

ON THE DOMAIN AND RANGE OF THE MAXIMAL OPERATOR

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Abstract. We give a detailed survey, known and new results on the domain and the range of the maximal operator. In particular we employ the grand L^p spaces and logarithmic Lebesgue spaces.

§1. Introduction

We shall systematically treat the problem of the domain and of the range of the maximal operator. Considering this problem has been motivated by questions how strong are assumptions about the maximal function, varying from the finiteness of the maximal function at a single point to L_r integrability, where $0 < r < 1$, and questions about the range of the maximal operator under minimal necessary hypothesis about the space on which it acts, employing spaces obtained by various extrapolation procedures, in particular, L^p spaces (Iwaniec and Sbordone [IS]) and the logarithmic Lebesgue spaces (Edmunds and Triebel [ET]).

We shall use standard notation L^p for the Lebesgue spaces; we shall denote by $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}^N$ and by $B_R(x)$ the open ball centered at x with radius R .

For $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, $N \geq 1$ we consider the (*global, centered*) *maximal function*

$$Mf(x) = \sup_{R>0} \frac{1}{|B_R(x)|} \int_{B_R(x)} |f(y)| dy, \quad x \in \mathbb{R}^N,$$

and we denote

$$\mathbb{D} = \{f \in L^1_{\text{loc}}(\mathbb{R}^N) ; Mf \neq \infty\}.$$

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In view of Theorem 2.2 and the sublinearity of the maximal operator, \mathbb{D} is a linear subspace of $L^1_{\text{loc}}(\mathbb{R}^N)$. We shall refer to it as the *domain* of the maximal operator.

Let us note that not every function in $L^1_{\text{loc}}(\mathbb{R}^N)$ is in \mathbb{D} : it suffices to consider, for instance, the function norm in \mathbb{R}^N .

It is well known that using, more generally, balls containing x or cubes (centered or off-centered with respect to x) with sides parallel to coordinate axes, or families of regular sets instead of balls in the definition of the maximal function yields an equivalent operator ([DS]).

For simplicity and without loss of generality we shall work only with non-negative functions so that we shall omit the absolute values everywhere.

In Section 2 we study the domain of the maximal operator, proving some necessary and sufficient conditions; after recalling that $L^1(\mathbb{R}^N) \subset \mathbb{D}$, examples are given showing that $L^1_{\text{loc}}(\mathbb{R}^N) \cap \bigcup_{0 < r < 1} L^r(\mathbb{R}^N) \not\subset \mathbb{D} \not\subset L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. In Section 3 we shall recall some known properties of Mf when $f \in \mathbb{D}$, and we shall fix our attention to the properties of Mf when $f \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. Among the consequences of our results, we get for instance that for every $0 < r, s < 1$, $\alpha > 0$, we have

$$f \in L^r(\mathbb{R}^N) \cap L^1_{\text{loc}}(\mathbb{R}^N) \text{ and } Mf \in L^s(\{Mf > \alpha\}) \implies f \in L^1(\mathbb{R}^N).$$

In Section 4 we study the local maximal operator $M_\Omega f$ on open bounded sets Ω in \mathbb{R}^N . This operator is a powerful tool in many areas of Analysis, for instance, in the theory of quasi-conformal mappings ([BI]), when proving the regularity of solutions of P.D.E. or minimizers of functionals of the Calculus of Variations (for instance, see [FS1], [FS2]), or, recently, when dealing with homogenization problems without regularity in the coefficients [CP]. We shall put in evidence the different properties with respect to the maximal operator in \mathbb{R}^N ; we shall prove a characterization of the domain \mathbb{D}_Ω of the local maximal operator and we shall see some examples, for instance, functions in $\mathbb{D}_\Omega \setminus L^1_{\text{loc}}(\Omega)$. In our treatment the grand L^p spaces (introduced in [IS]) and logarithmic Lebesgue spaces (see [ET]) will be involved.

§2. The domain of the maximal operator

In 1939 Wiener [Wie] found a subspace of $L^1_{\text{loc}}(\mathbb{R}^N)$ contained in \mathbb{D} ; namely, he proved the following

THEOREM 2.1. *If $f \in L^1(\mathbb{R}^N)$, then $Mf < \infty$ a.e. in \mathbb{R}^N .*

Plainly $L^\infty(\mathbb{R}^N) \subset \mathbb{D}$ so that by the sublinearity of the maximal operator $L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \subset \mathbb{D}$. Therefore the Lebesgue $L^p(\mathbb{R}^N)$ ($1 \leq p \leq \infty$), Orlicz $L_A(\mathbb{R}^N)$ (A Young function) and Lorentz spaces are subsets of \mathbb{D} . We shall show that \mathbb{D} is effectively larger than $L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, but it does not contain any space of the type $L^1_{loc}(\mathbb{R}^N) \cap (L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N))$ where $0 < r < 1$. We begin with the following characterization of \mathbb{D} .

THEOREM 2.2. *Let $f \in L^1_{loc}(\mathbb{R}^N)$. Then the following statements are equivalent:*

- (i) *there exists $x_0 \in \mathbb{R}^N$ such that $Mf(x_0) < \infty$;*
- (ii) *there exists $x_0 \in \mathbb{R}^N$ such that*

$$\limsup_{R \rightarrow \infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y) dy < \infty;$$

- (iii) *there exists $K > 0$ such that*

$$\limsup_{R \rightarrow \infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y) dy = K < \infty$$

for every $x_0 \in \mathbb{R}^N$;

- (iv) *$Mf(x) < \infty$ a.e. in \mathbb{R}^N ;*
- (v) *$f \in \mathbb{D}$.*

Proof. The implications (iv) \Rightarrow (v) \Rightarrow (i) are obvious. Also, if (i) holds, then the same x_0 satisfies (ii). We shall prove (ii) \Rightarrow (iii) \Rightarrow (iv).

