There exist } \eta_1, \eta_2 \in (0, 1) \text{ satisfying } \gamma_i(\eta_1^{-1}, \eta_2^{-1}) = 1 \text{ for all } i = 0, 1, 2, +.$$

The next theorem is immediate from this lemma and Theorem 3.1, which gives the conditions of structure-reversibility in terms of modeling parameters. Thus, the theorem is a main result of this paper.

THEOREM 4.1: *Suppose the conditions (i) and (ii). Then,  $\{\mathbf{Z}_\ell\}$  is structure-reversible if and only if the conditions (a1)–(a3) and (a5) hold.*

Remark 4.1: (a5) implies the condition (iii). So, it is more convenient than (a4), which needs (iii).

Remark 4.2: Latouche and Miyazawa [6] derive necessary and sufficient conditions for the stationary distribution of the two-dimensional reflecting random walk to have a product form solution (see Theorem 3.9 of [6]). Similarly to the condition (a5), their conditions have geometric interruptions. We can see that the equations,

$$\gamma_+(\eta_1^{-1}, \eta_2^{-1}) = 1, \quad \gamma_1(\eta_1^{-1}, \eta_2^{-1}) = 1, \quad \gamma_2(\eta_1^{-1}, \eta_2^{-1}) = 1$$

are equivalent to (3.13), (3.14) and (3.17) of [6]. But it is also notable that the reflecting random walk having a product form stationary distribution may not be structure reversible. Such an example is given in Appendix D. Moreover, the stationary distribution may not have a product form under the structure-reversibility (see Corollary 3.1). Thus, those two classes of the stationary distributions are slightly different. However, they are identical under some extra conditions. For example, if the conditions:

$$p_{11}^{(+)} = p_{11}^{(0)} = p_{11}^{(1)} = p_{11}^{(2)}, \tag{4.3}$$

$$p_{10}^{(0)} = p_{10}^{(1)}, \quad p_{01}^{(0)} = p_{01}^{(2)}, \quad p_{1j}^{(2)} = p_{1j}^{(+)}, \quad p_{j1}^{(1)} = p_{j1}^{(+)}, \quad j = 0, -1 \tag{4.4}$$

are satisfied, then (a1), (a2) and (a3) hold. Thus, if (a5) holds, then this reflecting random walk is structure reversible. Moreover, we have  $c^{(10)} = c^{(20)} = 1$ , and therefore,  $\gamma_0(\eta_1^{-1}, \eta_2^{-1}) = 1$  is equivalent to (3.36) of [6]. Thus, its stationary distribution has a product form solution (see Theorem 3.12 in [6]). Namely, under the conditions (4.3) and (4.4), the condition (a5) is just identical with the product form condition.

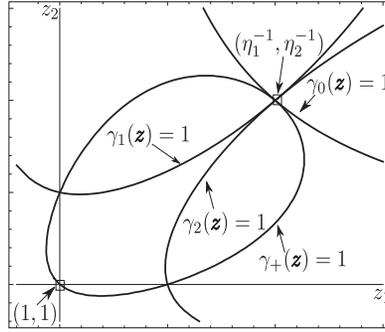


FIGURE 2. Example curves of  $\Gamma_i$ .

We briefly explain the condition (a5). Obviously,  $\gamma_+(1, 1) = 1$ . Moreover, the subset  $\Gamma_i = \{z \in \mathbb{R}^2; \gamma_i(z) = 1\}$  is a non-negative-directed convex (see, e.g., [5]). This means that  $\Gamma_i$  describes a locally convex curve on the non-negative quadrant for  $i = 0, \pm 1, +$ . In Figure 2, we illustrate the curve  $\Gamma_i$  that the condition (a5) holds.

We are now ready to prove Lemma 4.1.

PROOF OF LEMMA 4.1: We first prove the necessity of (a5). From the condition (iii), we have the following stationary equations.

$$\pi(0, 0) = p_{00}^{(0)} \pi(0, 0) + p_{(-1)0}^{(1)} \pi(1, 0) + p_{0(-1)}^{(2)} \pi(0, 1) + p_{(-1)(-1)}^{(+)} \pi(1, 1), \tag{4.5}$$

$$\pi(n, 0) = \sum_{i=-1}^1 \left( p_{i0}^{(1)} \pi(n-i, 0) + p_{i(-1)}^{(+)} \pi(n-i, 1) \right), \quad n \geq 2, \tag{4.6}$$

$$\pi(0, n) = \sum_{j=-1}^1 \left( p_{0j}^{(2)} \pi(0, n-j) + p_{(-1)j}^{(+)} \pi(1, n-j) \right), \quad n \geq 2, \tag{4.7}$$

$$\pi(n_1, n_2) = \sum_{i=-1}^1 \sum_{j=-1}^1 p_{ij}^{(+)} \pi(n_1-i, n_2-j). \quad n_1, n_2 \geq 2. \tag{4.8}$$

Substituting (3.4) into (4.8), we have  $\gamma_+(\eta_1^{-1}, \eta_2^{-1}) = 1$ . If  $c^{(10)} > 0$ , from (3.2)–(3.7), (4.6) and (4.7),

$$\begin{aligned} 1 &= p_{00}^{(1)} + p_{10}^{(1)} \eta_1^{-1} + p_{(-1)0}^{(1)} \eta_1 + c^{(1+)} \left( p_{1(-1)}^{(+)} \eta_1^{-1} \eta_2 + p_{0(-1)}^{(+)} \eta_2 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \right), \\ \frac{c^{(10)} c^{(1+)}}{c^{(2+)}} &= \frac{c^{(10)} c^{(1+)}}{c^{(2+)}} \left( p_{00}^{(2)} + p_{01}^{(2)} \eta_2^{-1} + p_{0(-1)}^{(2)} \eta_2 \right) \\ &\quad + c^{(10)} c^{(1+)} \left( p_{(-1)1}^{(+)} \eta_1 \eta_2^{-1} + p_{(-1)0}^{(+)} \eta_1 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \right), \end{aligned}$$

and we also have  $\gamma_1(\eta_1^{-1}, \eta_2^{-1}) = 1$  and  $\gamma_2(\eta_1^{-1}, \eta_2^{-1}) = 1$ . If  $c^{(20)} > 0$ , using (3.8)–(3.10),  $\gamma_1(\eta_1^{-1}, \eta_2^{-1}) = 1$  and  $\gamma_2(\eta_1^{-1}, \eta_2^{-1}) = 1$  are verified in a similar way. From Remark 3.1, either  $c^{(10)} > 0$  or  $c^{(20)} > 0$  holds, and therefore, we obtain  $\gamma_1(\eta_1^{-1}, \eta_2^{-1}) = 1$  and  $\gamma_2(\eta_1^{-1}, \eta_2^{-1}) = 1$  for all cases.

We finally verify  $\gamma_0(\eta_1^{-1}, \eta_2^{-1}) = 1$ . Using (3.2)–(3.7), (4.1) and (4.5), if  $c^{(10)} > 0$ , then we have

$$\begin{aligned} 1 &= p_{00}^{(0)} + c^{(10)} p_{(-1)0}^{(1)} \eta_1 + \frac{c^{(10)} c^{(1+)}}{c^{(2+)}} p_{0(-1)}^{(2)} \eta_2 + c^{(10)} c^{(1+)} p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ &= p_{00}^{(0)} + \frac{c^{(0)}}{c^{(1+)}} p_{(-1)0}^{(1)} \eta_1 + \frac{c^{(0)}}{c^{(2+)}} p_{0(-1)}^{(2)} \eta_2 + c^{(0)} p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ &= \gamma_0(\eta_1^{-1}, \eta_2^{-1}). \end{aligned}$$

The case  $c^{(20)} > 0$  is similarly proved, which is completed the proof of the necessity.

