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TWO-SIDED ESTIMATES FOR POSITIVE SOLUTIONS OF SUPERLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract

We give two-sided estimates for positive solutions of the superlinear elliptic problem $-\Delta u = a(x)|u|^{p-1}u$ with zero Dirichlet boundary condition in a bounded Lipschitz domain. Our result improves the well-known *a priori* L^∞-estimate and provides information about the boundary decay rate of solutions.

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1. Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 3)$. We investigate the boundary behaviour of positive weak solutions of the superlinear elliptic boundary value problem

$$\begin{cases} -\Delta u = a(x)|u|^{p-1}u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$ and $a(x)$ is a nonnegative function in $L^\infty(\Omega) \setminus \{0\}$. A weak solution of (1.1), or simply a solution of (1.1), is a function $u \in W_0^{1,2}(\Omega)$ satisfying $|u|^{p-1}u \in (W_0^{1,2}(\Omega))^*$ and

$$\int_\Omega \nabla u(x) \cdot \nabla \phi(x)\, dx = \int_\Omega a(x)|u(x)|^{p-1}u(x)\phi(x)\, dx \quad \text{for all } \phi \in W_0^{1,2}(\Omega).$$

First, let us recall some results concerning *a priori* estimates for this problem. The well-known result due to Brezis–Turner [4] states that all positive weak solutions of (1.1) are bounded in $L^\infty(\Omega)$ when $\partial\Omega$ is smooth and $1 < p < p_{\mathrm{BT}} := (n+1)/(n-1)$ (see also [13, Section 11]). Later, the validity of this statement for $1 < p < p_{\mathrm{S}} := (n+2)/(n-2)$ was shown by Gidas–Spruck [6] and de Figueiredo–Lions–Nussbaum [5] under some additional assumptions on $a(x)$. For bounded Lipschitz domains, the *a priori* L^∞-estimate was obtained by McKenna–Reichel [12] who introduced a new

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critical exponent corresponding to the Brezis–Turner exponent p_{BT} (see Remark 1.2). They actually discussed positive 'very' weak solutions and the optimality of the range of p. Note that these results show that every positive (very) weak solution has a continuous representative that belongs to $C^1(\Omega)$. If Ω has a C^2-boundary, $a(x) \equiv 1$ and $1 < p < p_{\mathrm{S}}$, then it is known that every positive (very) weak solution u of (1.1) belongs to $C^2(\overline{\Omega})$, that is, u and its first and second partial derivatives on Ω have continuous extensions to $\overline{\Omega}$, and therefore, by the mean value theorem,

$$u(x) \leq C\delta_\Omega(x) \quad \text{for all } x \in \Omega, \tag{1.2}$$

where $\delta_\Omega(x)$ stands for the distance from a point x to the boundary $\partial\Omega$. Note here that the constant C may depend on u itself because *a priori* bounds of $\|\nabla u\|_\infty$ are unknown. We can see its actual dependence from a result of Bidaut-Véron and Vivier [2], where it is shown that (1.2) holds with a constant C depending only on p, n and Ω if we restrict the range of p to $1 < p < p_{\mathrm{BT}}$. However, a lower estimate and an alternative upper estimate in a nonsmooth domain are unknown. We are interested in studying how positive continuous solutions of (1.1) behave near $\partial\Omega$. By developing the *a priori* L^∞-estimate, we give two-sided estimates, including information about the boundary decay rate of solutions. Let $x_0 \in \Omega$ be fixed and let

$$g_\Omega(x) := \min\{G_\Omega(x, x_0), 1\},$$

where G_Ω is the (Dirichlet) Green's function on Ω for the Laplacian. Our main result is the following theorem.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ ($n \geq 3$), let $a(x)$ be a nonnegative function in $L^\infty(\Omega) \setminus \{0\}$, let $p > 1$ and let $M > 0$. Then there exists $C = C(p, \|a\|_\infty, M, n, \Omega) > 1$ such that, for any positive continuous solution of (1.1) with $\|u\|_\infty \leq M$,*

$$\frac{1}{C} g_\Omega(x) \leq u(x) \leq C g_\Omega(x) \quad \textit{for all } x \in \Omega.$$

Moreover, the ratio u/g_Ω can be extended continuously up to $\partial\Omega$.