Let (ii) hold and choose a sequence $\{\varrho_n\}_{n \geq 1}$ so that

$$\lim_{n \rightarrow \infty} \frac{1}{|B_{\varrho_n}(x_0)|} \int_{B_{\varrho_n}(x_0)} f(y) dy = \limsup_{R \rightarrow \infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y) dy$$

and let \bar{x} be any point in $\mathbb{R}^N \setminus \{x_0\}$. Since

$$B_{\varrho_n + |\bar{x} - x_0|}(\bar{x}) \supset B_{\varrho_n}(x_0)$$

we have (ω_N denotes the measure of the unit ball in \mathbb{R}^N)

$$\frac{1}{|B_{\varrho_n + |\bar{x} - x_0|}(\bar{x})|} \int_{B_{\varrho_n + |\bar{x} - x_0|}(\bar{x})} f(y) dy$$

$$\begin{aligned}
 &= \frac{1}{\omega_N(\varrho_n + |\bar{x} - x_0|)^N} \int_{B_{\varrho_n + |\bar{x} - x_0|}(\bar{x})} f(y) \, dy \\
 &\geq \frac{1}{\omega_N(\varrho_n + |\bar{x} - x_0|)^N} \int_{B_{\varrho_n}(x_0)} f(y) \, dy \\
 &= \left(\frac{\varrho_n}{\varrho_n + |\bar{x} - x_0|} \right)^N \frac{1}{\omega_N \varrho_n^N} \int_{B_{\varrho_n}(x_0)} f(y) \, dy.
 \end{aligned}$$

This yields

$$(2.1) \quad \limsup_{R \rightarrow \infty} \frac{1}{|B_R(\bar{x})|} \int_{B_R(\bar{x})} f(y) \, dy \geq \limsup_{R \rightarrow \infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y) \, dy.$$

Replacing \bar{x} by x_0 and vice versa in the above argument gives equality in (2.1), which proves (iii).

Let (iii) hold. Denote $A_n = \{n-1 < |x| < n\}$, $n \in \mathbb{N}$. Then $f_n = f \chi_{A_n} \in L^1$. By virtue of Theorem 2.1, Mf_n is finite a.e. in \mathbb{R}^N . Let $E_n \subset A_n$ be a set of zero measure such that $Mf_n(x) < \infty$ for all $x \in A_n \setminus E_n$ and denote $E = \bigcup_{i=1}^\infty E_i \cup (\bigcup_{n \geq 0} \{|x| = n\})$. We claim that $Mf(x) < \infty$ for all $x \in \mathbb{R}^N \setminus E$. Let $x_0 \in \mathbb{R}^N \setminus E$, then $x_0 \in A_n \setminus E_n$ for some n . Let r_0 satisfy $B_{r_0}(x_0) \subset A_n$ and fix R_0 such that $\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y) \, dy < 2K$ for all $R \geq R_0$. If $R \leq r_0$, then $\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y) \, dy \leq Mf_n(x_0) < \infty$; further, if $r_0 < R < R_0$, then $\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y) \, dy \leq \frac{1}{|B_{r_0}(x_0)|} \int_{B_{r_0}(x_0)} f(y) \, dy$, and, finally, if $R \geq R_0$, then $\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f(y) \, dy < 2K$. Hence $Mf(x_0) < \infty$ and (iv) is true. \square

Remark 2.3. Let Ω be an open set in \mathbb{R}^N . The John-Nirenberg space $\text{BMO}(\Omega)$ is defined as the space of the measurable functions f such that

$$\|f\|_{\text{BMO}(\Omega)} = \sup_{Q \subset \Omega} \frac{1}{|Q|} \int_Q \left| f(x) - \frac{1}{|Q|} \int_Q f(y) \, dy \right| dx < +\infty$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes.

We remark that the implication (i) \Rightarrow (iv) from the previous theorem has been shown to hold for $f \in \text{BMO}(\mathbb{R}^N) \cap \mathbb{D}$ in the book by Bennett-Sharpely (cf. [BS, pp. 399–400]), by using the inequality

$$(2.2) \quad \frac{1}{|Q|} \int_Q Mf(x) \, dx \leq c \|f\|_{\text{BMO}(\mathbb{R}^N)} + \inf_{x \in Q} Mf(x),$$

true for every Q cube in \mathbb{R}^N , $f \in \text{BMO}(\mathbb{R}^N)$. Our proof here states this interesting property generally for $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and it is much simpler than that using (2.2). Notice that $\mathbb{D} \not\subset \text{BMO}(\mathbb{R}^N)$ (see next Example 2.8). Also a rather sophisticated proof of (i) \Rightarrow (iv) for $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ was given by Wik [Wik].

Finally, we remark that it is also $\text{BMO}(\mathbb{R}^N) \not\subset \mathbb{D}$; it suffices to consider $f = |\log |x||$ (cf. [BS, p. 400]).

We present two examples showing that if $0 < r < 1$, then we have $L^1_{\text{loc}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \not\subset \mathbb{D} \not\subset L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. Notice that the functions f we are going to consider are supported on sets of finite measure, therefore, every level set $\{f(x) > \alpha\}$, $\alpha \geq 0$, is of finite measure.

EXAMPLE 2.4. Let $A_n = \{n - 1 < |x| < n\}$, $n \in \mathbb{N}$, and let F_n be any measurable subset of A_n such that $|F_n| = 2^{-n}$. Put $f = \sum_{n=1}^\infty a_n \chi_{F_n}$ with $a_n = 2^n$. Then $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, $f \notin L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. At the same time $f \in \mathbb{D}$, in fact,

$$\frac{1}{|B_\varrho(0)|} \int_{B_\varrho(0)} f(y) dy \leq \frac{1}{\omega_N \varrho^N} \sum_{n=1}^{[\varrho+1]} a_n |F_n|, \quad \varrho > 0,$$

where $[\cdot]$ denotes here the integer part and therefore we have

$$\limsup_{\varrho \rightarrow \infty} \frac{1}{|B_\varrho(0)|} \int_{B_\varrho(0)} f(y) dy < \infty,$$

from which $f \in \mathbb{D}$ by Theorem 2.2. Notice that since $\int_{\mathbb{R}^N} (f(y))^r dy = \sum_{n=1}^\infty a_n^r |F_n| < \infty$, we have $f \in L^r(\mathbb{R}^N)$ for all $0 < r < 1$.

