ANZIAM J. 42(2001), 445–450

FAST DIFFUSION WITH LOSS AT INFINITY—ADDITIONAL SOLUTIONS

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(Received 9 May 1996; revised 26 June 2000)

Abstract

The paper presents some additional solutions of the diffusion equation

$$\frac{\partial \theta}{\partial t} = r^{1-s}\frac{\partial}{\partial r}\left(r^{s-1}\theta^m \frac{\partial \theta}{\partial r}\right)$$

for the case $s = 2$, $m = -1$, a case that was left open in a previous discussion. These solutions behave in a manner that is physically acceptable as the time, t, increases and as the radial coordinate, r, tends to infinity.

1. Introduction

This note may be regarded as an addendum to a paper by Philip [6] in which he discussed solutions of the nonlinear diffusion equation

$$\frac{\partial \theta}{\partial t} = r^{1-s}\frac{\partial}{\partial r}\left(r^{s-1}\theta^m \frac{\partial \theta}{\partial r}\right), \tag{1.1}$$

where θ, t and r denote respectively concentration, time and the radial space coordinate, with s as the number of dimensions and with the diffusion rate taken as θ^m. This equation occurs in numerous physical contexts, as far apart as percolation of liquid through soil and the transport of cosmic rays in interplanetary space [1, 4, 7]. In many applications m is positive but problems where m is negative also occur. For example, in discussing the expansion of a thermalised electron cloud Lonngren and Hirose [5] obtain a standardised equation which corresponds to (1.1) with $s = 1$ and $m = -1$. King [3] cites applications where $m = -1/2$, at the end of a paper in which he discusses similarity solutions of (1.1) and he includes solutions for a number of

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https://doi.org/10.1017/S1446181100012050 Published online by Cambridge University Press

cases where m is negative. Edwards and Broadbridge [2] examined solutions of the diffusion-conductivity equation, using Lie group symmetry analysis, and their paper includes some solutions which have $m < 0$, especially for $s = 2$ and $s = 3$.

In Philip's paper, he placed no restriction on m and examined cases where m is negative and there is loss of material at infinity, subject to suitable physical constraints on the behaviour of the solution, namely

(1) the total amount of the concentrate must be finite,

(2) for any fixed value of t,

$$\lim_{r\to\infty} r^{s-1}\theta^m \frac{\partial\theta}{\partial r} = -A, \tag{1.2}$$

where A is a finite real positive function of t,

(3)

$$\lim_{r\to\infty} \theta(r,t) = 0 \quad \text{for each relevant value of } t. \tag{1.3}$$

He considered similarity solutions of three different types and examined the restrictions on s and m that were imposed by the physical constraints. From this he deduced that physically acceptable solutions could exist for

(a) $0 < s < 2$, $-(2/s) < m \le -1$,

(b) $s > 2$, $-1 < m < -(2/s)$,

and noted that the border-line case $s = 2$, $m = -1$ needed further investigation. The purpose of the present paper is to show that a physically acceptable solution is possible when $s = 2$ and $m = -1$.

2. One form of solution for $s = 2$ and $m = -1$

In Section 6 of Philip's paper he considers a similarity solution of the form

$$\theta(r,t) = \Theta(\rho)[1-(t/T)]^{\alpha}, \qquad \rho = rT^{-1/2}[1-(t/T)]^{(\alpha-\beta)/s}, \tag{2.1}$$

with $0 < T < \infty$, $0 < \alpha < \infty$, $0 < \beta < \infty$. With this form of solution, $\theta(r, T) = 0$ and θ is taken as zero for $t > T$, leaving $0 \le t \le T$ as the relevant range of t for (2.1). To save frequent repetition of the restriction $0 \le t \le T$ we shall assume that the same convention applies in the subsequent discussion, that is, $\theta(r,t) = 0$ for $t > T$ and any statements about non-zero solutions are valid only for $0 \le t \le T$. For (2.1) Philip notes that similarity requires

$$\alpha - 1 = \alpha(1+m) + (2/s)(\alpha - \beta). \tag{2.2}$$

If $m = -1$ and $\alpha = 1$, we must have $\beta = \alpha = 1$.