We next verify the sufficiency of (a5). From Remark 3.1 again, either  $c^{(10)} > 0$  or  $c^{(20)} > 0$  holds. We only prove Lemma 4.1 for the case  $c^{(10)} > 0$  since the case  $c^{(20)} > 0$  is similarly proved. Thus, suppose that the conditions (a1)–(a3), (a5) and  $c^{(10)} > 0$  hold. Let  $\pi$  be a function from  $\mathcal{S}$  to  $[0, 1]$  satisfying (3.2)–(3.7). To verify the condition (iii) and (a4), it suffices to prove that this  $\pi$  is a finite measure and satisfies the stationary equation, that is,

$$\sum_{\mathbf{n} \in \mathcal{S}} \pi(\mathbf{n}) < \infty, \tag{4.9}$$

$$\sum_{\mathbf{n}' \in \mathcal{S}} \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{Z}_{\ell+1} = \mathbf{n} | \mathbf{Z}_\ell = \mathbf{n}') = 1, \quad \mathbf{n} \in \mathcal{S}, \tag{4.10}$$

since  $\pi(0, 0)$  is arbitrarily given. It is easy to verify (4.9) since  $\eta_1, \eta_2 \in (0, 1)$ . To verify (4.10), we consider the following six cases.

$$\mathbf{n} = (0, 0), (1, 0), (1, 1), (n_1, 0), (n_1, 1), (n_1, n_2), \quad n_1, n_2 \geq 2.$$

We omit the other cases that  $\mathbf{n} = (0, 1), (0, n_2), (1, n_2)$  because these cases are symmetric to  $(1, 0), (n_1, 0), (n_1, 1)$ , respectively. For  $\mathbf{n} = (0, 0)$ , we have  $\mathbf{n}' \in \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ , and therefore,

$$\begin{aligned} &\sum_{\mathbf{n}' \in \mathcal{S}} \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{Z}_{\ell+1} = \mathbf{n} | \mathbf{Z}_\ell = \mathbf{n}') \\ &= p_{00}^{(0)} + \frac{\pi(1, 0)}{\pi(0, 0)} p_{(-1)0}^{(1)} + \frac{\pi(0, 1)}{\pi(0, 0)} p_{0(-1)}^{(2)} + \frac{\pi(1, 1)}{\pi(0, 0)} p_{(-1)(-1)}^{(+)} \\ &= p_{00}^{(0)} + c^{(10)} p_{(-1)0}^{(1)} \eta_1 + \frac{c^{(10)} c^{(1+)}}{c^{(2+)}} p_{0(-1)}^{(2)} \eta_2 + c^{(10)} c^{(1+)} p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ &= p_{00}^{(0)} + \frac{c^{(0)}}{c^{(1+)}} p_{(-1)0}^{(1)} \eta_1 + \frac{c^{(0)}}{c^{(2+)}} p_{0(-1)}^{(2)} \eta_2 + c^{(0)} p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ &= \gamma_0(\eta_1^{-1}, \eta_2^{-1}) = 1, \end{aligned}$$

where the second and third equalities are obtained by (3.5)–(3.7) and (4.1). The other cases are similarly proved, but we give their detailed proofs in Appendix E for the reader. This completes the proof. ■

### 5. APPLICATION TO A QUEUEING NETWORK

In this section, we construct a discrete-time queueing network whose stationary distribution is not of product form but has closed form, using structure-reversibility. For this, we modify

a discrete-time Jackson network, which is introduced below. We define the reflecting random walk by

$$\begin{aligned}
 p_{11}^{(i)} &= 0, & p_{(-1)(-1)}^{(i)} &= 0, & p_{10}^{(i)} &= \lambda_1, & p_{01}^{(i)} &= \lambda_2, & i &= 0, 1, 2, +, \\
 p_{(-1)0}^{(j)} &= \mu_1 r_{10}, & p_{(-1)1}^{(j)} &= \mu_1 r_{12}, & j &= 1, +, \\
 p_{0(-1)}^{(k)} &= \mu_2 r_{20}, & p_{1(-1)}^{(k)} &= \mu_2 r_{21}, & k &= 2, +, \\
 p_{00}^{(+)} &= \mu_1 r_{11} + \mu_2 r_{22}, & p_{11}^{(1)} &= \mu_1 r_{11} + \mu_2, \\
 p_{00}^{(2)} &= \mu_1 + \mu_2 r_{22}, & p_{11}^{(0)} &= \mu_1 + \mu_2,
 \end{aligned}$$

where all constants are positive and

$$\begin{aligned}
 \lambda_1 + \lambda_2 + \mu_1 + \mu_2 &= 1, \\
 r_{10} + r_{11} + r_{12} &= 1, \\
 r_{20} + r_{21} + r_{22} &= 1.
 \end{aligned}$$

This random walk is called a discrete-time Jackson network. For  $i = 1, 2$ , let  $\alpha_i$  be a solution of the following traffic equations.

$$\alpha_1 = \lambda_1 + \alpha_2 r_{21} + \alpha_1 r_{11}, \quad \alpha_2 = \lambda_2 + \alpha_1 r_{12} + \alpha_2 r_{22}. \tag{5.1}$$

The stability condition for this model is given by

$$\rho_1 \equiv \frac{\alpha_1}{\mu_1} < 1, \quad \rho_2 \equiv \frac{\alpha_2}{\mu_2} < 1.$$

It is well known that the stationary distribution of the discrete-time Jackson network has a product form solution, that is, stationary distribution is given by

$$\pi(n_1, n_2) = (1 - \rho_1)(1 - \rho_2)\rho_1^{n_1}\rho_2^{n_2}, \quad n_1, n_2 \geq 0.$$

In addition, it is easy to verify the conditions (a1)–(a3) and (a5), and therefore, the discrete-time Jackson network is structure-reversible.

We modify the Jackson network introduced above. Let us consider the reflecting random walk whose transition probabilities in the interior of the quadrant are given by

$$\begin{aligned}
 p_{11}^{(+)} &= 0, & p_{(-1)(-1)}^{(+)} &= 0, & p_{10}^{(+)} &= \lambda_1, & p_{01}^{(+)} &= \lambda_2, \\
 p_{(-1)0}^{(+)} &= \mu_1 r_{10}, & p_{(-1)1}^{(+)} &= \mu_1 r_{12}, & p_{0(-1)}^{(+)} &= \mu_2 r_{20}, \\
 p_{1(-1)}^{(+)} &= \mu_2 r_{21}, & p_{00}^{(+)} &= \mu_1 r_{11} + \mu_2 r_{22},
 \end{aligned}$$

and  $p_{11}^{(0)} = p_{11}^{(1)} = p_{11}^{(2)} = 0$ . Note that the other transition probabilities may be arbitrarily given. We refer to this reflecting random walk as a queueing network flexible at the boundary with no simultaneous movement. For simplicity, we omit “with no simultaneous movement” in what follows.

For this queueing network, it is easy to see that the structure-reversibility conditions (a1) and (a2) in the Theorem 4.1 are simplified to the following two conditions.

$$\text{(b1)} \quad \frac{p_{01}^{(1)}}{p_{(-1)1}^{(1)}} = \frac{\lambda_2}{\mu_1 r_{12}} \quad \text{and} \quad \frac{p_{10}^{(2)}}{p_{1(-1)}^{(2)}} = \frac{\lambda_1}{\mu_2 r_{21}}.$$

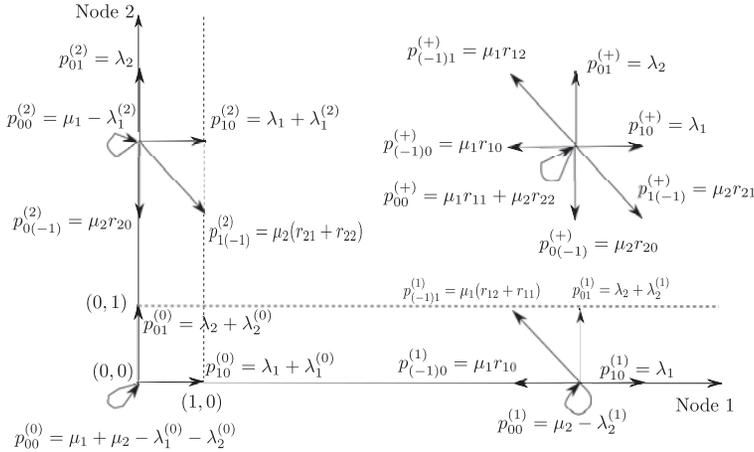


FIGURE 3. Transition diagram.

(b2)  $p_{10}^{(1)} = 0$  if and only if  $p_{10}^{(0)} = 0$ . Similarly,  $p_{01}^{(2)} = 0$  if and only if  $p_{01}^{(0)} = 0$ .

Namely, the queueing network flexible at the boundary is structure-reversible if and only if the conditions (b1), (b2), (a3) and (a5) are satisfied.