EXAMPLE 2.5. Let us put $a_n = (2^{(r+1)/2r})^n$ with some fixed $0 < r < 1$ in the previous example. Then $f \in L^1_{\text{loc}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, $f \notin \mathbb{D}$. Note that $f \notin L^1(\mathbb{R}^N)$.

Remark 2.6. Let Ω be an open set in \mathbb{R}^N . The space $\text{BLO}(\Omega)$ is defined as the subspace of $\text{BMO}(\Omega)$ of the functions f such that

$$\sup_{Q \subset \Omega} \left(\frac{1}{|Q|} \int_Q f(x) dx - \text{ess inf}_Q f \right) < +\infty.$$

It is possible to prove that (cf. [BS, (8.38) p. 400]; see also [T, p. 204])

$$M : \mathbb{D} \cap \text{BMO}(\mathbb{R}^N) \longrightarrow \text{BLO}(\mathbb{R}^N) \subset \text{BMO}(\mathbb{R}^N),$$

and that such operator is surjective, i.e.,

$$(2.3) \quad \text{BLO}(\mathbb{R}^N) = M(\mathbb{D} \cap \text{BMO}(\mathbb{R}^N)) + L^\infty(\mathbb{R}^N)$$

Remark 2.7. The Sobolev spaces are in \mathbb{D} , but also functions in $L^1_{\text{loc}}(\mathbb{R}^N)$ with weak derivatives in $L^p(\mathbb{R}^N)$, $1 \leq p < N$ are in \mathbb{D} since they, up to a constant, belong to some Lebesgue or Orlicz space (cf. [Sc, p. 40], or [Ma, p. 424]).

EXAMPLE 2.8. Functions in \mathbb{D} can be very bad indeed, for instance, the measure of every level set can be infinite, hence these functions cannot be rearranged (for general reference about rearrangements see for instance [B], [Mo], [K]). Let us have a look at the following one-dimensional example: Put

$$f(t) = \sum_{n=1}^{\infty} n\chi_{(n^3, n^3+1)}(t), \quad t \in \mathbb{R}^1.$$

Clearly $f \in L^1_{\text{loc}}(\mathbb{R}^1)$ and

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow \infty} \int_{-t}^t f(y) dy \leq \lim_{n \rightarrow \infty} \frac{1}{2n^3} \int_{-n^3-1}^{n^3+1} f(y) dy \\ &= \lim_{n \rightarrow \infty} \frac{n(n+1)}{4n^3} = 0, \end{aligned}$$

therefore, $f \in \mathbb{D}$ by Theorem 2.2. Notice that the measure of every level set of f is infinite, and, since $f \leq Mf$, also the measure of every level set of Mf is infinite. Finally, notice that $f \notin \text{BMO}(\mathbb{R}^1)$.

§3. The range of the maximal operator

We shall first survey known properties of the maximal operator. Let $f \in \mathbb{D}$. Then Mf is measurable lower semicontinuous function. By the Lebesgue Differentiation Theorem, $f \leq Mf$, and if $N > 2$ an equality can hold even without f being a constant (this result is true for the *spherical* maximal operator; see [F1]). In terms of membership in various function spaces we recall that if $f \not\equiv 0$ a.e. in \mathbb{R}^N then $Mf \notin L^r(\mathbb{R}^N) \forall 0 < r \leq 1$ and $\log(1 + Mf) \notin L^1(\mathbb{R}^N)$ (since there is $c > 0$ such that $Mf(x) \geq c/|x|^N$ for all $|x| > 1$, cf., e.g., [BS, p. 117]). The behaviour of M in Lebesgue spaces has been studied thoroughly: M acts continuously from $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ if $1 < p \leq \infty$ and from $L^1(\mathbb{R}^N)$ into the weak- $L^1(\mathbb{R}^N)$. Passing to

finer scales of spaces, it is possible to find criteria for the boundedness of M in Lorentz and Orlicz spaces, in particular then in Lorentz-Zygmund spaces. A curious phenomenon occurs, namely, a deterioration of the behaviour of M near L^1 , expressed in the classical form of the Zygmund inequality, a feature common to other classical operators. This is now well understood also in a more abstract setting as a general property of operators of weak type $(1, 1)$ or its generalizations. From numerous references let us recall at least [Gu], [GCRF], [T], [BR], [BS], [KK].

Now let us begin by studying the properties of Mf . Starting only from the assumption $f \in \mathbb{D}$ it may happen that $Mf \notin L^1_{loc}(\mathbb{R}^N)$; it suffices to consider the following well-known

EXAMPLE 3.1. Let us put $f(x) = \chi_{]0, 1/2[}(x)/(x \log^2 x)$, $x \in \mathbb{R}^1$ a.e. (cf. [BS, p. 118], or [T, p. 77]). Then $f \in \mathbb{D}$, $Mf \notin L^1_{loc}(\mathbb{R}^N)$. We observe that $Mf \in L^1(\log L)^{-1}(]0, 1/2[)$.

Example 3.1 shows that also starting from $f \in L^1(\mathbb{R}^N)$ generally we have not $Mf \in L^1_{loc}(\mathbb{R}^N)$. As we are going to see, in this case it is possible to find optimal spaces to which Mf belongs.

