Can. J. Math., Vol. XXIV, No. 1, 1972, pp. 1–4

A CLASS OF NON-CENTRAL E-FUNCTORS

G. R. CHAPMAN

1. Introduction. We refer the reader to [**1**, Chapters 1 and 2] for the notions of E-functor and centrality. Let $R_1 \subseteq R$ be the integral group rings of the groups $G_1 \subseteq G$. Butler and Horrocks [**1**, Chapter 26] have shown that on the category $\mathfrak{C}$ of left, unitary R-modules the Hochschild E-functor determined by R_1 is central. There are no examples of non-central Hochschild E-functors, and our purpose is to establish the existence of a class of such E-functors.

Take G to be finite, non-abelian and let S be the centre of R. Denote by Φ the Hochschild E-functor determined by S. We obtain a necessary condition for the centrality of Φ in terms of the group structure of G. Let G^* denote the subgroup of G generated by the commutators of G together with the set $\{g^{h(g)}: g \in G\}$, where $h(g)$ is the class number of g in G. The main result is

THEOREM. *If $G/G^* \neq \{e\}$, then Φ is non-central.*

We note that $G/G^* \neq \{e\}$ when G is a non-abelian p-group.

2. Preliminaries. Let Z denote the integers with trivial G-module action, $\mathrm{Ext}_{\mathfrak{C}} = \mathrm{Ext}$, $\mathrm{Tor}^{\mathfrak{C}} = \mathrm{Tor}$, and $A, B \in \mathfrak{C}$. If $0 \to L \to P \to A \to 0$, $0 \to B \to Q \to M \to 0$ denote, respectively, a Φ-projective representation of A and a Φ-injective representation of B, then we have the exact sequences:

$$\ldots \to \mathrm{Ext}^1(P, B) \to \mathrm{Ext}^1(L, B) \xrightarrow{\delta_1} \mathrm{Ext}^2(A, B) \to \ldots$$

$$\ldots \to \mathrm{Ext}^1(A, Q) \to \mathrm{Ext}^1(A, M) \xrightarrow{\delta_2} \mathrm{Ext}^2(A, B) \to \ldots$$

We note that $\mathrm{Ext} \cdot \Phi(A, B) = \mathrm{Im}\, \delta_1$; $\Phi \cdot \mathrm{Ext}(A, B) = \mathrm{Im}\, \delta_2$ [**1**, Chapter 10].

3. Proof of the theorem. We use the result quoted in the preliminaries, with $A = B = Z$ and show that $\Phi \cdot \mathrm{Ext}(Z, Z) = 0$, $\mathrm{Ext} \cdot \Phi(Z, Z) = G/G^*$, whence the theorem follows. To avoid confusion, in the early stages of the proof B is taken to be a finitely generated, torsion free, G-trivial G-module. We prove that if $0 \to J \to R \otimes_S Z \to Z \to 0$ denotes the standard Φ-projective representation of Z, then $|G| \cdot J = 0$. From this, it follows that B is Φ-projective (so $\Phi \cdot \mathrm{Ext}(-, B) = 0$) and that $\mathrm{Ext}^1 (J, B) \cong \mathrm{Ext}_Z{}^1 (Z \otimes_G J, B)$. We use the last fact to show that $\mathrm{Ext} \cdot \Phi(Z, B) \cong \mathrm{Ext}^1 (J, B)$. When $B = Z$, this is G/G^*, and the proof is complete.

Received November 10, 1970

LEMMA 1. *J has exponent $|G|$.*

Proof. Let I be the difference ideal of G. Define $f:R \to R \otimes_S Z$ by $f(r) = r \otimes_S 1$ $(r \in R)$; then we have the exact commutative diagram of G-modules and G-module homomorphisms:

$$(a) \qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & I & \to & R & \xrightarrow{\epsilon} & Z & \to & 0 \\
 & & \downarrow & & \downarrow f & & \| & & \\
0 & \to & J & \to & R \otimes_S Z & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since S is the centre of R, if $s \in S$, then $s \cdot (R \otimes_S Z) = \epsilon(s) \cdot (R \otimes_S Z)$. Let N_G denote the sum of the elements of G. Then $N_G \in S$, so $N_G \cdot (R \otimes_S Z) = |G| \cdot (R \otimes_S Z)$. But J is a G-submodule of $R \otimes_S Z$, so $N_G \cdot J = |G| \cdot J$.