We next consider a special case of the queueing network flexible at the boundary for constructing an example such that it is structure-reversible but does not have a product form stationary distribution. For this, we put the transition probabilities at the boundary of the queueing network as follows.

$$\begin{aligned}
 p_{10}^{(1)} &= p_{10}^{(+)} = \lambda_1, & p_{01}^{(2)} &= p_{01}^{(+)} = \lambda_2, & p_{(-1)0}^{(1)} &= p_{(-1)0}^{(+)} = \mu_1 r_{10}, & p_{0(-1)}^{(2)} &= p_{0(-1)}^{(+)} = \mu_2 r_{20}, \\
 p_{01}^{(1)} &= \lambda_2 + \lambda_2^{(1)}, & p_{10}^{(2)} &= \lambda_1 + \lambda_1^{(2)}, & p_{10}^{(0)} &= \lambda_1 + \lambda_1^{(0)}, & p_{01}^{(0)} &= \lambda_2 + \lambda_2^{(0)}, \\
 p_{(-1)1}^{(1)} &= \mu_1(r_{12} + r_{11}), & p_{1(-1)}^{(2)} &= \mu_2(r_{21} + r_{22}), & p_{00}^{(1)} &= \mu_2 - \lambda_2^{(1)}, \\
 p_{00}^{(2)} &= \mu_1 - \lambda_1^{(2)}, & p_{00}^{(0)} &= \mu_1 + \mu_2 - \lambda_1^{(0)} - \lambda_2^{(0)},
 \end{aligned}$$

where we assume  $0 \leq \lambda_2^{(1)} \leq \mu_2$ ,  $0 \leq \lambda_1^{(2)} \leq \mu_1$  and  $0 \leq \lambda_1^{(0)} + \lambda_2^{(0)} \leq \mu_1 + \mu_2$ . We refer to this queueing network as a discrete-time Jackson network with extra arrivals at empty nodes. In Figure 3, we depict the transition diagram of this queueing network.

For the discrete-time Jackson network with extra arrivals at empty nodes, it is easy to verify that the condition (b2) is satisfied. Assume that

$$\frac{\lambda_2 + \lambda_2^{(1)}}{\lambda_2} = \frac{r_{12} + r_{11}}{r_{12}}, \quad \frac{\lambda_1 + \lambda_1^{(2)}}{\lambda_1} = \frac{r_{21} + r_{22}}{r_{21}}. \tag{5.2}$$

Then,

$$\frac{p_{01}^{(1)}}{p_{(-1)1}^{(1)}} = \frac{\lambda_2 + \lambda_2^{(1)}}{\mu_1(r_{12} + r_{11})} = \frac{\lambda_2}{\mu_1 r_{12}}, \quad \frac{p_{10}^{(2)}}{p_{1(-1)}^{(2)}} = \frac{\lambda_1 + \lambda_1^{(2)}}{\mu_2(r_{21} + r_{22})} = \frac{\lambda_1}{\mu_2 r_{21}}.$$

Hence, the condition (b1) is satisfied. Moreover, we have

$$c^{(1+)} = 1 + \frac{\lambda_2^{(1)}}{\lambda_2}, \quad c^{(2+)} = 1 + \frac{\lambda_1^{(2)}}{\lambda_1}, \quad c^{(10)} = 1 + \frac{\lambda_1^{(0)}}{\lambda_1}, \quad c^{(20)} = 1 + \frac{\lambda_2^{(0)}}{\lambda_2}. \tag{5.3}$$

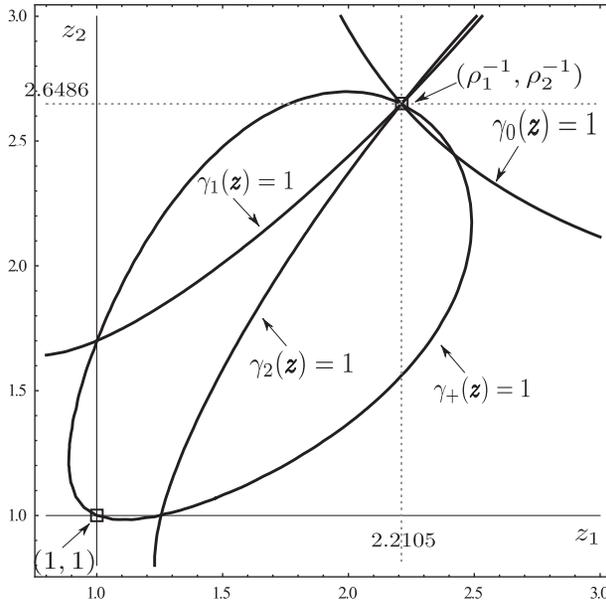


FIGURE 4. The curves of  $\Gamma_i$  satisfying (5.10).

We also assume the condition (a3), that is,  $c^{(10)}c^{(1+)} = c^{(20)}c^{(2+)}$  which is equivalent to

$$\left(1 + \frac{\lambda_1^{(0)}}{\lambda_1}\right) \left(1 + \frac{\lambda_2^{(1)}}{\lambda_2}\right) = \left(1 + \frac{\lambda_2^{(0)}}{\lambda_2}\right) \left(1 + \frac{\lambda_1^{(2)}}{\lambda_1}\right). \tag{5.4}$$

Moreover, for the condition (a5), we assume the following conditions.

$$\lambda_1 = \lambda_2 \equiv \lambda, \tag{5.5}$$

$$\frac{r_{12}}{r_{10}} = \frac{r_{21}}{r_{20}}. \tag{5.6}$$

Then, we confirm that the condition (a5) is satisfied with  $\eta_1 = \rho_1$  and  $\eta_2 = \rho_2$  (see, Appendix F). Thus, under the conditions (5.2)–(5.6), the discrete-time Jackson network with extra arrivals at empty nodes is structure-reversible.

From (5.2), (5.4) and (5.5), we have

$$\frac{\lambda + \lambda_1^{(0)}}{\lambda + \lambda_2^{(0)}} = \frac{\lambda + \lambda_1^{(2)}}{\lambda + \lambda_2^{(1)}}, \tag{5.7}$$

$$\lambda_1^{(2)} = \frac{r_{22}}{r_{21}} \lambda, \quad \lambda_1^{(2)} = \frac{r_{11}}{r_{12}} \lambda. \tag{5.8}$$

These imply that  $\lambda_1^{(2)}$ ,  $\lambda_2^{(1)}$ ,  $\lambda_1^{(0)}$  and  $\lambda_2^{(0)}$  are determined by  $\lambda$ ,  $r_{11}$ ,  $r_{12}$ ,  $r_{21}$  and  $r_{22}$ . It is easy to see that the condition (5.4) is satisfied if

$$\lambda_1^{(2)} = \lambda_1^{(0)}, \quad \lambda_2^{(1)} = \lambda_2^{(0)}, \tag{5.9}$$

and we obviously have (3.23) and (3.24). Thus, from Corollary 3.1, under the structure-reversibility conditions (5.2)–(5.6), the stationary distribution of this network has a product

form solution if and only if (5.9) is satisfied. From (5.7) and (5.8), there may be the case where (5.9) does not hold while (5.2)–(5.6) are satisfied. We give such an example below.

$$\begin{aligned} \lambda_1 = \lambda_2 = 0.0667, \quad \mu_1 = 0.4000, \quad \mu_2 = 0.4667, \quad r_{10} = 0.368, \quad r_{11} = r_{12} = 0.3158, \\ r_{20} = 0, 3784, \quad r_{21} = 0.3243, \quad r_{22} = 0.2973, \quad \lambda_2^{(1)} = 0.0667, \quad \lambda_1^{(2)} = 0.0611, \\ \lambda_1^{(0)} = 0.2104, \quad \lambda_2^{(0)} = 0.2225. \end{aligned} \quad (5.10)$$

Thus, the discrete-time Jackson network with additional arrivals at empty nodes may not have a product form solution when it is structure-reversible. For this example, we also have  $\rho_1^{-1} = 2.2105$  and  $\rho_2^{-1} = 2.6486$  (see Figure 4).