First of all, by virtue of the Kolmogorov inequality (cf., e.g., [Sa, Thm. 6.1, p. 190], or [S3, p. 43]),

$$(3.1) \quad \|Mf\|_{L^r(A)}^r \leq \frac{c(N)|A|^{1-r}}{1-r} \|f\|_{L^1(\mathbb{R}^N)}^r \quad \forall f \in L^1(\mathbb{R}^N), r \in]0, 1[,$$

true for every set $A \subset \mathbb{R}^N$, $|A| < \infty$, the function Mf is in every $L^r(A)$ with $0 < r < 1$ provided $f \in L^1(\mathbb{R}^N)$, where A is any set of \mathbb{R}^N with finite measure. Furthermore, Mf belongs to any space on A resulting from an extrapolation procedure based on (3.1), which preserves a finite (quasi-)norm on the right hand side. The extrapolation theory as developed up to now offers two reasonable candidates. The first is given by using an analogue of the approach considered for $p > 1$ in [ET],

$$(3.2) \quad \int_0^{\varepsilon_0} \varepsilon^{\sigma-1} \|Mf\|_{L^{1-\varepsilon}(A)} d\varepsilon \leq c(N, |A|, \varepsilon_0, \sigma) \|f\|_{L^1(\mathbb{R}^N)},$$

where $\varepsilon_0 \in (0, 1)$ is chosen arbitrarily, $\sigma > 1$ is a parameter, or by using its discrete variant, the Σ -method due to Milman [Mi]. Going along the lines of the proof in [ET], where spaces $L^p(\log L)^{-\sigma}(A)$ with $1 < p < \infty$ are considered, it is not difficult to prove that the quasinorm on the left hand

side of (3.2) is equivalent to the quasinorm in the (generalized) Orlicz space $L^1(\log L)^{-\sigma}(A)$. Hence we have the following

PROPOSITION 3.2. *If $f \in L^1(\mathbb{R}^N)$, then $Mf \in \bigcap_{\sigma>1} L^1(\log L)^{-\sigma}(A)$ for every $A \subset \mathbb{R}^N$ of finite measure.*

This assertion is optimal in the scale of logarithmic Lebesgue spaces, as we are going to see from the following

EXAMPLE 3.3. Let us consider the function

$$f(x) = \frac{1}{x |\log x| \log^2 |\log x|} \chi_{]0,a[}(x), \quad x \in \mathbb{R}^1 \setminus \{0, 1, e\},$$

where $a = \exp(-\exp(1))$. It is $f \in L^1(\mathbb{R}^N)$ because

$$\int_0^a f(x) dx = \left[\frac{1}{\log |\log x|} \right]_{x \rightarrow 0}^{x=a} = \frac{1}{\log |\log a|}.$$

On the other hand we have

$$Mf\left(\frac{x}{2}\right) \geq \frac{1}{x} \int_0^x f(t) dt = \frac{1}{x \log |\log x|}, \quad x \in]0, \frac{a}{2}[,$$

and therefore, since $\log |\log x| > 1 \forall x \in]0, a/2[$, we get

$$\begin{aligned} \frac{Mf\left(\frac{x}{2}\right)}{\log\left(e + Mf\left(\frac{x}{2}\right)\right)} &\geq \frac{\frac{1}{x \log |\log x|}}{\log\left(e + \frac{1}{x \log |\log x|}\right)} \geq \frac{1}{x \log |\log x| \log\left(e + \frac{1}{x}\right)} \\ &\geq \frac{1}{x \log |\log x| \log \frac{2}{x}} \geq \frac{1}{2x |\log x| \log |\log x|}, \end{aligned}$$

from which $Mf \notin L^1(\log L)^{-1}(]0, a[)$.

The second approach follows from another estimate consequence of (3.1), namely, a bound for the quasinorm of Mf in $L^1(A)$, the grand L^1 space, which is a particular case of spaces introduced by Iwaniec and Sbordone [IS] and investigated in details, e.g., in Greco [Gr]:

$$\|Mf\|_{L^1(A)} \leq c(N, |A|) \|f\|_{L^1(\mathbb{R}^N)}$$

Let us recall that the quasinorm in $L^1(A)$ is given by

$$\|f\|_{L^1(A)} = \sup_{0 < \varepsilon < \varepsilon_0} \left(\varepsilon \frac{1}{|A|} \int_A |f(y)|^{1-\varepsilon} dy \right)^{1/(1-\varepsilon)},$$

where $\varepsilon_0 \in (0, 1)$; observe that only values of ε near zero are relevant. Hence we have the following

PROPOSITION 3.4. *If $f \in L^1(\mathbb{R}^N)$, then $Mf \in L^1(A)$ for every $A \subset \mathbb{R}^N$ of finite measure.*

Let us remark that this approach is better in terms of inclusions of functions spaces, because it is

$$L^1(\log L)^{-1}(A) \subset L^1(A) \subset \bigcap_{\sigma > 1} L^1(\log L)^{-\sigma}(A)$$

for every A of finite measure.

In Example 2.8 we showed that if $f \in \mathbb{D}$, then the measure of every level set of Mf can be infinite, considering a function f having the same property. The next theorem shows among others that such phenomenon may occur even if the measure of every level set of f is finite (namely, when $f \notin L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, as in the Example 2.4.

THEOREM 3.5. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\varphi : [0, \infty[\rightarrow [0, \infty[$, φ strictly increasing, $\varphi(\infty) = \infty$, $\lim_{t \rightarrow \infty} \varphi(t)/t^s = 0$ for some $0 < s < 1$. Then the following statements are equivalent:*

- (i) $f \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;
- (ii) there is $\alpha > 0$ such that $f \in L^1(\{f > \alpha\})$;
- (iii) there is $\alpha > 0$ such that $|\{Mf > \alpha\}| < \infty$;
- (iv) there is $\alpha > 0$ and $0 < r < 1$ such that $Mf \in L^r(\{Mf > \alpha\})$;
- (v) there is $\alpha > 0$ such that $Mf \in L^r(\{Mf > \alpha\})$ for all $0 < r < 1$;
- (vi) there is $0 < r < 1$ such that $Mf \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;
- (vii) $Mf \in \bigcap_{0 < r < 1} L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;
- (viii) $\varphi(Mf) \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;

(ix) there is $\alpha > 0$ such that $|\{Mf > \alpha\}| < \infty$ and

$$Mf \in L^1(\{Mf > \alpha\}) + L^\infty(\mathbb{R}^N);$$

(x) there is $\alpha > 0$ such that $|\{Mf > \alpha\}| < \infty$ and

$$Mf \in \bigcap_{\sigma > 1} L^1(\log L)^{-\sigma}(\{Mf > \alpha\}) + L^\infty(\mathbb{R}^N).$$

Proof. We shall prove (i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i) and (iii) \Rightarrow (v) \Rightarrow (viii) \Rightarrow (iii) and (v) \Rightarrow (vii) \Rightarrow (vi) \Rightarrow (iv) \Rightarrow (iii) and (iii) \Rightarrow (ix) \Rightarrow (x) \Rightarrow (vi).

