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LINEAR TRANSFORMATIONS PRESERVING THE REAL ORTHOGONAL GROUP

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1. Introduction. Let K be a field and $M_n(K)$ denote the vector space of $n \times n$ matrices over K. Marcus [**4**] posed the following general problem: Let W be a subspace of $M_n(K)$ and S a subset of W. Describe the set $L(S, W)$ of all linear transformations T on W such that $T(S)$ is contained in S.

Dieudonne [**1**] took W to be $M_n(K)$ and S to be the set of matrices of determinant zero. He proved that if $T \in L(S, W)$ is non-singular then it has the form

(1) $T(X) = UXV, \quad \text{all } X \in M_n(K),$

or

(2) $T(X) = UX'V, \quad \text{all } X \in M_n(K),$

where U, V belong to $GL_n(K)$ and X' is the transpose of X. Marcus and Moyls [**6**] took W to be $M_n(K)$ where K is an algebraically closed field of characteristic zero. They let S be equal to the set of matrices of rank 1. Then they showed that $L(S, W)$ consists precisely of those linear transformations of the forms (1) or (2) with $U, V \in GL_n(K)$. We note that this result does not assume *a priori* T is non-singular. Neither does the following result of Marcus [**3**]. He proved that if T is a linear map on n-square complex matrices taking the unitary group into itself, then it has the form (1) or (2) with U, V being unitary. For a comprehensive survey of this problem and preservers of other invariants, see Marcus [**5**].

In the same article [**5**], Marcus conjectured that if T is a linear map on $M_n(R)$, where R denotes the real field, such that T maps the orthogonal group $O_n(R)$ into itself, then T has the form (1) or (2) with U, V being orthogonal matrices. It is the purpose of this paper to show that this conjecture holds except for $n = 2, 4$, or 8 and that in the exceptional cases there exist singular maps. To a certain extent, we will determine the structures of those singular maps as well. We accomplish this by enlisting the aid of some results of Radon [**7**] and Hurwitz [**2**].

2. Statement of result. We define on $M_n(R)$ the following linear maps: For $U, V \in O_n(R)$ we let

$$M(U, V)(X) = UXV, \quad \text{all } X \in M_n(R), \text{ and}$$
$$tp(X) = X', \quad \text{all } X \in M_n(R).$$

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Let G_n denote the group generated by the above two types. If n, m are positive integers, we let $L(n, m)$ denote the set of all R-linear maps $T : M_n(R) \to M_m(R)$ such that $T(O_n(R))$ is contained in $O_m(R)$. It is clear that G_n is contained in $L(n, n)$.

We next define what we call pairwise skew symmetric matrices. First, for $A, B \in M_n(R)$ we set

$$\{A, B\} = AB' + BA'.$$

It is clear $\{\ ,\ \}$ is symmetric and bilinear.

Definition. Let $A_1, \ldots, A_t$ be (not necessarily distinct) elements of $M_n(R)$. They are said to be *pairwise skew symmetric* (henceforth abbreviated PSS) if $\{A_i, A_j\} = \{A_i', A_j'\} = 0$ for all $i \neq j$, $i, j = 1, \ldots, t$.

Suppose $A_1, \ldots, A_n \in O_n(R)$ are PSS. We define a linear map $E(A_1, \ldots, A_n)$ on $M_n(R)$ by setting

$$E(A_1, \ldots, A_n)(E_{ij}) = \delta_{i1}A_j$$

where E_{ij} is the matrix with 1 at the (i, j) position and zero elsewhere. If $U = (u_{ij}) \in M_n(R)$ and $V = E(A_1, \ldots, A_n)(U)$, then

$$\begin{aligned} VV' &= \sum_{j=1}^{n} u_{1j}^2 A_j A_j' + \sum_{i<j} u_{1i}u_{1j}\{A_i, A_j\} \\ &= \sum_{j=1}^{n} u_{1j}^2 I. \end{aligned}$$

Hence V is a multiple of an orthogonal matrix. In particular if U is orthogonal, then V is orthogonal. Hence $E(A_1, \ldots, A_n) \in L(n, n)$. We remark that $E(A_1, \ldots, A_n)$ is a singular map of nullity $n^2 - n$.

THEOREM. (i) *$L(n, m)$ is empty if $1 \leqq m < n$.*

(ii) *$L(n, n) = G_n$ if $n \neq 2, 4$ or 8.*

(iii) *If $n = 2, 4$ or 8 and $T \in L(n, n)$, then $T \in G_n$ or $T = T_1 \circ T_2 \circ T_3$ where $T_1, T_3 \in G_n$ and $T_2 = E(A_1, \ldots, A_n)$ for some PSS orthogonal matrices $A_1, \ldots, A_n$.*

We exhibit some examples of PSS matrices $B_1, \ldots, B_n$ in $O_n(R)$ for $n = 2, 4, 8$. First we set

$$J_2 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

and let J_{2k} be the direct sum of k copies of J_2.