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## APPENDIX A. SINGULAR REFLECTING RANDOM WALK

In this section, we obtain a structure-reversibility condition in the special case. For this, we assume the following conditions.

$$p_{ij}^{(+)} = 0, \quad i = 0, \pm 1, \quad j = \pm 1, \quad (\text{A1})$$

$$p_{i0}^{(+)} > 0, \quad i = 0, \pm 1. \quad (\text{A2})$$

Then, it is easy to see that the random walk  $\{\mathbf{Y}_\ell\}$  is not irreducible. This reflecting random walk is referred to as a singular reflecting random walk, which is introduced by Fayolle, Iasnogorodski

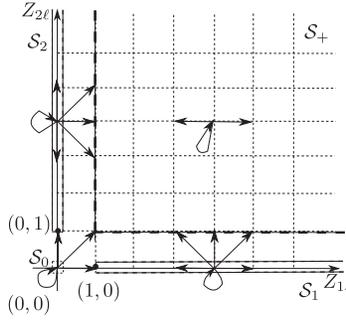


FIGURE A.1. Singular reflecting random walk satisfying the irreducible condition (i).

and Malyshev [3]. From irreducibility condition (i), we must have, for some  $i, j = 0, 1$

$$p_{10}^{(1)} > 0, \quad p_{(-1)0}^{(1)} > 0, \tag{A3}$$

$$p_{i1}^{(2)} > 0, \quad p_{j(-1)}^{(2)} > 0. \tag{A4}$$

In Figure A.1, we depict transition diagram of the reflecting random walk satisfying the conditions (A1)–(A4).

This reflecting random walk is structure-reversible if and only if the following conditions hold.

$$\begin{aligned} \pi(n_1, n_2) &= \eta_1^{n_1-1} \eta_2^{n_2-1} \pi(1, 1), \quad n_1 \geq 1, n_2 \geq 1, \\ \pi(n_1, 0) &= \alpha_1^{n_1-1} \pi(1, 0), \quad n_1 \geq 1, \\ \pi(0, n_2) &= \eta_2^{n_2-1} \pi(0, 1), \quad n_2 \geq 1, \\ \pi(1, 0) &= c^{(10)} \alpha_1 \pi(0, 0), \\ \pi(0, 1) &= c^{(20)} \eta_2 \pi(0, 0), \\ \pi(1, 1) &= c^{(20)} c^{(2+)} \eta_1 \eta_2 \pi(0, 0), \\ p_{i1}^{(1)} &= 0, \quad p_{1j}^{(2)} = 0, \quad p_{11}^{(0)} = 0, \quad i = 0, \pm 1, j = \pm 1, \end{aligned}$$

where  $\eta_1, \eta_2, \alpha_1 \in (0, 1)$  and  $c^{(2+)}, c^{(10)}, c^{(20)} > 0$  are given by

$$\begin{aligned} \eta_1 &= \frac{p_{10}^{(+)}}{p_{(-1)0}^{(+)}}, \quad \eta_2 = \frac{p_{01}^{(2)}}{p_{0(-1)}^{(2)}}, \quad \alpha_1 = \frac{p_{10}^{(1)}}{p_{(-1)0}^{(1)}}, \\ c^{(2+)} &= \frac{p_{10}^{(2)}}{p_{10}^{(+)}}, \quad c^{(10)} = \frac{p_{10}^{(0)}}{p_{10}^{(1)}}, \quad c^{(20)} = \frac{p_{01}^{(0)}}{p_{01}^{(2)}}. \end{aligned}$$

In what follows, we will derive these conditions.

We assume that  $\{Z_\ell\}$  is structure-reversible. Then, for  $i, j = 0, \pm 1$  and  $(n_1, n_2) \in S_k$ , we can define the following probability function  $\tilde{p}_{ij}^{(k)}$ .

$$\tilde{p}_{ij}^{(k)} = \mathbb{P}(\tilde{Z}_{\ell+1} = (n_1 + i, n_2 + j) | \tilde{Z}_\ell = (n_1, n_2)).$$

From (2.2), we have

$$\tilde{p}_{10}^{(+)} = \mathbb{P}(\tilde{Z}_{\ell+1} = (n_1 + 1, n_2) | \tilde{Z}_\ell = (n_1, n_2)) = \frac{\pi(n_1 + 1, n_2)}{\pi(n_1, n_2)} p_{(-1)0}^{(+)} > 0, \quad (n_1, n_2) \in S_+.$$

Thus, if  $\{\mathcal{Z}_\ell\}$  is structure-reversible, then we must have

$$\pi(n_1 + 1, n_2) = \eta_1 \pi(n_1, n_2), \quad (n_1, n_2) \in \mathcal{S}_+, \tag{A5}$$

for some  $\eta_1 \in (0, 1)$ . Similarly, from (A3), we have

$$\tilde{p}_{10}^{(1)} = \mathbb{P}(\tilde{\mathcal{Z}}_{\ell+1} = (n_1 + 1, 0) | \tilde{\mathcal{Z}}_\ell = (n_1, 0)) = \frac{\pi(n_1 + 1, 0)}{\pi(n_1, 0)} p_{(-1)0}^{(1)} > 0, \quad (n_1, 0) \in \mathcal{S}_1.$$

Thus, for some  $\alpha_1 \in (0, 1)$ ,

$$\pi(n_1 + 1, 0) = \alpha_1 \pi(n_1, 0), \quad (n_1, 0) \in \mathcal{S}_1. \tag{A6}$$

On the other hand, we have, for  $i = 0, \pm 1$  and  $(n_1 - i, n_2 - 1), (n_1, n_2) \in \mathcal{S}_+$ ,

$$\tilde{p}_{(-i)(-1)}^{(+)} = \mathbb{P}(\tilde{\mathcal{Z}}_{\ell+1} = (n_1 - i, n_2 - 1) | \tilde{\mathcal{Z}}_\ell = (n_1, n_2)) = \frac{\pi(n_1 - i, n_2 - 1)}{\pi(n_1, n_2)} p_{i1}^{(+)} = 0,$$

since we assume (A1). For  $n_1 > 1$ ,

$$\tilde{p}_{(-i)(-1)}^{(+)} = \mathbb{P}(\tilde{\mathcal{Z}}_{\ell+1} = (n_1 - i, 0) | \tilde{\mathcal{Z}}_\ell = (n_1, 1)) = \frac{\pi(n_1 - i, 0)}{\pi(n_1, 1)} p_{i1}^{(1)}.$$

Thus, we must have  $p_{i1}^{(1)} = 0$  for any  $i = 0, \pm 1$ . Similarly, we have, for  $j = \pm 1$  and  $(n_1 - 1, n_2 - j), (n_1, n_2) \in \mathcal{S}_+$ ,

$$\tilde{p}_{(-1)(-j)}^{(+)} = \mathbb{P}(\tilde{\mathcal{Z}}_{\ell+1} = (n_1 - 1, n_2 - j) | \tilde{\mathcal{Z}}_\ell = (n_1, n_2)) = \frac{\pi(n_1 - 1, n_2 - j)}{\pi(n_1, n_2)} p_{1j}^{(+)} = 0,$$

from (A1). Since

$$\tilde{p}_{(-1)(-j)}^{(+)} = \mathbb{P}(\tilde{\mathcal{Z}}_{\ell+1} = (0, n_2 - j) | \tilde{\mathcal{Z}}_\ell = (1, n_2)) = \frac{\pi(0, n_2 - j)}{\pi(1, n_2)} p_{1j}^{(2)},$$

we also have  $p_{1j}^{(2)} = 0$  for  $j = \pm 1$ , and from (A4),  $p_{01}^{(2)} > 0$  and  $p_{0(-1)}^{(2)} > 0$ . Moreover, for  $n_2 \geq 1$ ,

$$\tilde{p}_{01}^{(2)} = \mathbb{P}(\tilde{\mathcal{Z}}_{\ell+1} = (0, n_2 + 1) | \tilde{\mathcal{Z}}_\ell = (0, n_2)) = \frac{\pi(0, n_2 + 1)}{\pi(0, n_2)} p_{0(-1)}^{(2)} > 0.$$