Step 1, (i) \Rightarrow (iii): If $f \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, then there is $\alpha > 0$ such that $f = f\chi_{\{f > \alpha\}} + f\chi_{\{f \leq \alpha\}} \equiv f_1 + f_2$ with $f_1 \in L^1(\mathbb{R}^N)$. We have $Mf \leq Mf_1 + Mf_2 \leq Mf_1 + \alpha$ and thus $\{Mf > \beta\} \subset \{Mf_1 + \alpha > \beta\} = \{Mf_1 > \beta - \alpha\}$ for all $\beta \geq \alpha$.

Step 2, (iii) \Rightarrow (ii): This is an immediate consequence of the reverse inequality of weak type (1, 1) of M , namely, there exist constants $c_1, c_2 > 0$ depending on N such that (see [S3, p. 43], for instance):

$$|\{x \in \mathbb{R}^N : Mf(x) > c_1\alpha\}| \geq \frac{c_2}{\alpha} \int_{|f| > \alpha} |f| dx, \quad \forall \alpha > 0.$$

Step 3, (ii) \Rightarrow (i): It suffices to write $f = f\chi_{\{f > \alpha\}} + f\chi_{\{f \leq \alpha\}}$.

Step 4, (iii) \Rightarrow (v): Fix $0 < r < 1$. By Step 2 there exists $\alpha > 0$ such that $f\chi_{\{f > \alpha\}} \in L^1(\mathbb{R}^N)$, therefore

$$\begin{aligned} \int_{Mf > \alpha} (Mf)^r dx &\leq \int_{Mf > \alpha} [M(f\chi_{\{f > \alpha\}})]^r dx \\ &\quad + \int_{Mf > \alpha} [M(f\chi_{\{f \leq \alpha\}})]^r dx < \infty. \end{aligned}$$

The first term on the right hand side is finite in view of the Kolmogorov inequality since $\{Mf > \alpha\}$ has a finite measure.

Step 5, (v) \Rightarrow (viii): Let t_s ($0 < s < 1$) be such that $\varphi(t) \leq t^s, t \geq t_s$ and put $K = \max(t_s, \alpha)$. Then

$$Mf = (Mf)\chi_{\{Mf > K\}} + (Mf)\chi_{\{Mf \leq K\}}.$$

Step 6, (viii) \Rightarrow (iii): Let $\alpha > 0$ be such that $\varphi(Mf) \in L^1(\{Mf > \alpha\})$. If $|\{Mf > \alpha\}| = \infty$, then

$$\int_{\{Mf > \alpha\}} \varphi(Mf) dx > \varphi(\alpha)|\{Mf > \alpha\}| = \infty,$$

which is a contradiction.

Step 7, (v) \Rightarrow (vii): It suffices to write $Mf = (Mf)(\chi_{\{Mf>\alpha\}}) + (Mf)(\chi_{\{Mf\leq\alpha\}})$.

Step 8, (vii) \Rightarrow (vi): This is obvious.

Step 9, (vi) \Rightarrow (iv): It suffices to apply (i) \Rightarrow (ii) to $(Mf)^r$.

Step 10, (iv) \Rightarrow (iii): It suffices to argue as in Step 6, replacing $\varphi(t)$ by t^r .

Step 11, (iii) \Rightarrow (ix): Arguing as in Step 4 and using the quantitative form of the Kolmogorov inequality we get, for $0 < r < 1$,

$$\begin{aligned} \int_{Mf>\alpha} (Mf)^r dx &\leq \int_{Mf>\alpha} [M(f\chi_{\{f>\alpha\}})]^r dx + \int_{Mf>\alpha} [M(f\chi_{\{f\leq\alpha\}})]^r dx \\ &\leq \frac{c}{1-r} |\{Mf > \alpha\}|^{1-r} \|f\chi_{\{f>\alpha\}}\|_{L^1(\{Mf>\alpha\})}^r + \alpha^r |\{Mf > \alpha\}|, \end{aligned}$$

for some constant c independent of r . Hence

$$\begin{aligned} \sup_{0 < r < 1-\varepsilon_0} \left(\frac{1-r}{|\{Mf > \alpha\}|} \int_{Mf>\alpha} (Mf)^r dx \right)^{1/r} \\ \leq c_1 \left(\frac{\|f\chi_{\{f>\alpha\}}\|_{L^1(\{Mf>\alpha\})}}{|\{Mf > \alpha\}|} + 1 \right) \end{aligned}$$

for some c_1 independent of r , which yields $Mf \in L^1(\{Mf > \alpha\})$. Hence $Mf \in L^1(\{Mf > \alpha\}) + L^\infty(\mathbb{R}^N)$.

Step 12, (ix) \Rightarrow (x): This follows by the known inclusion properties of L^1 (see [Gr]).

Step 13, (x) \Rightarrow (vi): This follows by the known inclusion theorems of Orlicz spaces. □

Remark 3.6. The condition (ii) in Theorem 3.5 says that f is integrable over a special set of a finite measure. Examples 2.4 and 2.5 show that this cannot be replaced by integrability of f over any set of finite measure. Furthermore, the condition (iii) in Theorem 3.5 implies that all level sets of the maximal functions in the examples recalled are of infinite measure.