Remark 1.2. As stated above, McKenna–Reichel [12] showed the existence of *a priori* bounds for all positive very weak solutions of (1.1) when Ω is a bounded Lipschitz domain and

$$1 < p < \frac{n + \alpha_\Omega}{n + \alpha_\Omega - 2},$$

where

$$\alpha_\Omega := \inf\left\{\alpha > 0 : \inf_{x \in \Omega} \frac{g_\Omega(x)}{\delta_\Omega(x)^\alpha} > 0\right\}. \tag{1.3}$$

Therefore, for such p, the conclusion of Theorem 1.1 holds for all positive continuous solutions of (1.1).

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Remark 1.3. Theorem 1.1 shows that every positive continuous solution u of (1.1) vanishes continuously on $\partial\Omega$ with the same speed as g_Ω. This suggests that $u \in C^1(\overline{\Omega})$ does not always hold, unlike in the case of smooth domains. Namely, the gradient of u is not necessarily continuous up to $\partial\Omega$. For example, let ω be an open connected subset of the unit sphere in $\mathbb{R}^n$ that is strictly bigger than a unit hemisphere and assume that $\Omega \cap B = \{x \in \mathbb{R}^n \setminus \{0\} : x/\|x\| \in \omega\} \cap B$, where B is some ball centred at the origin in $\mathbb{R}^n$. Then $g_\Omega(x)$ vanishes more slowly than $\delta_\Omega(x)$ as $x \to 0$ nontangentially. Therefore we see from the mean value theorem that $\|\nabla u\|$ blows up at the origin.

Using an estimate in [7, pages 37–38], we can obtain the following gradient estimate from Theorem 1.1.

Corollary 1.4. *The assumptions are the same as in Theorem 1.1. Then there exists $C = C(p, \|a\|_\infty, M, n, \Omega) > 0$ such that, for any positive solution $u \in C^2(\Omega)$ of (1.1) with $\|u\|_\infty \le M$,*

$$\|\nabla u(x)\| \le C \frac{g_\Omega(x)}{\delta_\Omega(x)} \quad \text{for all } x \in \Omega.$$

If Ω has a $C^{1,1}$-boundary, then g_Ω is comparable to the distance function δ_Ω and the ratio g_Ω/δ_Ω has a positive and finite nontangential limit at each boundary point. Theorem 1.1 and *a priori* estimates obtained by Gidas–Spruck [6] and McKenna–Reichel [12] yield the following corollary.

Corollary 1.5. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ $(n \ge 3)$ and let $a(x)$ be a nonnegative function in $L^\infty(\overline{\Omega}) \setminus \{0\}$. Assume either:*

(a) $1 < p < p_S$ *and $a(x)$ is a continuous function on $\overline{\Omega}$ with $\min_{\overline{\Omega}} a > 0$; or*
(b) $1 < p < p_{BT}$.

Then there exists $C = C(p, a(x), n, \Omega) > 1$ such that, for any positive solution $u \in C^1(\Omega)$ of (1.1),

$$\frac{1}{C}\delta_\Omega(x) \le u(x) \le C\delta_\Omega(x) \quad \text{for all } x \in \Omega.$$

Moreover, the ratio u/δ_Ω can be extended continuously up to $\partial\Omega$.

In Section 3, we give a proof of Theorem 1.1 based on the integral representation of (1.1), careful estimates of the Green's function and iteration arguments.

2. Preliminaries

In the rest of this paper, we suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \ge 3)$. As in the previous section, we use the symbol C to denote an absolute positive constant whose value may vary at each occurrence. Writing $C(a, b, \ldots)$ means that the constant C may depend on the parameters $a, b, \ldots$. In particular, $C(\Omega)$ means that C depends on Lipschitz constants of functions defining $\partial\Omega$, the diameter of Ω and $\delta_\Omega(x_0)$, where x_0 is a fixed point in Ω. Also, for two positive functions f and g, we write $f \lesssim g$ if $f(x) \le Cg(x)$ for some positive constant C independent of x. If $f \lesssim g$ and $g \lesssim f$, then we write $f \approx g$. A constant appearing in this relation is called a constant of comparison. We recall some estimates for the Green's function $G_\Omega(x, y)$. As stated in [8], there exists $C = C(\Omega) > 1$ such that, for any pair of points $x, y \in \Omega$, the set

$$\mathcal{B}(x, y) := \Big\{ b \in \Omega : \frac{1}{C} \max\{\|x-b\|, \|y-b\|\} \le \|x-y\| \le C\delta_\Omega(b) \Big\}$$

is nonempty. The following estimate can be found in [3, 8].