Thus, for  $\eta_2 \in (0, 1)$ ,

$$\pi(0, n_2 + 1) = \eta_2 \pi(0, n_2), \quad (0, n_2) \in \mathcal{S}_2.$$

We next have, for any  $n_2 \geq 1$  and  $n_1 \geq 1$ ,

$$\tilde{p}_{(-1)0}^{(+)} = \mathbb{P}(\tilde{\mathcal{Z}}_{\ell+1} = (n_1, n_2) | \tilde{\mathcal{Z}}_\ell = (n_1 + 1, n_2)) = \frac{\pi(n_1, n_2)}{\pi(n_1 + 1, n_2)} p_{10}^{(+)} = \eta_1^{-1} p_{10}^{(+)} > 0,$$

$$\tilde{p}_{(-1)0}^{(+)} = \mathbb{P}(\tilde{\mathcal{Z}}_{\ell+1} = (0, n_2) | \tilde{\mathcal{Z}}_\ell = (1, n_2)) = \frac{\pi(0, n_2)}{\pi(1, n_2)} p_{10}^{(2)} = \frac{\eta_2^{n_2-1} \pi(0, 1)}{\pi(1, n_2)} p_{10}^{(2)} = \eta_1^{-1} p_{10}^{(+)}.$$

Hence,

$$\pi(1, n_2) = \frac{p_{10}^{(2)}}{p_{10}^{(+)}} \eta_1 \eta_2^{n_2-1} \pi(0, 1), \quad n_2 \geq 1,$$

which implies

$$\begin{aligned} \pi(1, 1) &= \frac{p_{10}^{(2)}}{p_{10}^{(+)}} \eta_1 \pi(0, 1) = c^{(2+)} \eta_1 \pi(0, 1), \\ \pi(n_1, n_2) &= \eta_1^{n_1-1} \pi(1, n_2) = c^{(2+)} \eta_1^{n_1-1} \eta_2^{n_2-1} \pi(1, 1), \quad (n_1, n_2) \in \mathcal{S}_+. \end{aligned}$$

On the other hand,

$$\begin{aligned} \tilde{p}_{(-1)(-1)}^{(+)} &= \mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = (0, 0) | \tilde{\mathbf{Z}}_{\ell} = (1, 1)) = \frac{\pi(0, 0)}{\pi(1, 1)} p_{11}^{(0)} = 0, \\ \tilde{p}_{(-1)0}^{(1)} &= \mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = (0, 0) | \tilde{\mathbf{Z}}_{\ell} = (1, 0)) = \frac{\pi(0, 0)}{\pi(1, 0)} p_{10}^{(0)} = \frac{\pi(n_1 - 1, 0)}{\pi(n_1, 0)} p_{10}^{(1)} = \alpha_1^{-1} p_{10}^{(1)}, \\ \tilde{p}_{0(-1)}^{(2)} &= \mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = (0, 0) | \tilde{\mathbf{Z}}_{\ell} = (0, 1)) = \frac{\pi(0, 0)}{\pi(0, 1)} p_{01}^{(0)} = \frac{\pi(0, n_2 - 1)}{\pi(0, n_2)} p_{01}^{(2)} = \eta_2^{-1} p_{01}^{(2)}. \end{aligned}$$

It follows from the irreducible assumption (i), (A1),  $p_{i1}^{(1)} = 0$  for any  $i = 0, \pm 1$  and  $p_{1j}^{(2)} = 0$  for  $j = \pm 1$  that  $p_{10}^{(1)}, p_{01}^{(2)} > 0$ . Thus, we immediately have  $p_{11}^{(0)} = 0, p_{10}^{(0)}, p_{01}^{(0)} > 0$  and

$$\begin{aligned} \pi(1, 0) &= \frac{p_{10}^{(0)}}{p_{10}^{(1)}} \alpha_1 \pi(0, 0) = c^{(10)} \alpha_1 \pi(0, 0), \\ \pi(0, 1) &= \frac{p_{01}^{(0)}}{p_{01}^{(2)}} \eta_2 \pi(0, 0) = c^{(20)} \eta_2 \pi(0, 0), \\ \pi(1, 1) &= c^{(2+)} \eta_1 \pi(0, 1) = c^{(20)} c^{(2+)} \eta_1 \eta_2 \pi(0, 0). \end{aligned}$$

We finally obtain  $\eta_1, \eta_2$  and  $\alpha_1$ . We have the following stationary equations.

$$\begin{aligned} (p_{10}^{(+)} + p_{(-1)0}^{(+)}) \pi(n_1, n_2) &= p_{(-1)0}^{(+)} \pi(n_1 + 1, n_2) + p_{10}^{(+)} \pi(n_1 - 1, n_2), \\ (p_{10}^{(1)} + p_{(-1)0}^{(1)}) \pi(n_1, 0) &= p_{(-1)0}^{(1)} \pi(n_1 + 1, 0) + p_{10}^{(1)} \pi(n_1 - 1, 0). \end{aligned}$$

From (A5) and (A6), we have

$$\begin{aligned} (p_{10}^{(+)} + p_{(-1)0}^{(+)}) \pi(n_1, n_2) &= p_{(-1)0}^{(+)} \eta_1 \pi(n_1, n_2) + p_{10}^{(+)} \eta_1^{-1} \pi(n_1, n_2), \\ (p_{10}^{(1)} + p_{(-1)0}^{(1)}) \pi(n_1, 0) &= p_{(-1)0}^{(1)} \alpha_1 \pi(n_1, 0) + p_{10}^{(1)} \alpha_1 \pi(n_1 - 1, 0). \end{aligned}$$

Thus, we obtain

$$\eta_1 = \frac{p_{10}^{(+)}}{p_{(-1)0}^{(+)}} < 1, \quad \alpha_1 = \frac{p_{10}^{(1)}}{p_{(-1)0}^{(1)}} < 1.$$

Repeatedly, from the stationary equations,

$$(p_{01}^{(2)} + p_{0(-1)}^{(2)} + p_{10}^{(2)}) \pi(0, n_2) = p_{0(-1)}^{(2)} \pi(0, n_2 + 1) + p_{01}^{(2)} \pi(0, n_2 - 1) + p_{(-1)0}^{(+)} \pi(1, n_2),$$

we have

$$\begin{aligned}
 (p_{01}^{(2)} + p_{0(-1)}^{(2)} + p_{10}^{(2)}) &= p_{0(-1)}^{(2)} \eta_2 + p_{01}^{(2)} \eta_2^{-1} + p_{(-1)0}^{(+)} \frac{p_{10}^{(2)}}{p_{10}^{(+)}} \eta_1 \\
 &= p_{0(-1)}^{(2)} \eta_2 + p_{01}^{(2)} \eta_2^{-1} + p_{(-1)0}^{(+)} \frac{p_{10}^{(2)}}{p_{10}^{(+)}} \frac{p_{10}^{(+)}}{p_{(-1)0}^{(+)}} \\
 &= p_{0(-1)}^{(2)} \eta_2 + p_{01}^{(2)} \eta_2^{-1} + p_{10}^{(2)}.
 \end{aligned}$$

Hence,

$$\eta_2 = \frac{p_{01}^{(2)}}{p_{0(-1)}^{(2)}} < 1.$$

**APPENDIX B. PROOF OF LEMMA 3.3**

We first prove the condition (a1). For  $n_1 \geq 1$  and  $i = 0, \pm 1$ , substituting  $\mathbf{n} = (n_1, 1)$  and  $\mathbf{n}' = (n_1 - i, 0)$  into (3.1),

$$\mathbb{P}(\tilde{\mathbf{Z}}_{\ell+1} = (n_1 - i, 0) | \tilde{\mathbf{Z}}_{\ell} = (n_1, 1)) = \mathbb{P}(\tilde{\mathbf{X}}^{(+)} = -(i, 1)).$$

Since  $\mathbf{n}' \in \mathcal{S}_1$ , we have, using Lemma 3.1,

$$\begin{aligned}
 \mathbb{P}(\tilde{\mathbf{X}}^{(+)} = -(i, 1)) &= \frac{\pi(n_1 - i, 0)}{\pi(n_1, 1)} \mathbb{P}(\mathbf{Z}_{\ell+1} = (n_1, 1) | \mathbf{Z}_{\ell} = (n_1 - i, 0)) \\
 &= \frac{\pi(1, 0)}{\pi(1, 1)} p_{i1}^{(1)} \eta_1^{-i}, \quad i = 0, \pm 1.
 \end{aligned}$$