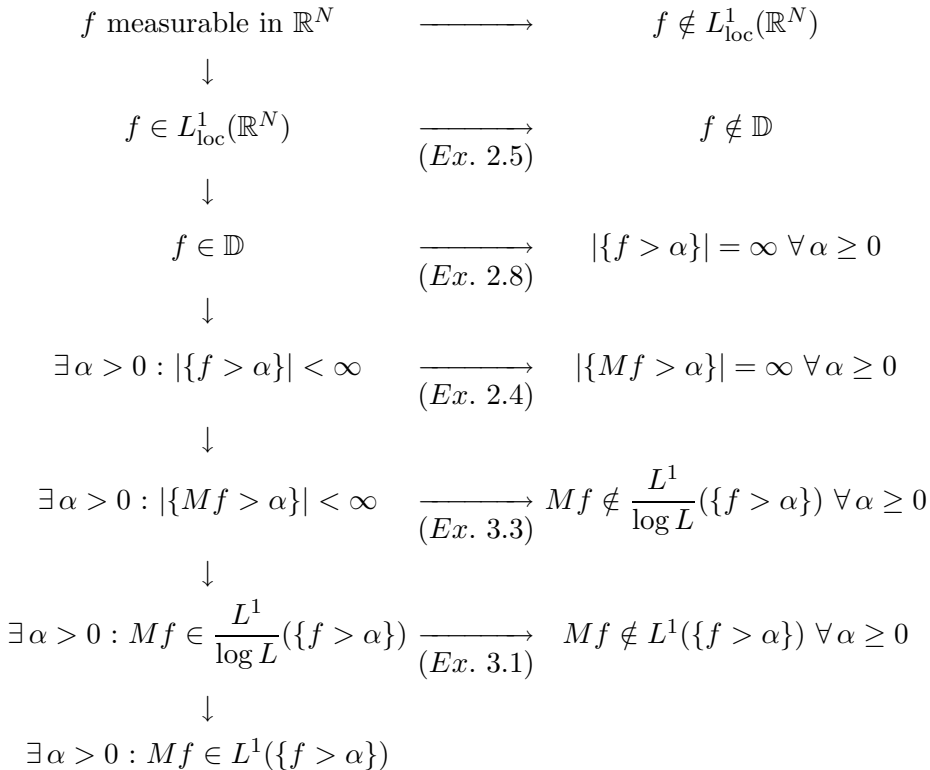
Remark 3.7. Notice that assuming $f \in L^{1-\varepsilon}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)$ for some $\varepsilon \in (0, 1)$ and $Mf \in L^r(\{Mf > \alpha\})$ for some $0 < r < 1$ and $\alpha > 0$, then in view of Theorem 3.5 we have $f \in L^1(\mathbb{R}^N)$.

Next theorem is a well-known variant of Stein’s $L \log L$ theorem ([S1], [S2]) for the maximal function. We state it here for completeness.

THEOREM 3.8. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\sigma \geq 1$. Then the following statements are equivalent:*

- (i) $f[\log(1 + f)]^\sigma \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;
- (ii) *there is $\alpha > 0$ such that $f[\log(1 + f)]^\sigma \in L^1(\{f > \alpha\})$;*
- (iii) *there is $\alpha > 0$ such that $Mf \in L^1(\log L)^{\sigma-1}(\{f > \alpha\})$;*
- (iv) $Mf \in L^1(\log L)^{\sigma-1}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$.

What follows is a scheme of some of the sets of functions we have considered up to now, ordered by the strength of the conditions involved:



§4. The local maximal function on open bounded sets

The symbol Ω will now stand for an open bounded subset of \mathbb{R}^N , functions in Ω will be assumed to be measurable and non-negative.

The local maximal function of f is defined by

$$M_\Omega f(x) = \sup_{\substack{Q \ni x \\ Q \subset \Omega \\ Q \text{ cube}}} \frac{1}{|Q|} \int_Q f(y) dy, \quad x \in \Omega,$$

where edges of cubes Q are parallel with coordinate axes. If $\Omega \neq \mathbb{R}^N$, then M_Ω preserves only some of the properties of M as we are going to see.

We shall first consider the case $\Omega = Q_0$, cube in \mathbb{R}^N . Putting

$$\bar{f} = \begin{cases} f & \text{in } Q_0, \\ 0 & \text{in } \mathbb{R}^N \setminus Q_0, \end{cases}$$

it is easy to prove that

$$M_{Q_0} f = (M_{\mathbb{R}^N} \bar{f})|_{Q_0}.$$

As a consequence of Theorem 2.2, we have

PROPOSITION 4.1. *It is $M_{Q_0} f < \infty$ a.e. in Q_0 if and only if $f \in L^1(Q_0)$.*

After the considerations of Section 3, we have some information about the range of M_{Q_0} : if $f \in L^1(Q_0)$, then we have $Mf \in L^1(Q_0)$ and also that $Mf \in \bigcap_{\sigma > 1} L^1(\log L)^{-\sigma}(Q_0)$; in particular, $Mf \in \bigcap_{0 < r < 1} L^r(Q_0)$. Also, Example 3.1 shows that Mf need not be locally integrable in Q_0 .

Let us recall some known results about the range of M_{Q_0} when f is “better” than $L^1(Q_0)$: by the Stein’s $L \log L$ result [S1], $f \in L \log L(Q_0)$ iff $M_{Q_0} f \in L^1(Q_0)$ and f belongs to the Orlicz space $L_A(Q_0)$, where $\inf_{t > 0} tA'(t)/A(t) > 1$, iff M_{Q_0} belongs to the same space. For other imbedding properties of the maximal operator see [T], [LZ], [GIM].

We shall turn our attention to the local maximal function with respect to a general domain Ω . In this case M_Ω is different from $(M_{\mathbb{R}^N} \bar{f})|_\Omega$. In general $M_\Omega f \leq (M_{\mathbb{R}^N} \bar{f})|_\Omega$ and these functions need not be equivalent. We shall give an example of this curious phenomenon.