Lemma 2.1. *For all $x, y \in \Omega$ and $b \in \mathcal{B}(x, y)$,*

$$G_\Omega(x, y) \approx \frac{g_\Omega(x) g_\Omega(y)}{g_\Omega(b)^2} \|x-y\|^{2-n},$$

where the constant of comparison depends only on n and Ω.

To estimate the Green's function, the following well-known facts are useful.

Lemma 2.2. *There exist positive constants α, β and C, depending only on n and Ω, with the following properties:*

(1) $\beta \le 1 \le \alpha$;

(2) *for all $x \in \Omega$,*

$$\frac{1}{C}\delta_\Omega(x)^\alpha \le g_\Omega(x) \le C\delta_\Omega(x)^\beta;$$

(3) *for each $x, y \in \Omega$ and $b \in \mathcal{B}(x, y)$,*

$$\max\{g_\Omega(x), g_\Omega(y)\} \le C g_\Omega(b);$$

(4) *for each $x, y \in \Omega$ and $b \in \mathcal{B}(x, y)$,*

$$g_\Omega(x) \le C\Big(\frac{\delta_\Omega(x)}{\|x-y\|}\Big)^\beta g_\Omega(b).$$

Proof. The existence of α and β satisfying (1) and (2) was proved by Maeda–Suzuki [11]. Statement (3) follows from the Carleson estimate (see [10, Lemma 4.4]) and the Harnack inequality for harmonic functions (see the argument below). Also, (4) can be proved easily by the use of [10, Lemmas 4.1 and 4.4]. For the reader's convenience, we sketch a proof of (4). Let $x, y \in \Omega$ and let $b \in \mathcal{B}(x, y)$. Take $\xi \in \partial\Omega$ with $\|x-\xi\| = \delta_\Omega(x)$. If $\delta_\Omega(x) \ge r_0$, then $\delta_\Omega(b) \ge \delta_\Omega(x) - \|x-b\| \ge \delta_\Omega(x) - C\delta_\Omega(b)$, and so $\delta_\Omega(b) \gtrsim r_0$. Therefore $g_\Omega(x) \approx g_\Omega(b)$ by the Harnack inequality. Since Ω is bounded, we can obtain (4) in this case. Consider the case $\delta_\Omega(x) < r_0$. If $\|x-y\| \le \delta_\Omega(x)$, then the Harnack inequality yields $g_\Omega(x) \approx g_\Omega(b)$, and so the conclusion follows. If $\|x-y\| > \delta_\Omega(x)$, then by [10, Lemmas 4.1 and 4.4],

$$g_\Omega(x) \lesssim \Big(\frac{\delta_\Omega(x)}{\|x-y\|}\Big)^\beta g_\Omega(\xi_{\|x-y\|}),$$

where $\xi_{\|x-y\|}$ is a nontangential point in $\Omega \cap \partial B(\xi, \|x-y\|)$. Since

$$\|\xi_{\|x-y\|} - b\| \le \|\xi_{\|x-y\|} - \xi\| + \|\xi - x\| + \|x-b\| \lesssim \|x-y\| \lesssim \min\{\delta_\Omega(\xi_{\|x-y\|}), \delta_\Omega(b)\},$$

it follows from the Harnack inequality that $g_\Omega(\xi_{\|x-y\|}) \approx g_\Omega(b)$. Thus (4) is proved. □

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Note that, from Lemmas 2.1 and 2.2(3),

$$G_\Omega(x, y) \lesssim \frac{g_\Omega(x)}{g_\Omega(b)} \|x - y\|^{2-n} \tag{2.1}$$

for all $x, y \in \Omega$ and $b \in \mathcal{B}(x, y)$. This will be used frequently in the next section.