On the other hand, from Lemma 3.1, for  $\mathbf{n}, \mathbf{n}' = \mathbf{n} - (i, 1) \in \mathcal{S}_+$ , we have

$$\begin{aligned}
 \mathbb{P}(\tilde{\mathbf{X}}^{(+)} = -(i, 1)) &= \frac{\pi(\mathbf{n} - (i, 1))}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{X}^{(+)} = (i, 1)) \\
 &= \eta_1^{-i} \eta_2^{-1} p_{i1}^{(+)}, \quad i = 0, \pm 1.
 \end{aligned}$$

Thus, we obtain, for all  $i = 0, \pm 1$ ,

$$\frac{\pi(1, 0)}{\pi(1, 1)} p_{i1}^{(1)} = \eta_2^{-1} p_{i1}^{(+)}.$$

Thus,  $p_{i1}^{(1)} = 0$  if and only if  $p_{i1}^{(+)} = 0$ . In addition, by the condition (ii), for some  $i = 0, \pm 1$  such that  $p_{i1}^{(+)} > 0$ ,

$$p_{i1}^{(1)} = \frac{\pi(1, 1)}{\pi(1, 0)} \eta_2^{-1} p_{i1}^{(+)} = c^{(1+)} p_{i1}^{(+)}. \tag{B.1}$$

Using a similar argument, we also have

$$p_{1i}^{(2)} = \frac{\pi(1, 1)}{\pi(0, 1)} \eta_1^{-1} p_{1i}^{(+)} = c^{(2+)} p_{1i}^{(+)}. \tag{B.2}$$

These complete the proof of the condition (a1).

We next obtain the conditions (a2), (a3) and (a4). Since  $\{\tilde{Z}_\ell\}$  has the homogeneous transitions as long as  $\tilde{Z}_\ell \in \mathcal{S}_1$ , we have, from (2.2), (3.1) and Lemma 3.1,

$$\mathbb{P}(\tilde{X}^{(1)} = (-1, 0)) = \frac{\pi(0, 0)}{\pi(1, 0)} p_{10}^{(0)} = \frac{\pi(n-1, 0)}{\pi(n, 0)} p_{10}^{(1)} = \eta_1^{-1} p_{10}^{(1)}, \quad \forall n \geq 1.$$

Similarly, for  $\tilde{X}^{(+)} = (-1, -1)$  and for any  $n \geq 1$ , we have

$$\mathbb{P}(\tilde{X}^{(+)} = (-1, -1)) = \frac{\pi(0, 0)}{\pi(1, 1)} p_{11}^{(0)} = \frac{\pi(n-1, 0)}{\pi(n, 1)} p_{11}^{(1)} = \frac{\pi(1, 0)}{\pi(1, 1)} \eta_1^{-1} p_{11}^{(1)}. \tag{B.3}$$

Hence, for any  $i = 0, 1$ , we must have  $p_{1i}^{(0)} = p_{1i}^{(1)} = 0$  if  $c^{(10)} = 0$ , and  $c^{(10)} p_{1i}^{(1)} = p_{1i}^{(0)}$  if  $c^{(10)} > 0$ . In addition, if  $c^{(10)} > 0$ , then we have, from (B.1)–(B.3)

$$\begin{aligned} \pi(1, 0) &= c^{(10)} \eta_1 \pi(0, 0), \\ \pi(1, 1) &= c^{(1+)} \eta_2 \pi(1, 0) = c^{(10)} c^{(1+)} \eta_1 \eta_2 \pi(0, 0), \\ \pi(0, 1) &= \frac{1}{c^{(2+)}} \eta_1^{-1} \pi(1, 1) = \frac{c^{(10)} c^{(1+)}}{c^{(2+)}} \eta_2 \pi(0, 0). \end{aligned}$$

Similarly, we obtain (3.8)–(3.10) if  $c^{(20)} > 0$ . We complete the proof. ■

### APPENDIX C. PROOF OF COROLLARY 3.1

We first note that the stationary distribution has a product form if and only if

$$\pi(n_1, n_2) = \nu_{n_1}^{(1)} \nu_{n_2}^{(2)}, \quad n_1, n_2 \in \mathbb{Z}_+, \tag{C.1}$$

for some  $\nu_{n_1}^{(1)}, \nu_{n_2}^{(2)} \in (0, 1)$ . Assume that the stationary distribution of  $\{Z_\ell\}$  satisfies (C.1). Since  $\{Z_\ell\}$  is structure-reversible, if  $c^{(20)} > 0$ , then from (3.8)–(3.10),

$$\nu_0^{(1)} \nu_0^{(2)} = \pi(0, 0), \tag{C.2}$$

$$\nu_1^{(1)} \nu_0^{(2)} = \frac{c^{(20)} c^{(2+)}}{c^{(1+)}} \eta_1 \pi(0, 0), \tag{C.3}$$

$$\nu_0^{(1)} \nu_1^{(2)} = c^{(20)} \eta_2 \pi(0, 0), \tag{C.4}$$

$$\nu_1^{(1)} \nu_1^{(2)} = c^{(20)} c^{(2+)} \eta_1 \eta_2 \pi(0, 0). \tag{C.5}$$

Substituting (C.2) and (C.3) into (C.4) and (C.5), we have

$$\nu_1^{(2)} = c^{(20)} \eta_2 \nu_0^{(2)} = c^{(1+)} \eta_2 \nu_0^{(2)}.$$

Hence, the condition (3.23) holds. We also have (3.24) in a similar way.

We next suppose that the condition (3.23) is satisfied. Then, we have

$$\pi(1, 0) = \frac{c^{(20)}c^{(2+)}}{c^{(1+)}}\eta_1\pi(0, 0) = c^{(2+)}\eta_1\pi(0, 0).$$

Thus, we can rewrite the stationary distribution of Theorem 3.1 as

$$\begin{aligned} \pi(n, 0) &= \eta_1^{n-1}\pi(1, 0) = c^{(2+)}\eta_1^n\pi(0, 0), \quad n \geq 1, \\ \pi(0, n) &= \eta_2^{n-1}\pi(0, 1) = c^{(20)}\eta_2^n\pi(0, 0), \quad n \geq 1, \\ \pi(n_1, n_2) &= \eta_1^{n_1-1}\eta_2^{n_2-1}\pi(1, 1) = c^{(20)}c^{(2+)}\eta_1^{n_1}\eta_2^{n_2}\pi(0, 0), \quad n_1, n_2 \geq 1. \end{aligned}$$

For  $n \in \mathbb{Z}_+$ , we put  $\nu_n^{(1)}$  and  $\nu_n^{(2)}$  as follows.

$$\begin{aligned} \nu_n^{(1)} &= c^{(2+)}\eta_1^n\nu_0^{(1)}, \quad n \geq 1, \\ \nu_n^{(2)} &= c^{(20)}\eta_2^n\nu_0^{(2)}, \quad n \geq 1, \\ \nu_0^{(1)}\nu_0^{(2)} &= \pi(0, 0), \end{aligned}$$

and we have

$$\begin{aligned} \nu_n^{(1)}\nu_0^{(2)} &= c^{(2+)}\eta_1^n\pi(0, 0) = \pi(n, 0), \quad n \geq 1, \\ \nu_0^{(1)}\nu_n^{(2)} &= c^{(20)}\eta_2^n\pi(0, 0) = \pi(0, n), \quad n \geq 1, \\ \nu_{n_1}^{(1)}\nu_{n_2}^{(2)} &= c^{(20)}c^{(2+)}\eta_1^{n_1}\eta_2^{n_2}\pi(0, 0) = \pi(n_1, n_2), \quad n_1, n_2 \geq 1. \end{aligned}$$

These equations imply (C.1). We similarly obtain (C.1) if the condition (3.24) holds. This completes the proof since we have either  $c^{(10)} > 0$  or  $c^{(20)} > 0$  (see Remark 3.1).