In this case,

$$\theta(r,t) = \Theta(\rho)(T-t)/T, \qquad \rho = r/\sqrt{T}, \tag{2.3}$$

so ρ is simply a scaled version of r and $\theta(r,t)$ has the form of a separable solution. If we take $s = 2$ and substitute (2.3) in (1.1), we get

$$-\rho\Theta(\rho) = \frac{d}{d\rho}\left(\frac{\rho}{\Theta}\frac{d\Theta}{d\rho}\right), \tag{2.4}$$

which corresponds to (4.9) of Philip's paper. We want to solve this equation with the initial conditions that $\Theta = \Theta_o$ and $d\Theta/d\rho = 0$ for $\rho = 0$. To do this, we can introduce $\Phi(\rho) = \int_0^\rho u\Theta(u)\,du$; hence

$$\Phi'(\rho) = \rho\Theta(\rho), \quad \Phi''(\rho) = \Theta(\rho) + \rho\Theta'(\rho) \tag{2.5}$$

and

$$\Phi(0) = 0, \quad \Phi'(0) = 0, \quad \Phi''(0) = \Theta_0. \tag{2.6}$$

Integrating (2.4) and using the initial conditions gives, in turn,

$$-\Phi(\rho) = \rho\frac{\Theta'(\rho)}{\Theta(\rho)} = \frac{\rho\Phi''(\rho) - \Phi'(\rho)}{\Phi'(\rho)},$$

$$-\frac{1}{2}\{\Phi(\rho)\}^2 = \rho\Phi'(\rho) - 2\Phi(\rho),$$

$$\Phi(\rho) = 4K\rho^2/(1+K\rho^2),$$

where K is a positive constant. Hence

$$\Theta(\rho) = \frac{64\Theta_0}{(8+\rho^2\Theta_0)^2}, \qquad \theta(r,t) = \frac{64T\Theta_0(T-t)}{(8T+r^2\Theta_0)^2}. \tag{2.7}$$

The total quantity of the concentrate at time t is

$$q(t) = \int_0^\infty 2\pi r\theta(r,t)\,dr = 8\pi(T-t),$$

which is finite and it is easy to check that the constraints (1.2) and (1.3) are satisfied, with $A = 4$. For $t = 0$, we have $\theta(r,0) = \Theta_0/\{1 + (r^2\Theta_0/8T)\}^2$, which gives the right-hand half of a bell-shaped curve, with a maximum Θ_0 at $r = 0$ and with $\theta \to 0$ as $r \to \infty$. As t increases, the graph retains the same profile as it sinks toward zero. This is clearly a solution which is rather specialised but physically acceptable.

3. Other forms of solution for $s = 2$ and $m = -1$

In Section 2 the similarity solution took the form of a separable solution and this suggests that we look to see if other separable solutions are available. If we take $s = 2$ and $m = -1$ in (1.1) and introduce $\phi(r, t) = r\theta(r, t)$, then

$$\frac{\partial \phi}{\partial t} = r\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial r}\left(\frac{r}{\theta}\frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{r}{\phi}\frac{\partial \phi}{\partial r}\right), \tag{3.1}$$

and a separable solution $\phi(r, t) = F(t)G(r)$ requires that

$$F'(t) = \frac{1}{G(r)}\frac{d}{dr}\left\{\frac{rG'(r)}{G(r)}\right\} = \text{constant}. \tag{3.2}$$

We want the factor $F(t)$ to decrease with time so we can take the constant in (3.2) to be -2λ, with $\lambda > 0$. With $F(t) = 2\lambda(T - t)$, we can ensure that θ and ϕ are zero at $t = T$ and assume that they remain zero for $t > T$. Then we can follow the same convention as before and take $0 \le t \le T$ as the relevant range for t in solving (3.2). The equation for G is now

$$\frac{d}{dr}\left\{rG'(r)/G(r)\right\} = -2\lambda G(r). \tag{3.3}$$

We can obtain a formal solution if we assume that, as $r \to 0$, $G(r) \sim r^c$, with $c > 1$. Then if $H(r) = \int_0^r G(u)\,du$, we can expect to have

$$H(r) \sim \frac{r^{c+1}}{c+1}, \quad H'(r) = G(r) \sim r^c, \quad H''(r) \sim cr^{c-1} \quad \text{as } r \to 0.$$

Equation (3.3) becomes

$$-2\lambda H'(r) = \frac{d}{dr}\left\{\frac{rH''(r)}{H'(r)}\right\}. \tag{3.4}$$