Now $N_G \cdot I = 0$, and since J is a G-homomorphic image of I, $N_G \cdot J = 0$. Thus, $|G| \cdot J = 0$, and the lemma is proved.

Note that J is finitely generated, so it is finite.

LEMMA 2. *B is a Φ-injective.*

Proof. Consider the standard Φ-injective representation of B

$$0 \to B \xrightarrow{i} \mathrm{Hom}_S(R, B) \to N \to 0.$$

Since B is G-trivial, $\mathrm{Hom}_S\,(R, B) \cong \mathrm{Hom}_Z\,(Z \otimes_S R, B)$, and since S is commutative we can take this to be $\mathrm{Hom}_Z\,(R \otimes_S Z, B)$. As an S-module (and hence as a group) $R \otimes_S Z$ is $Z \oplus J$. Now by Lemma 1, J is finite, so $\mathrm{Hom}_Z\,(J, B) = 0$ and $\mathrm{Hom}_S\,(R, B) \cong B$. This isomorphism provides a G-splitting map for i.

LEMMA 3. $\mathrm{Ext}^1\,(J, B) \cong \mathrm{Ext}_Z{}^1\,(Z \otimes_G J, B)$

Proof. If $0 \to M \to F \to J \to 0$ is a free presentation of J, we obtain exact sequences:

(1) $$0 \to \mathrm{Hom}_G\,(J, B) \to \mathrm{Hom}_G\,(F, B) \to \mathrm{Hom}_G\,(M, B) \to \mathrm{Ext}^1\,(J, B) \to \mathrm{Ext}^1\,(F, B) \to \ldots$$

$$\ldots \to \mathrm{Tor}_1\,(Z, F) \to \mathrm{Tor}_1\,(Z, J) \to Z \otimes_G M \to Z \otimes_G F \to Z \otimes_G J \to 0$$

https://doi.org/10.4153/CJM-1972-001-7 Published online by Cambridge University Press

In the latter sequence, $\mathrm{Tor}_1\,(Z, F) = 0$ and if we put

$$U = \mathrm{Ker}\,\{Z \otimes_G F \to Z \otimes_G J\},$$

there arise further exact sequences

$$(2)\quad 0 \to \mathrm{Hom}_Z\,(U, B) \xrightarrow{\xi} \mathrm{Hom}_Z\,(Z \otimes_G M, B) \to \mathrm{Hom}_Z\,(\mathrm{Tor}_1\,(Z, J), B) \to \dots$$

$$(3)\quad 0 \to \mathrm{Hom}_Z\,(Z \otimes_G J, B) \to \mathrm{Hom}_Z\,(Z \otimes_G F, B) \to \mathrm{Hom}_Z\,(U, B) \to \mathrm{Ext}_Z{}^1\,(Z \otimes_G J, B) \to \mathrm{Ext}_Z{}^1\,(Z \otimes_G F, B)$$

Now, $\mathrm{Ext}^1\,(F, B) = \mathrm{Ext}_Z{}^1\,(Z \otimes_G F, B) = 0$. Since J is finite (Lemma 1), so are $\mathrm{Tor}_1\,(J, Z)$ and $Z \otimes_G J$. However, B is torsion free, which means that $\mathrm{Hom}_Z\,(J, B) = \mathrm{Hom}_Z\,(\mathrm{Tor}_1\,(J, Z), B) = \mathrm{Hom}_Z\,(Z \otimes_G J, B) = 0$. Applying these facts to the sequences (1), (2), and (3) we can construct the exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_G\,(F, B) & \to & \mathrm{Hom}_G\,(M, B) & \to & \mathrm{Ext}^1\,(J, B) & \to 0 \\ & \wr\Vert & & \wr\Vert & & & \\ 0 \to & \mathrm{Hom}_Z\,(Z \otimes_G F, B) & \to & \mathrm{Hom}_Z\,(U, B) & \to & \mathrm{Ext}_Z{}^1\,(Z \otimes_G J, B) & \to 0, \end{array}$$

where the left hand isomorphism is an associativity isomorphism (B is G-trivial), and the right hand isomorphism is the composite of an associativity isomorphism and ξ. Hence, Lemma 3 follows.