EXAMPLE 4.2. Let $N = 2$,

$$\Omega = \{z = (x, y) ; |z - 1| < 1\},$$

and let $f = 1$ in $\Omega \cap \{y > 2/\sqrt{5}\}$ and 0 otherwise in Ω . Then $(M_\Omega f)|_{\Omega \cap \{y \leq 0\}} = 0$ while $(M_{\mathbb{R}^N} \bar{f})|_{\Omega \cap \{y \leq 0\}} > 0$.

Hence it turns out that the assumption $\bar{f} \in \mathbb{D}$ is a sufficient condition to get $f \in \mathbb{D}_\Omega$. Next example shows that such assumption is too strong when looking for the functions in \mathbb{D}_Ω . A necessary and sufficient condition for a function f to be in \mathbb{D}_Ω will be given in Theorem 4.5.

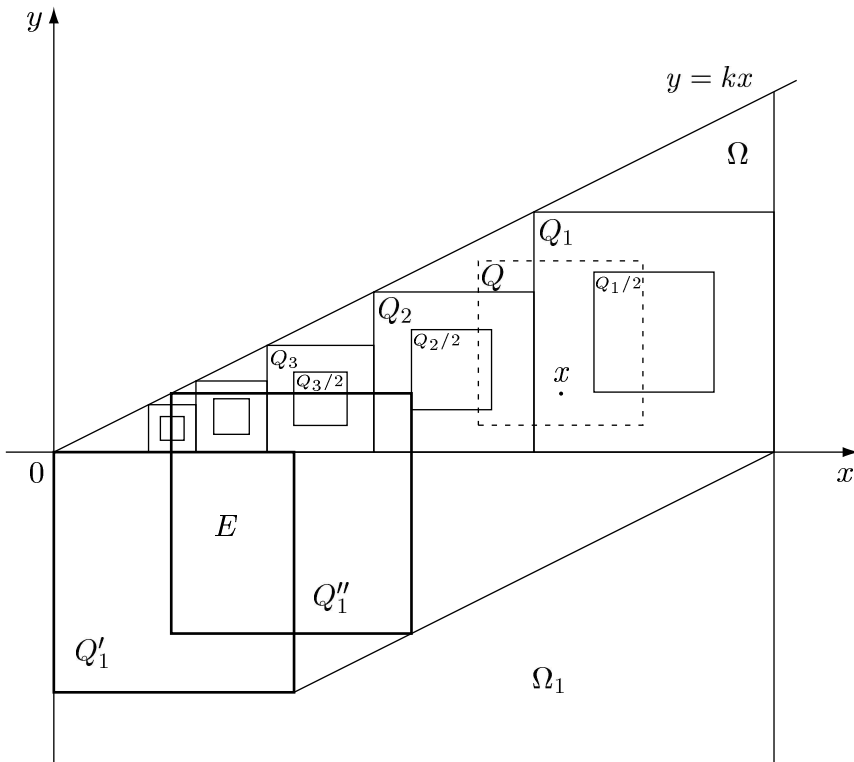


Figure 1.

EXAMPLE 4.3. Let us consider the situation illustrated by the upper part of Fig. 1, that is, let $N = 2$ and consider a sequence of open cubes Q_1, Q_2, \dots such that their bottom sides lay on the positive axis x , the right vertical side of Q_{k+1} is subset of the left vertical side of Q_k and the left upper corners of all Q_k lay on a line $y = kx$ with some $k > 0$. Let Ω be the triangle domain whose boundary is contained in the positive axis x , the line $y = kx$, and the line containing the right vertical side of Q_1 . Denote by $Q_1/2, Q_2/2, \dots$, concentric cubes with sidelength equal to the half of the sides of Q_1, Q_2, \dots . Let (a_i) , be any sequence of positive real numbers such

that

$$\sum_{i=1}^{\infty} a_i |Q_i| = \infty$$

and put

$$f = \sum_{i=1}^{\infty} a_i \chi_{Q_i/2}.$$

Then f is supported in a compact set and $f \notin L^1(\Omega)$. If we fix $x \in \Omega$, then every cube Q such that $Q \ni x$, $Q \subset \Omega$, intersect at most two of the cubes Q_i , thus M_Ω is finite a.e.

The preceding example shows that f need not be integrable over every compact subset of its support in order to have $M_\Omega f < \infty$ a.e. in Ω . Of course f must be integrable over cubes contained in Ω , this is, however, not sufficient for $M_\Omega f < \infty$ a.e. in Ω . Indeed, f can be integrable over cubes in Ω , and still M_Ω need not be a.e. finite, as shown by the following

EXAMPLE 4.4. Let us consider Ω_1 as Ω from Example 4.3 united with a rectangle pasted from below to Ω , with the left vertical side on the axe y and the upper horizontal side on the axe x and put

$$f = \sum_{i=1}^{\infty} \frac{i}{|Q_i|} \chi_{Q_i/2}.$$

Let Q'_1 be the translation of Q_1 having the left upper corner on the origin, and $Q''_1 \neq Q'_1$ be any fixed translation of Q_1 , contained in Ω_1 , such that the left upper corner of Q''_1 stays on the line $y = kx$ and such that the set $E = Q'_1 \cap Q''_1$ has positive measure. Then for every Q_i , i sufficiently large, there exists a translation of Q_1 containing $E \cup Q_i$, therefore, if $x \in E$, since

$$M_\Omega f(x) \geq \frac{1}{|Q_1|} \int_{Q_i} f(y) dy = \frac{i}{4|Q_1|},$$

we have $M_\Omega f = \infty$ in E . Hence $M_\Omega f$ is not finite a.e. while f is integrable over every cube contained in Ω .