3. Proof of Theorem 1.1

In the argument below, let u be a positive continuous solution of (1.1) with $\|u\|_\infty \le M$. Note that u has the representation

$$u(x) = \int_\Omega G_\Omega(x, y) a(y) u(y)^p \, dy \quad \text{for all } x \in \Omega. \tag{3.1}$$

In Section 3.1, we give a proof of the upper estimate $u(x) \lesssim g_\Omega(x)$ considering two cases $\alpha < 2$ and $\alpha \ge 2$ separately, where α is as in Lemma 2.2. The case $\alpha < 2$ follows easily from a simple estimation of the right-hand side in (3.1), but the other case needs iteration arguments to improve the estimates because $\int_\Omega G_\Omega(x, y) \, dy$ vanishes slowly at some boundary point. In Section 3.2, we give a proof of the lower estimate $g_\Omega(x) \lesssim u(x)$ by the use of the Harnack inequality for (1.1) and the uniform lower boundedness of $u(x_0)$. In Section 3.3, we prove that the ratio u/g_Ω has a continuous extension to $\overline{\Omega}$.

3.1. Upper estimate.

3.1.1. The case $\alpha < 2$. Let $x \in \Omega$. By (2.1) and Lemma 2.2(2),

$$G_\Omega(x, y) \lesssim g_\Omega(x) \|x - y\|^{2-n-\alpha} \quad \text{for all } y \in \Omega.$$

Using (3.1),

$$u(x) \le \|a\|_\infty M^p \int_\Omega G_\Omega(x, y) \, dy \lesssim \frac{\|a\|_\infty M^p}{2 - \alpha} g_\Omega(x).$$

3.1.2. The case $\alpha \ge 2$. For simplicity, we write

$$p_k := \sum_{j=0}^{k} p^j.$$

Let N be the smallest nonnegative integer such that $\alpha < 2p_N$. Then $N \ge 1$. We claim that, for each $k \in \{0, \dots, N - 1\}$,

$$u(x) \le C g_\Omega(x)^{2p_k/\alpha} \quad \text{for all } x \in \Omega, \tag{3.2}$$

where C depends only on $\|a\|_\infty$, M, p, n and Ω. We prove this by induction. Let $x \in \Omega$. It is easy to see that

$$\int_{B(x, \delta_\Omega(x)/2)} G_\Omega(x, y) \, dy \lesssim \delta_\Omega(x)^2 \lesssim g_\Omega(x)^{2/\alpha}. \tag{3.3}$$

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To estimate the integral over $\Omega \setminus B(x, \delta_\Omega(x)/2)$, we take γ_0 with

$$\frac{\alpha - 2}{\alpha} < \gamma_0 < \frac{\alpha - 2}{\alpha - \beta}.$$

Note that $0 < \gamma_0 < 1$. Then, by (2.1) and Lemma 2.2(2),(4), for all $y \in \Omega \setminus B(x, \delta_\Omega(x)/2)$ and $b \in \mathcal{B}(x, y)$,

$$\begin{aligned} G_\Omega(x, y) &\lesssim \left(\frac{g_\Omega(x)}{g_\Omega(b)}\right)^{\gamma_0}\left(\frac{g_\Omega(x)}{g_\Omega(b)}\right)^{1-\gamma_0}\|x - y\|^{2-n} \\ &\lesssim \left(\frac{\delta_\Omega(x)}{\|x - y\|}\right)^{\beta\gamma_0}\left(\frac{g_\Omega(x)}{\|x - y\|^\alpha}\right)^{1-\gamma_0}\|x - y\|^{2-n} \\ &\lesssim \delta_\Omega(x)^{\beta\gamma_0} g_\Omega(x)^{1-\gamma_0}\|x - y\|^{2-n+(\alpha-\beta)\gamma_0-\alpha}. \end{aligned}$$

By the choice of γ_0, this yields

$$\int_{\Omega\setminus B(x,\delta_\Omega(x)/2)} G_\Omega(x, y)\, dy \lesssim \delta_\Omega(x)^{2+\alpha(\gamma_0-1)} g_\Omega(x)^{1-\gamma_0} \lesssim g_\Omega(x)^{2/\alpha}. \tag{3.4}$$