### APPENDIX D. PRODUCT FORM BUT NOT STRUCTURE-REVERSIBILITY

We give an example of the reflecting random walk to have a product form stationary distribution but not to be structure-reversible. For transition probabilities of the reflecting random walk, let

$$\begin{aligned} p_{00}^{(0)} &= 0.0840821, & p_{10}^{(0)} &= 0.49716, & p_{01}^{(0)} &= 0.188503, & p_{11}^{(0)} &= 0.230255, \\ p_{00}^{(1)} &= 0.0840821, & p_{10}^{(1)} &= 0.126693, & p_{01}^{(1)} &= 0.216346, & p_{11}^{(1)} &= 0.15534, \\ p_{(-1)0}^{(1)} &= 0.1205, & p_{(-1)1}^{(1)} &= 0.297039, & p_{00}^{(2)} &= 0.363151, & p_{10}^{(2)} &= 0.267565, \\ p_{01}^{(2)} &= 0.0246397, & p_{11}^{(2)} &= 0.025552, & p_{0(-1)}^{(2)} &= 0.223309, & p_{1(-1)}^{(2)} &= 0.0957827, \\ p_{00}^{(+)} &= 0.469511, & p_{10}^{(+)} &= 0.0449179, & p_{01}^{(+)} &= 0.00497654, & p_{11}^{(+)} &= 0.012212, \\ p_{(-1)0}^{(+)} &= 0.398019, & p_{(-1)1}^{(+)} &= 0.0045338, & p_{0(-1)}^{(+)} &= 0.0380278, & p_{1(-1)}^{(+)} &= 0.0278023, \\ p_{(-1)(-1)}^{(+)} &= 0. \end{aligned}$$

Then, we can see that (3.13), (3.14), (3.17) and (3.24)–(3.29) of [6] hold with  $\eta_1 = 0.3$  and  $\eta_2 = 0.2$ , and therefore, the stationary distribution of this random walk has a product form. For this random

walk, we have

$$\frac{p_{(-1)1}^{(1)}}{p_{(-1)1}^{(+)}} = 65, 5165, \quad \frac{p_{01}^{(1)}}{p_{01}^{(+)}} = 43.4732, \quad \frac{p_{11}^{(1)}}{p_{11}^{(+)}} = 12.7203.$$

Hence, (a1) does not hold, and therefore, this random walk is not structure-reversible.

**APPENDIX E. PROOF OF (4.10)**

Recall that  $c^{(10)} > 0$ . Similar to the case  $\mathbf{n} = (0, 0)$ , using the conditions (a1)–(a3), (3.2)–(3.7), and (4.1), we compute the left-hand side of (4.10) in the following ways.

- For  $\mathbf{n} = (1, 0)$ , we have  $\mathbf{n}' \in \{(i, j); i = 0, 1, 2, j = 0, 1\}$ .

$$\begin{aligned} & \sum_{\mathbf{n}' \in \mathcal{S}} \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{Z}_{\ell+1} = \mathbf{n}' | \mathbf{Z}_{\ell} = \mathbf{n}') = p_{00}^{(1)} + \frac{\pi(0, 0)}{\pi(1, 0)} p_{10}^{(0)} + \frac{\pi(2, 0)}{\pi(1, 0)} p_{(-1)0}^{(1)} \\ & \quad + \frac{\pi(0, 1)}{\pi(1, 0)} p_{1(-1)}^{(2)} + \frac{\pi(1, 1)}{\pi(1, 0)} p_{0(-1)}^{(+)} + \frac{\pi(2, 1)}{\pi(1, 0)} p_{(-1)(-1)}^{(+)} \\ & = p_{00}^{(1)} + \frac{1}{c^{(10)}} p_{10}^{(0)} \eta_1^{-1} + p_{(-1)0}^{(1)} \eta_1 \\ & \quad + \frac{c^{(1+)}}{c^{(2+)}} p_{1(-1)}^{(2)} \eta_1^{-1} \eta_2 + c^{(1+)} p_{0(-1)}^{(+)} \eta_2 + c^{(1+)} p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ & = p_{00}^{(1)} + p_{10}^{(1)} \eta_1^{-1} + p_{(-1)0}^{(1)} \eta_1 + c^{(1+)} \left( p_{1(-1)}^{(+)} \eta_1^{-1} \eta_2 + p_{0(-1)}^{(+)} \eta_2 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \right) \\ & = \gamma_1 (\eta_1^{-1}, \eta_2^{-1}) = 1. \end{aligned}$$

- For  $\mathbf{n} = (1, 1)$ , we have  $\mathbf{n}' \in \{(i, j); i, j = 0, 1, 2\}$ .

$$\begin{aligned} & \sum_{\mathbf{n}' \in \mathcal{S}} \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{Z}_{\ell+1} = \mathbf{n}' | \mathbf{Z}_{\ell} = \mathbf{n}') \\ & = p_{00}^{(+)} + \frac{\pi(0, 1)}{\pi(1, 1)} p_{10}^{(2)} + \frac{\pi(0, 2)}{\pi(1, 1)} p_{1(-1)}^{(2)} + \frac{\pi(1, 2)}{\pi(1, 1)} p_{0(-1)}^{(+)} + \frac{\pi(2, 2)}{\pi(1, 1)} p_{(-1)(-1)}^{(+)} \\ & \quad + \frac{\pi(2, 1)}{\pi(1, 1)} p_{(-1)0}^{(+)} + \frac{\pi(2, 0)}{\pi(1, 1)} p_{(-1)1}^{(1)} + \frac{\pi(1, 0)}{\pi(1, 1)} p_{01}^{(1)} + \frac{\pi(0, 0)}{\pi(1, 1)} p_{11}^{(0)} \\ & = p_{00}^{(+)} + \frac{1}{c^{(2+)}} p_{10}^{(2)} \eta_1^{-1} + \frac{1}{c^{(2+)}} p_{1(-1)}^{(2)} \eta_1^{-1} \eta_2 + p_{0(-1)}^{(+)} \eta_2 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ & \quad + p_{(-1)0}^{(+)} \eta_1 + \frac{1}{c^{(1+)}} p_{(-1)1}^{(1)} \eta_1 \eta_2^{-1} + \frac{1}{c^{(1+)}} p_{01}^{(1)} \eta_2^{-1} + \frac{1}{c^{(10)} c^{(1+)}} p_{11}^{(0)} \eta_1^{-1} \eta_2^{-1} \\ & = p_{00}^{(+)} + p_{10}^{(+)} \eta_1^{-1} + p_{1(-1)}^{(+)} \eta_1^{-1} \eta_2 + p_{0(-1)}^{(+)} \eta_2 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 + p_{(-1)0}^{(+)} \eta_1 \\ & \quad + p_{(-1)1}^{(+)} \eta_1 \eta_2^{-1} + p_{01}^{(+)} \eta_2^{-1} + p_{11}^{(+)} \eta_1^{-1} \eta_2^{-1} \\ & = \gamma_+ (\eta_1^{-1}, \eta_2^{-1}) = 1. \end{aligned}$$

- For  $\mathbf{n} = (n_1, 0)$ , we have  $\mathbf{n}' \in \{(n_1 + i, j), i = 0, \pm 1, j = 0, 1\}$ .

$$\begin{aligned} & \sum_{\mathbf{n}' \in \mathcal{S}} \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{Z}_{\ell+1} = \mathbf{n} | \mathbf{Z}_\ell = \mathbf{n}') \\ &= p_{00}^{(1)} + \frac{\pi(n_1 - 1, 0)}{\pi(n_1, 0)} p_{10}^{(1)} + \frac{\pi(n_1 + 1, 0)}{\pi(n_1, 0)} p_{(-1)0}^{(1)} \\ & \quad + \frac{\pi(n_1 - 1, 1)}{\pi(n_1, 0)} p_{1(-1)}^{(+)} + \frac{\pi(n_1, 1)}{\pi(n_1, 0)} p_{0(-1)}^{(+)} + \frac{\pi(n_1 + 1, 1)}{\pi(n_1, 0)} p_{(-1)(-1)}^{(+)} \\ &= p_{00}^{(1)} + p_{10}^{(1)} \eta_1^{-1} + p_{(-1)0}^{(1)} \eta_1 + c^{(1+)} \left( p_{1(-1)}^{(+)} \eta_1^{-1} \eta_2 + p_{0(-1)}^{(+)} \eta_2 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \right) \\ &= \gamma_1 (\eta_1^{-1}, \eta_2^{-1}) = 1. \end{aligned}$$

- For  $\mathbf{n} = (n_1, 1)$ , we have  $\mathbf{n}' \in \{(n_1 + i, j) \mid i = 0, \pm 1, j = 0, 1, 2\}$ .