Taking a closer look at the foregoing two examples we see that in both cases there exists a sequence of cubes such that the averages of f blow up. In the former case the edges tend to zero and this is not necessarily in the latter example. It turns out that this property is in fact crucial for \mathbb{D}_Ω ; indeed, we have

THEOREM 4.5. *Let $f \in L^1(Q)$ for all $Q \subset \Omega$. Then the following statements are equivalent:*

- (i) $M_\Omega f < \infty$ a.e. in Ω ;
- (ii) $\sup_{\substack{|Q| > \varepsilon \\ Q \text{ cube} \\ Q \subset \Omega}} \frac{1}{|Q|} \int_Q f(y) dy < \infty$ for all $\varepsilon > 0$.

Proof. Let (ii) hold and assume that M_Ω is not finite a.e. in Ω . Then there exists a cube $C \subset \Omega$ such that $(M_\Omega f)|_C$ is not finite a.e. Let

$$C_\varepsilon = \{x \in C ; \text{dist}(x, \partial C) > \varepsilon\}$$

and choose ε so that $|C \setminus C_\varepsilon| < |\{(M_\Omega f)|_C = \infty\}|$. Since $f \in L^1(C_\varepsilon)$ we have $M_{C_\varepsilon} f < \infty$ a.e. in C_ε and there exists $x \in C_\varepsilon$ such that $M_\Omega f(x) = \infty$ and $M_{C_\varepsilon} f(x) < \infty$. Consequently, one can find cubes $Q_n \subset \Omega$ such that $x \in \cap Q_n$, $|Q_n|^{-1} \int_{Q_n} f(y) dy \rightarrow \infty$, and at the same time $Q_n \not\subset C_\varepsilon$ for every sufficiently large n , which yields $|Q_n| > \varepsilon$. This contradicts (ii), thus (i) holds.

Let us suppose that (i) is true and choose ε_0 such that

$$\sup_{\substack{|Q| > \varepsilon_0 \\ Q \text{ cube} \\ Q \subset \Omega}} \frac{1}{|Q|} \int_Q f(x) dx = \infty;$$

further, let $Q_n \subset \Omega$, $n \in \mathbb{N}$, satisfy $|Q_n|^{-1} \int_{Q_n} f(x) dx \rightarrow \infty$, $|Q_n| > \varepsilon_0$. Denote by x_n the centre of Q_n . Since Ω is bounded there exists a subsequence (x_{n_k}) convergent to some $x \in \mathbb{R}^N$. But $x \in \Omega$ because $|Q_n| > \varepsilon_0$ for all n and cubes Q_{n_k} with sufficiently large k contain the cube $Q(x, \varepsilon_0^{1/N})$ centered at x and with sidelength $\varepsilon_0^{1/N}$; hence $M_\Omega f(y) = \infty$ for all $y \in Q(x, \varepsilon_0^{1/N})$. This contradicts (i) and the proof is complete. □

Remark 4.6. If Ω is a cube Q_0 , then (ii) is equivalent to $f \in L^1(Q_0)$. Furthermore, Theorem 4.5 remains to be true for $\Omega = \mathbb{R}^N$ by virtue of Theorem 2.2.

Let us observe that the proof of Theorem 4.5 can be modified, admitting unbounded Ω .

We turn our attention to the problem of the range of the local maximal function. If $f \in \mathbb{D}_\Omega$, then M_Ω trivially enjoys the standard properties,

namely, it is lower semicontinuous and $f \leq M_\Omega f$ a.e. in Ω . On the other hand, in contrast to the behaviour of $M_{\mathbb{R}^N}$ and M_{Q_0} , it is not generally true that $M_\Omega f \in L^r(\Omega)$. Let us consider, for instance, Example 4.3 with $a_i = i \exp(1/|Q_i|^2)$. Then

$$\int_\Omega f^r dx = \sum_{i=1}^\infty \frac{|Q_i|}{4} i^r \exp\left(\frac{r}{|Q_i|^2}\right) = \infty, \quad 0 < r < 1,$$

therefore $M_\Omega f \notin L^r(\Omega)$. This example indicates that the integrability properties of M_Ω depend heavily on the geometric properties of Ω . In particular, if Ω is a cube Q_0 , then $M_{Q_0} f \in L^r(Q_0)$ for all $0 < r < 1$.

Let us remark also that if Ω is a cube Q_0 , in contrast to the behaviour of $M_{\mathbb{R}^N}$, $\text{BMO}(Q_0) \subset \mathbb{D}_{Q_0}$ because $\text{BMO}(Q_0) \subset L^1(Q_0)$, and, modulo bounded functions, $\text{BLO}(Q_0)$ is exactly the range of M_{Q_0} on $\text{BMO}(Q_0)$: in fact we have the following analogous of (2.3) (see [BS, p. 390])

$$\text{BLO}(Q_0) = M_{Q_0}(\text{BMO}(Q_0)) + L^\infty(Q_0).$$

In spite of this we have the following easy analogue of (x) of Theorem 3.5:

THEOREM 4.7. *Let $f \in L^1(\Omega)$. Then $M_\Omega f \in L^1(\Omega)$, therefore, $M_\Omega f \in \bigcap_{\sigma>1} L^1(\log L)^{-\sigma}(\Omega)$*

Proof. Let $Q_0 \supset \Omega$ be a cube and put $\bar{f}(x) = f(x)$ for $x \in \Omega$ and $\bar{f}(x) = 0$ in $\mathbb{R}^N \setminus \Omega$. The claims now follow since $M_\Omega f \leq M_{Q_0} \bar{f} = (M_{\mathbb{R}^N} \bar{f})|_{Q_0}$ and $f \in L^1(Q_0)$.

Remark 4.8. If Ω is a cube Q_0 and $M_{Q_0} f \in L^1(\log L)^{-1}(Q_0)$, then the L^1 norm of f can be estimated as follows ([F2]):

$$\int_{Q_0} f(x) dx \leq 2^{N+1} \int_{Q_0} \frac{M_{Q_0} f(x)}{\log\left(e + \frac{M_{Q_0} f(x)}{|M_{Q_0} f(x)|_{Q_0}}\right)} dx,$$

where $|M_{Q_0} f(x)|_{Q_0} = |Q_0|^{-1} \int_{Q_0} M_{Q_0} f(x) dx$.

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