It follows from (3.1), (3.3) and (3.4) that $u(x) \lesssim \|a\|_\infty M^p g_\Omega(x)^{2/\alpha}$, which implies that (3.2) holds for $k = 0$. Next, we assume that (3.2) holds for some $k \in \{0, \ldots, N - 2\}$. Then, for all $x \in \Omega$,

$$\int_{B(x,\delta_\Omega(x)/2)} G_\Omega(x, y)u(y)^p\, dy \lesssim g_\Omega(x)^{2(p_{k+1}-1)/\alpha}\delta_\Omega(x)^2 \lesssim g_\Omega(x)^{2p_{k+1}/\alpha}, \tag{3.5}$$

where, in the first inequality, we used the Harnack inequality: $g_\Omega(y) \lesssim g_\Omega(x)$ for all $y \in B(x, \delta_\Omega(x)/2)$. Take γ_k with

$$\frac{\alpha - 2p_{k+1}}{\alpha} < \gamma_k < \min\left\{\frac{\alpha - 2(p_{k+1} - 1)}{\alpha}, \frac{\alpha - 2p_{k+1}}{\alpha - \beta}\right\}.$$

Since

$$\begin{aligned} G_\Omega(x, y)g_\Omega(y)^{2(p_{k+1}-1)/\alpha} &\lesssim \left(\frac{g_\Omega(x)}{g_\Omega(b)}\right)^{\gamma_k}\frac{g_\Omega(x)^{1-\gamma_k}}{g_\Omega(b)^{1-\gamma_k-2(p_{k+1}-1)/\alpha}}\|x - y\|^{2-n} \\ &\lesssim \delta_\Omega(x)^{\beta\gamma_k} g_\Omega(x)^{1-\gamma_k}\|x - y\|^{2-n-\beta\gamma_k-(1-\gamma_k)\alpha+2(p_{k+1}-1)} \end{aligned}$$

by (2.1) and Lemma 2.2, it follows from the choice of γ_k that

$$\int_{\Omega\setminus B(x,\delta_\Omega(x)/2)} G_\Omega(x, y)u(y)^p\, dy \lesssim \delta_\Omega(x)^{(\gamma_k-1)\alpha+2p_{k+1}} g_\Omega(x)^{1-\gamma_k} \lesssim g_\Omega(x)^{2p_{k+1}/\alpha}. \tag{3.6}$$

Therefore we obtain from (3.1), (3.5) and (3.6) that $u(x) \lesssim g_\Omega(x)^{2p_{k+1}/\alpha}$ for all $x \in \Omega$. Thus (3.2) holds.

Let us apply (3.2) with $k = N - 1$ to show $u(x) \lesssim g_\Omega(x)$. Let $x \in \Omega$. Note that, for all $y \in \Omega$ and $b \in \mathcal{B}(x, y)$,

$$G_\Omega(x, y)g_\Omega(y)^{2(p_N-1)/\alpha} \lesssim \frac{g_\Omega(x)}{g_\Omega(b)^{1-2(p_N-1)/\alpha}}\|x - y\|^{2-n}.$$

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If $1 - 2(p_N - 1)/\alpha \le 0$, then

$$\int_\Omega G_\Omega(x, y)u(y)^p \, dy \lesssim g_\Omega(x).$$

If $1 - 2(p_N - 1)/\alpha > 0$, then

$$\int_\Omega G_\Omega(x, y)u(y)^p \, dy \lesssim g_\Omega(x) \int_\Omega \|x - y\|^{-n-\alpha+2p_N} \, dy \lesssim g_\Omega(x)$$

by our choice of N. Hence $u(x) \lesssim g_\Omega(x)$ in all cases. This completes the proof of the upper estimate.

3.2. Lower estimate. In a previous paper [9, Section 5], we proved the Harnack inequality for positive classical solutions of the Lane–Emden equation $-\Delta v = |v|^{p-1}v$ with $1 < p < (n + 2)/(n - 2)$, but the argument given there is applicable to a positive continuous function v on Ω with a distributional Laplacian that satisfies $0 \le -\Delta v \le C\delta_\Omega(x)^{-2}v$ in Ω. Since the distributional Laplacian of our object u satisfies $0 \le -\Delta u = a(x)u^p \le \|a\|_\infty M^{p-1}u$ in Ω, we can obtain the following Harnack inequality.