Integrating this equation and using the behaviour as $r \to 0$ to evaluate the constants of integration we obtain in turn

$$\begin{aligned} -2\lambda H(r)H'(r) + (1+c)H'(r) &= \frac{d}{dr}\left\{rH'(r)\right\}, \\ -\lambda\left\{H(r)\right\}^2 + (1+c)H(r) &= rH'(r), \\ H(r) &= \left\{(1+c)Dr^{1+c}\right\}/\left\{1 + \lambda Dr^{1+c}\right\}, \end{aligned} \tag{3.5}$$

where D is a positive constant. This gives

$$G(r) = H'(r) = \left\{(1+c)^2 Dr^c\right\}/\left\{1 + \lambda Dr^{1+c}\right\}^2, \tag{3.6}$$

$$\theta(r, t) = (1/r)G(r)F(t) = \left\{2\lambda(T-t)(1+c)^2 Dr^{c-1}\right\}/\left\{1 + \lambda Dr^{1+c}\right\}^2. \tag{3.7}$$

With this form for $\theta(r, t)$ it can be checked that $q(t) = 4\pi(1+c)(T-t)$, so the total quantity of concentrate is finite. Also, as $r \to \infty$, conditions (1.2) and (1.3) are satisfied, with $A = c + 3$. Thus the constraints (1), (2) and (3) of Section 1 are satisfied. Note that the solution for $\theta(r, t)$ in (2.7) can be regarded as a limiting case of the solution given by (3.7), obtained by letting $c \to 1$ and taking $\lambda D = \Theta_0/(8T)$.

However, there is an additional requirement which reduces the number of acceptable solutions. For example, if we put $c = 2$ in (3.7), $\theta(r, t) \sim r$ as $r \to 0$ and $\partial\theta/\partial r$ approaches a non-zero constant, whereas we want radial symmetry with $\partial\theta/\partial r$ zero at $r = 0$. This difficulty can be removed if we take $c = 2n + 1$, with n a positive integer. With this restriction

$$\theta(r, t) = \left\{8\lambda D(T-t)(n+1)^2 r^{2n}\right\} / \left\{1 + \lambda D r^{2n+2}\right\}^2, \tag{3.8}$$

which makes θ an even function of r, with $\partial\theta/\partial r = 0$ at $r = 0$. A pointer in this direction is that if we write $\theta(r, t) = \psi(u, v)$, with $u = r^2$ and $v = 4t$, then the equation for ψ is

$$\frac{\partial\psi}{\partial v} = \frac{\partial}{\partial u}\left(\frac{u}{\psi}\frac{\partial\psi}{\partial u}\right), \tag{3.9}$$

which is of the same form as the equation for $\phi(r, t)$ (Equation (3.1)). So we can write

$$\begin{aligned}\theta(r, t) &= \psi(u, v) = \phi(r^2,\ 4t) = F(4t)G(r^2),\\ &= \left\{2\lambda D(T-4t)(1+c)^2 r^{2c}\right\} / \left\{1 + \lambda D r^{2+2c}\right\}^2,\end{aligned} \tag{3.10}$$

and this is essentially of the same form as (3.8). To indicate what these solutions look like we can use $n = 1$ in (3.8). Then

$$\theta(r, 0) = (32\lambda DT)r^2 / \left(1 + \lambda D r^4\right)^2 \tag{3.11}$$

and this function of r has a minimum value (zero) at $r = 0$, with a maximum at $r = r^* = (3\lambda D)^{-1/4}$. For $r > r*$, $\Theta(r, 0)$ decreases and approaches zero as $r \to \infty$. The behaviour for $n = 2, 3, \dots$ is similar, that is, $\theta(r, 0)$ has a minimum at $r = 0$ and a maximum for a single positive value of r. In the two-dimensional picture, the concentration is a maximum for a ring at distance r^* from the centre of symmetry. As t increases, $\theta(r, t) = (1 - t/T)\theta(r, 0)$ so the profile remains the same as θ declines toward zero.

4. Acknowledgements

This work was carried out while working as a Visiting Fellow in the Research School of Physical Sciences and Engineering, Australian National University, and I am grateful to the Department of Theoretical Physics for the facilities it has provided. I should like to acknowledge also a helpful discussion of this work with the late Dr J. R. Philip.

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