(1) $n = 2$. Let $B_1 = I$, $B_2 = J_2$.

(2) $n = 4$. Let $B_1 = I$, $B_2 = J_4$,

$$B_3 = \begin{vmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{vmatrix}, \quad B_4 = \begin{vmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{vmatrix}.$$

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(3) $n = 8$. Let $B_1 = I$, $B_2 = J_2 \oplus -J_2 \oplus J_4$,

$$B_3 = \begin{vmatrix} & & 1 & & & & & \\ & & & 1 & & & & \\ -1 & & & & & & & \\ & -1 & & & & & & \\ & & & & & & & 1 \\ & & & & & & 1 & \\ & & & & & -1 & & \\ & & & & -1 & & & \end{vmatrix},$$

$$B_4 = \begin{vmatrix} & & & 1 & & & & \\ & & -1 & & & & & \\ & 1 & & & & & & \\ -1 & & & & & & & \\ & & & & & & -1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & -1 & & \end{vmatrix},$$

$$B_5 = \begin{vmatrix} & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & & -1 \\ & & & & & & 1 & \\ -1 & & & & & & & \\ & 1 & & & & & & \\ & & & -1 & & & & \\ & & 1 & & & & & \end{vmatrix},$$

$$B_6 = \begin{vmatrix} & & & & & 1 & & \\ & & & & 1 & & & \\ & & & & & & -1 & \\ & & & & & & & -1 \\ & -1 & & & & & & \\ -1 & & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{vmatrix},$$

$$B_7 = \begin{vmatrix} & & & & & & 1 & \\ & & & & & & & -1 \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ -1 & & & & & & & \\ & 1 & & & & & & \end{vmatrix},$$

$$B_8 = \begin{vmatrix} & & & & & & & 1 \\ & & & & & & 1 & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ & -1 & & & & & & \\ -1 & & & & & & & \end{vmatrix},$$

The entries not shown in these matrices are assumed to be zero.

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If we set $E_n = E(B_1, \ldots, B_n)$, then part (iii) of the theorem may be improved on as follows: If $n = 2$ or 4 and $T \in L(n, n)$ is singular then $T = T_1 \circ E_n \circ T_2$ where $T_1, T_2 \in G_n$. It is an open question whether or not this statement holds for $n = 8$. The matrices $B_1, \ldots, B_n$ have combinatoric significances as well. If $x_1, \ldots, x_n$ are indeterminants, then $\sum_{i=1}^{n} x_i B_i$ gives rise to an n-letter Hadamard design of Williamson type [**8**].

3. PSS matrices. It is evident from the statement of the theorem that PSS matrices play a central role in our analysis. We first list some immediate consequences of the definition. In what follows, $A_1, \ldots, A_t$ are assumed to be n-square PSS matrices:

(i) If $A_1 = I$, then $A_2, \ldots, A_t$ are skew symmetric.

(ii) $UA_1V, \ldots, UA_tV$ are PSS for all $U, V \in O_n(R)$.

(iii) $A_1', \ldots, A_t'$ are PSS.

(iv) If A is a linear combination of $A_2, \ldots, A_t$, then A, A_1 are PSS.

(v) If $A_i = B_i \oplus C_i$ where $B_i \in M_k(R)$ and $C_i \in M_{n-k}(R)$ for all $i = 1, \ldots, t$, then $B_1, \ldots, B_t$ are PSS and so are $C_1, \ldots, C_t$.

(vi) If for some $i \neq j$, $A_i = A_j$ then $A_i = A_j = 0$.

The following is a sufficient condition for matrices to be PSS, the proof of which is the same as a similar lemma given in [**3**, Lemma 2].

LEMMA 1. *If the matrices* $A_1, \ldots, A_t \in M_n(R)$ *satisfy*

$$\sum_{i=1}^{t} \epsilon(i) A_i \in O_n(R)$$

for all functions ϵ *from* $\{1, \ldots, t\}$ *into* $\{1, -1\}$, *then they are PSS. Furthermore,*

$$\sum_{i=1}^{t} A_i A_i' = \sum_{i=1}^{t} A_i' A_i = I.$$

We give another easy lemma without proof.