LEMMA 4. $\mathrm{Ext} \cdot \Phi(Z, B) \cong \mathrm{Ext}^1\,(J, B)$

Proof. From the preliminaries, it is sufficient to show that

$$\mathrm{Ext}^1\,(R \otimes_S Z, B) = 0$$

in the exact sequence

$$\dots \to \mathrm{Ext}^1\,(Z, B) \to \mathrm{Ext}^1\,(R \otimes_S Z, B) \to \mathrm{Ext}^1\,(J, B) \xrightarrow{\delta_1} \mathrm{Ext}^2\,(Z, B) \to \dots$$

Now, $\mathrm{Ext}^1\,(Z, B) = 0$, and from Lemmas 1 and 3, $\mathrm{Ext}^1\,(J, B)$ is finite, so $\mathrm{Ext}^1\,(R \otimes_S Z, B)$ is finite. Consider the exact sequences

$$0 \to \mathrm{Hom}_G\,(Z, B) \to \mathrm{Hom}_G\,(R \otimes_S Z, B) \to \mathrm{Hom}_G\,(J, B) \to \dots$$

$$0 \to \mathrm{Hom}_G\,(R \otimes_S Z, B) \xrightarrow{[f]} \mathrm{Hom}_G\,(R, B) \to \mathrm{Hom}_G\,(K, B) \to \mathrm{Ext}^1\,(R \otimes_S Z, B) \to \mathrm{Ext}^1\,(R, B) \to \dots$$

In the former, $\mathrm{Hom}_G\,(J, B) = 0$, which means that Coker $[f]$ is finite. Thus, $\mathrm{Hom}_G\,(K, B)$ is finite. However, since B is G-trivial and torsion free, $\mathrm{Hom}_G\,(K, B)$ is free, whence it is zero. It now follows from the latter exact sequence that $\mathrm{Ext}^1\,(R \otimes_S Z, B) = 0$, and Lemma 4 is proved.

Now take $B = Z$. From Lemmas 3 and 4, $\operatorname{Ext}\cdot\Phi(Z, Z) \cong Z \otimes_G J$. Considering the diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & I^2 & & \\
 & & \downarrow & & \\
0 \rightarrow K \rightarrow & & I & \rightarrow & J \rightarrow 0 \\
 & & \downarrow & & \downarrow \\
 & & Z \otimes_G I & \rightarrow & Z \otimes_G J \rightarrow 0 \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array},$$

we see that $Z \otimes_G J \cong I/(I^2 + K)$.

From diagram (a), $K = R(I \cap S)$. Let $C_1, \ldots, C_k$ denote the distinct conjugacy classes of G, and let h_i, $\hat{c}_i$ denote, respectively, the number of and the sum of the elements in $C_i (1 \leqq i \leqq k)$. Then a basis for K is

$$\{\hat{c}_i - h_i : 1 \leqq i \leqq k\}.$$

Now, the map $\eta: I/I^2 \rightarrow G/G'$ given by $\eta((g - 1) + I^2) = gG'$ is an isomorphism between the additive group I/I^2 and the multiplicative group G/G'. Since $\eta((g_1 - 1) + I^2) = \eta((gg_1g^{-1} - 1) + I^2)$, for all $g, g_1 \in G$, it follows that $\eta((\hat{c}_i - h_i) + I^2) = (g_iG')^{h_i}$, where g_i is some element in C_i, $1 \leqq i \leqq k$. Hence, the additive group $I/(I^2 + K)$ is isomorphic to the multiplicative group G/G^*, where G^* is the subgroup of G generated by the commutators of G together with the set $\{g_i^{h_i} : g_i \in C_i, 1 \leqq i \leqq k\}$, and the theorem is proved.

COROLLARY. *If G is a non-abelian p-group, then Φ is non-central.*

Proof. From the class equation for G, it follows that $G^* \subseteq G' \cdot C(G) \cdot G^p$, where $G^p = \{g^p : g \in G\}$. For a p-group, $G' \cdot G^p = \phi(G)$, the Frattini subgroup of G, so there exists an epimorphism $G/G^* \rightarrow G/\{\phi(G) \cdot C(G)\}$. But $G \neq \phi(G) \cdot C(G)$, since otherwise, G would be abelian. Hence, $G/G^* \neq \{e\}$, and the result follows from the theorem.

REFERENCE

1. M. C. R. Butler and G. Horrocks, *Classes of extensions and resolutions*, Philos. Trans. Roy. Soc. London Ser. A *254* (1961), 155–222.

University of Guelph
Guelph, Ontario