Lemma 3.1. *There exists $\kappa = \kappa(\|a\|_\infty, M, p, n) \in (0, 1)$ such that*

$$u(x) \le 2u(y)$$

for any pair of points $x, y \in \Omega$ satisfying $\|x - y\| \le \kappa \min\{\delta_\Omega(x), \delta_\Omega(y)\}$.

To guarantee that all solutions take their maximum values apart from the boundary, we need the following lemma.

Lemma 3.2. *There exists $C = C(\|a\|_\infty, M, p, n, \Omega) > 0$ such that*

$$u(x) \le Cu(x_0) \quad \textit{for all } x \in \Omega.$$

Proof. From the discussion in the previous subsection, we see that there exists $\gamma > 0$ such that, for all $x \in \Omega$,

$$u(x) = \int_\Omega G_\Omega(x, y)a(y)u(y)^p \, dy \lesssim M^{p-1}\|a\|_\infty\|u\|_\infty g(x)^\gamma.$$

Therefore, we find $\delta = \delta(\|a\|_\infty, M, p, n, \Omega) > 0$ such that $u(x) \le \|u\|_\infty/2$ for all $x \in \Omega$ satisfying $\delta_\Omega(x) \le \delta$. This implies that u attains its maximum at some point $x_1 \in \Omega$ with $\delta_\Omega(x_1) \ge \delta$. By Lemma 3.1, $u(x) \le u(x_1) \lesssim u(x_0)$ for all $x \in \Omega$, as required. □

Let us show that $g_\Omega(x) \lesssim u(x)$ for all $x \in \Omega$. Let $E := \{x \in \Omega : G_\Omega(x, x_0) \ge 1\}$. Then E is compact in Ω. By Lemma 3.1, we have $g_\Omega(x) = 1$ and $u(x_0) \lesssim u(x)$ for all $x \in E$, and so

$$g_\Omega(x)u(x_0) \lesssim u(x) \quad \text{on } E.$$

By the minimum principle for superharmonic functions, we see that this inequality holds on the whole of Ω. Therefore it suffices to show that

$$u(x_0) \ge C > 0. \tag{3.7}$$

Since

$$\sup_{x\in\Omega}\int_\Omega G_\Omega(x,y)\,dy \le C(n,\operatorname{diam}\Omega),$$

we have $\|u\|_\infty \le C\|a\|_\infty\|u\|_\infty^p$. This, together with Lemma 3.2, yields

$$(C\|a\|_\infty)^{-1/(p-1)} \le \|u\|_\infty \lesssim u(x_0),$$

which gives (3.7). Thus the lower estimate is proved.

3.3. Continuous extension. Let $\xi \in \partial\Omega$. Note that

$$\lim_{x\to\xi}\frac{G_\Omega(x,y)}{G_\Omega(x,x_0)} = M_\Omega(y,\xi),$$

since the Martin boundary of a bounded Lipschitz domain is identical to the Euclidean boundary (see [1]). By the upper estimate $u(x) \lesssim g_\Omega(x)$, (2.1) and Lemma 2.2,

$$\frac{G_\Omega(x,y)}{G_\Omega(x,x_0)}a(y)u(y)^p \lesssim g_\Omega(y)^{p-1}\|x-y\|^{2-n} \lesssim \|x-y\|^{2-n}$$

for all $x, y \in \Omega$. It follows from a version of Lebesgue's dominated convergence theorem that

$$\lim_{x\to\xi}\frac{u(x)}{g_\Omega(x)} = \lim_{x\to\xi}\int_\Omega \frac{G_\Omega(x,y)}{G_\Omega(x,x_0)}a(y)u(y)^p\,dy = \int_\Omega M_\Omega(y,\xi)a(y)u(y)^p\,dy.$$

Hence u/g_Ω has a continuous extension to $\overline{\Omega}$. This completes the proof of Theorem 1.1.

Remark 3.3. If α_Ω defined by (1.3) is greater than 2, then $\int_\Omega M_\Omega(x,\xi)\,dx$ may diverge for some $\xi \in \partial\Omega$. Therefore we need the upper estimate $u(x) \lesssim g_\Omega(x)$ to show the existence of boundary limits of u/g_Ω.

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