LEMMA 2. *Let* $A, B \in M_m(R)$ *be PSS. Suppose* A *is a diagonal matrix of the form*

$$A = \bigoplus_{i=1}^{k} a_i I_{s_i}$$

where $a_i \geqq 0$, $a_i \neq a_j$ *if* $i \neq j$, *and* $\sum_{i=1}^{k} s_i = m$. *Partition* B *into* $B = (B_{ij})$ *so that* B_{ij} *is* $s_i \times s_j$. *Then* $B_{ij} = 0$ *for all* $i \neq j$, $i, j = 1, \ldots, k$.

The next lemma states a normal form for PSS matrices.

LEMMA 3. *If* $A_1, \ldots, A_t \in M_m(R)$ *are* PSS, *then there exist* $U, V \in O_m(R)$ *such that*

$$UA_iV = \bigoplus_{j=1}^{k} a(i,j) B(i\ j)$$

where $a(i, j) \in R$, $B(i, j) \in O_{s_j}(R) \cup \{0\}$, $i = 1, \ldots, t$, $j = 1, \ldots, k$ *and for each* j, $B(1, j), \ldots, B(t, j)$ *are* PSS.

Before proving Lemma 3 we remark that if $a(i, j)B(i, j) = 0$ for some (i, j), then we assume $a(i, j) = 0$ and $B(i, j) = 0$. Moreover we can assume whenever necessary that for a fixed (i, j) if $B(i, j) \neq 0$ then $B(i, j) = I_{s_j}$.

Proof of Lemma 3. We induct on m. If $m = 1$, then there is nothing to prove. Assume $m > 1$. If each A_i is a multiple of an orthogonal matrix, then again there is nothing to prove. Hence we can assume A_1 is not. By the polar decomposition theorem there exist $U, V \in O_m(R)$ such that

$$UA_1V = \bigoplus_{i=1}^{k} a_i I_{s_i}$$

where $a_i \geqq 0$, $a_i \neq a_j$ if $i \neq j$, $i, j = 1, \ldots, k$ and $k \geqq 2$. Let $B_i = UA_iV$, $i = 1, \ldots, t$. $B_1, \ldots, B_t$ are PSS. Using Lemma 2 we get

$$B_i = \bigoplus_{j=1}^{k} C(i, j)$$

where $C(i, j) \in M_{s_j}(R)$, $i = 1, \ldots, t$, $j = 1, \ldots, k$. Now for each j, $C(1, j)$, $\ldots C(t, j)$ are PSS and $s_j < m$ since $k \geqq 2$. The lemma follows by induction.

In view of this lemma, we see that problems concerning PSS matrices in general may be reduced to problems concerning PSS orthogonal matrices. We next state a result due to Radon [7] that is crucial to our cause.

Let $\nu(n) = \max t$ *where* t *ranges over the cardinality of all sets of matrices* $A_1, \ldots, A_t$ *in* $O_n(R)$ *which are* PSS. *Express* n *uniquely as* $n = 16^p.2^q.r$ *where* p *is some non-negative integer*, $q = 0, 1, 2$ *or* 3, *and* r *is odd. Then* $\nu(n) = 8p + 2^q$.

The number $\nu(n)$ is known in the literature as the Radon-Hurwitz Number. For the purpose of this paper we shall only need the following easy consequence.

LEMMA 4. $\nu(n) \leqq n$ *with equality if and only if* $n = 1, 2, 4$ *or* 8.

LEMMA 5. *Suppose* $n = 4$ *or* 8 *and* $A_1, \ldots, A_n \in O_n(R)$ *are* PSS. *If* $A \in O_n(R)$ *is such that* $A_1, \ldots, A_{n-2}, A$ *are* PSS *also, then* A *is a linear combination of* A_{n-1} *and* A_n.

Proof. (i) $n = 4$. Since the property of being PSS is invariant under pre and post multiplication by orthogonal matrices, we can assume $A_1 = I$ and $A_2 = J_4$. Easy computations then show that we must have

$$A = \begin{vmatrix} & & c & d \\ & 0 & d & -c \\ -c & -d & & 0 \\ -d & c & & \end{vmatrix}, \quad A_i = \begin{vmatrix} & & c_i & d_i \\ & 0 & d_i & -c_i \\ -c_i & -d_i & & 0 \\ -d_i & c_i & & \end{vmatrix}, \quad i=3, 4,$$

where $c^2 + d^2 = 1$ and $c_i^2 + d_i^2 = 1$. Furthermore we have $c_3c_4 + d_3d_4 = 0$

and $c_3d_4 - d_3c_4 = \pm 1$. Now post multiply all matrices by A_3' and bring them to the following forms:

$$A_3 = I,\ A_4 = \pm J_4,\ A = \begin{vmatrix} a & b \\ -b & a \end{vmatrix} \oplus \begin{vmatrix} a & b \\ -b & a \end{vmatrix}$$

where $a = cc_3 + dd_3$ and $b = cd_3 - dc_3$. It is clear $A = aA_3 \pm bA_4$.