$$\begin{aligned} & \sum_{\mathbf{n}' \in \mathcal{S}} \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{Z}_{\ell+1} = \mathbf{n} | \mathbf{Z}_\ell = \mathbf{n}') \\ &= p_{00}^{(+)} + \frac{\pi(n_1 - 1, 1)}{\pi(n_1, 1)} p_{10}^{(+)} + \frac{\pi(n_1 - 1, 2)}{\pi(n_1, 1)} p_{1(-1)}^{(+)} \\ & \quad + \frac{\pi(n_1, 2)}{\pi(n_1, 1)} p_{0(-1)}^{(+)} + \frac{\pi(n_1 + 1, 2)}{\pi(n_1, 1)} p_{(-1)(-1)}^{(+)} + \frac{\pi(n_1 + 1, 1)}{\pi(n_1, 1)} p_{(-1)0}^{(+)} \\ & \quad + \frac{\pi(n_1 + 1, 0)}{\pi(n_1, 1)} p_{(-1)1}^{(1)} + \frac{\pi(n_1, 0)}{\pi(n_1, 1)} p_{01}^{(1)} + \frac{\pi(n_1 - 1, 0)}{\pi(n_1, 1)} p_{11}^{(1)} \\ &= p_{00}^{(+)} + p_{10}^{(+)} \eta_1^{-1} + p_{1(-1)}^{(+)} \eta_1^{-1} \eta_2 + p_{0(-1)}^{(+)} \eta_2 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ & \quad + p_{(-1)0}^{(+)} \eta_1 + \frac{1}{c^{(1+)}} p_{(-1)1}^{(1)} \eta_1 \eta_2^{-1} + \frac{1}{c^{(1+)}} p_{01}^{(1)} \eta_2^{-1} + \frac{1}{c^{(1+)}} p_{11}^{(1)} \eta_1^{-1} \eta_2^{-1} \\ &= p_{00}^{(+)} + p_{10}^{(+)} \eta_1^{-1} + p_{1(-1)}^{(+)} \eta_1^{-1} \eta_2 + p_{0(-1)}^{(+)} \eta_2 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ & \quad + p_{(-1)0}^{(+)} \eta_1 + p_{(-1)1}^{(1)} \eta_1 \eta_2^{-1} + p_{01}^{(1)} \eta_2^{-1} + p_{11}^{(1)} \eta_1^{-1} \eta_2^{-1} \\ &= \gamma_+ (\eta_1^{-1}, \eta_2^{-1}) = 1. \end{aligned}$$

- For  $\mathbf{n} = (n_1, n_2)$ , we have  $\mathbf{n}' \in \{(n_1 + i, n_2 + j); i, j = 0, \pm 1\}$ .

$$\begin{aligned} & \sum_{\mathbf{n}' \in \mathcal{S}} \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} \mathbb{P}(\mathbf{Z}_{\ell+1} = \mathbf{n} | \mathbf{Z}_\ell = \mathbf{n}') \\ &= p_{00}^{(+)} + \frac{\pi(n_1 - 1, n_2)}{\pi(n_1, n_2)} p_{10}^{(+)} + \frac{\pi(n_1 - 1, n_2 + 1)}{\pi(n_1, n_2)} p_{1(-1)}^{(+)} \\ & \quad + \frac{\pi(n_1, n_2 + 1)}{\pi(n_1, n_2)} p_{0(-1)}^{(+)} + \frac{\pi(n_1 + 1, n_2 + 1)}{\pi(n_1, n_2)} p_{(-1)(-1)}^{(+)} + \frac{\pi(n_1 + 1, n_2)}{\pi(n_1, n_2)} p_{(-1)0}^{(+)} \\ & \quad + \frac{\pi(n_1 + 1, n_2 - 1)}{\pi(n_1, n_2)} p_{(-1)1}^{(+)} + \frac{\pi(n_1, n_2 - 1)}{\pi(n_1, n_2)} p_{01}^{(+)} + \frac{\pi(n_1 - 1, n_2 - 1)}{\pi(n_1, n_2)} p_{11}^{(+)} \\ &= p_{00}^{(+)} + p_{10}^{(+)} \eta_1^{-1} + p_{1(-1)}^{(+)} \eta_1^{-1} \eta_2 + p_{0(-1)}^{(+)} \eta_2 + p_{(-1)(-1)}^{(+)} \eta_1 \eta_2 \\ & \quad + p_{(-1)0}^{(+)} \eta_1 + p_{(-1)1}^{(+)} \eta_1 \eta_2^{-1} + p_{01}^{(+)} \eta_2^{-1} + p_{11}^{(+)} \eta_1^{-1} \eta_2^{-1} \\ &= \gamma_+ (\eta_1^{-1}, \eta_2^{-1}) = 1. \end{aligned}$$

## APPENDIX F. PROOF OF $\gamma_i(\rho_1^{-1}, \rho_2^{-1}) = 1$ FOR THE DISCRETE TIME JACKSON NETWORK WITH EXTRA ARRIVALS

Under the assumption (5.5), we rewrite the traffic Eq. (5.1) as follows.

$$\alpha_1 = \lambda + \alpha_2 r_{21} + \alpha_1 r_{11}, \quad \alpha_2 = \lambda + \alpha_1 r_{12} + \alpha_2 r_{22}. \quad (\text{F.1})$$

The solutions of these equations are given by

$$\alpha_1 = \frac{\lambda(1 - r_{22}) + \lambda r_{21}}{1 - r_{11} - r_{22} - r_{12}r_{21} + r_{11}r_{22}}, \quad \alpha_2 = \frac{\lambda(1 - r_{11}) + \lambda r_{12}}{1 - r_{11} - r_{22} - r_{12}r_{21} + r_{11}r_{22}}.$$

We note that  $1 - r_{11} = r_{12} + r_{10}$  and  $1 - r_{22} = r_{21} + r_{20}$ . From the assumption (5.6),

$$(1 - r_{22})r_{12} = (1 - r_{11})r_{21},$$

and therefore, we have  $\alpha_1 r_{12} = \alpha_2 r_{21}$ . Substituting this equation into (F.1), we have  $\lambda = \alpha_1 r_{10} = \alpha_2 r_{20}$ . For  $i = 0, 1, +$ ,

$$\begin{aligned} \gamma_0(\rho_1^{-1}, \rho_2^{-1}) &= \mu_1 + \mu_2 - \lambda_1^{(0)} - \lambda_2^{(0)} + \left(1 + \frac{\lambda_1^{(0)}}{\lambda}\right) \mu_1 r_{10} \rho_1 + \left(1 + \frac{\lambda_2^{(0)}}{\lambda}\right) \mu_2 r_{20} \rho_2 \\ &= \mu_1 + \mu_2 - \lambda_1^{(0)} - \lambda_2^{(0)} + \left(1 + \frac{\lambda_1^{(0)}}{\lambda}\right) \alpha_1 r_{10} + \left(1 + \frac{\lambda_2^{(0)}}{\lambda}\right) \alpha_2 r_{20} \\ &= \mu_1 + \mu_2 + \lambda + \lambda = 1, \end{aligned}$$

$$\begin{aligned} \gamma_1(\rho_1^{-1}, \rho_2^{-1}) &= \mu_2 - \lambda_2^{(1)} + \rho_1^{-1} \lambda + \mu_1 r_{10} \rho_1 + \left(1 + \frac{\lambda_2^{(1)}}{\lambda}\right) (\mu_2 r_{21} \rho_1^{-1} \rho_2 + \mu_2 r_{20} \rho_2) \\ &= \mu_2 - \frac{r_{11}}{r_{12}} \lambda + \frac{\mu_1}{\alpha_1} \lambda + \lambda + \left(1 + \frac{r_{11}}{r_{12}}\right) \left(\frac{\alpha_2}{\alpha_1} r_{21} \mu_1 + r_{20} \alpha_2\right) \\ &= \mu_2 - \frac{r_{11}}{r_{12}} \lambda + \mu_1 r_{10} + \lambda + \mu_1 r_{12} + \lambda + \mu_1 r_{11} + \frac{r_{11}}{r_{12}} \lambda = 1, \end{aligned}$$

$$\gamma_+(\rho_1^{-1}, \rho_2^{-1}) = \mu_1 r_{11} + \mu_2 r_{22} + \mu_1 r_{10} + \mu_2 r_{20} + \lambda + \mu_2 r_{21} + \lambda + \mu_1 r_{12} = 1.$$

By symmetry of  $\gamma_1(\rho_1^{-1}, \rho_2^{-1}) = 1$ , we have  $\gamma_2(\rho_1^{-1}, \rho_2^{-1}) = 1$ .