(ii) $n = 8$. We use a technique due to Hurwitz [**2**]. It clearly suffices to prove that there exist complex unitary matrices U, V such that UAV is a real linear combination of UA_7V and UA_8V. To this end we set $B_j = -iA_1'A_j$, $j = 1, \ldots, 8$ and $B = -iA_1'A$. The following equations are easily computable:

(1) $B_j^2 = B^2 = I, \qquad j = 2, \ldots, 8$

(2) $B_jB_k + B_kB_j = 0, \quad j \neq k, j, k = 2, \ldots, 8$
$BB_j + B_jB = 0, \qquad j = 2, \ldots, 6.$

We note that the above equations are invariant under unitary similarity transformations. Now B_2 is unitary, hence it is unitarily diagonalizable. $B_2^2 = I$ means the eigenvalues of B_2 are 1 and -1. Furthermore from (2) we get $B_3^*B_2B_3 = -B_2$ which implies B_2 and $-B_2$ have the same eigenvalues. Hence we can assume $B_2 = I_4 \oplus (-I_4)$. Using (1) and (2) we get that

$$B = \begin{vmatrix} 0 & C \\ C^* & 0 \end{vmatrix}, \quad B_j = \begin{vmatrix} 0 & C_j \\ C_j^* & 0 \end{vmatrix}, \quad j = 3, \ldots, 8$$

where C and C_j are 4×4 unitary. Now let all matrices undergo similarity transformation by $I_4 \oplus C_3^*$. We get that

$$B_3 = \begin{vmatrix} 0 & I_4 \\ I_4 & 0 \end{vmatrix}, \quad B = \begin{vmatrix} 0 & iD \\ -iD & 0 \end{vmatrix}, \quad B_j = \begin{vmatrix} 0 & iD_j \\ -iD_j & 0 \end{vmatrix}, \qquad j = 4, \ldots, 8,$$

and the matrices $D_4, \ldots, D_8$ and D satisfy equations (1) and (2). Hence we can duplicate the above argument and get

$$D = \begin{vmatrix} 0 & iH \\ -iH & 0 \end{vmatrix}, \quad D_j = \begin{vmatrix} 0 & iH_j \\ -iH_j & 0 \end{vmatrix}, \quad j = 6, 7, 8;$$

and H_6, H_7, H_8 and H satisfy (1) and (2). Repeating the process again, we get

$$H = \begin{vmatrix} 0 & h \\ \bar{h} & 0 \end{vmatrix}, \quad H_7 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad H_8 = \pm \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix}.$$

It is clear H is a real linear combination of H_7 and H_8, whence B is that of B_7 and B_8.

4. Proof of theorem. Suppose $T \in L(n, m)$. We let F_{ij} denote the image of E_{ij} under T. We need the following lemma:

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LEMMA 6. *If* $1 \leqq s, t, k, h \leqq n$ *and* $s \neq k$, $t \neq h$, *then the following equations hold:*
(i) $\{F_{st}, F_{kt}\} = \{F_{sh}, F_{kh}\}$,
(ii) $\{F_{st}', F_{kt}'\} = \{F_{sh}', F_{kh}'\}$,
(iii) $\{F_{st}, F_{sh}\} = \{F_{kt}, F_{kh}\}$,
(iv) $\{F_{st}', F_{sh}'\} = \{F_{kt}', F_{kh}'\}$.

Proof. There exist a permutation σ in the symmetric group of degree n such that $\sigma(s) = t$ and $\sigma(k) = h$. Choose $a, b \in R$ such that $ab \neq 0$ and $a^2 + b^2 = 1$. Then the matrices,

$$A = a(F_{st} + F_{kh}), B = b(F_{sh} - F_{kt}), F_{r\sigma(r)}, r \neq s, r \neq k \text{ and } 1 \leqq r \leqq n,$$

are PSS by Lemma 1. Similarly the matrices,

$$C = a(F_{st} - F_{kh}), D = b(F_{sh} + F_{kt}), F_{r\sigma(r)}, r \neq s, r \neq k, \text{ and } 1 \leqq r \leqq n,$$

are PSS. Expanding the equations $\{A, B\} = 0$ and $\{C, D\} = 0$ and adding, we get $\{F_{st}, F_{sh}\} = \{F_{kh}, F_{kt}\}$, which is (iii). (i), (ii) and (iv) are proved similarly.

We mention that if s, t, k, h are as in Lemma 6 and if one of the matrices, $F_{st}, F_{kt}, F_{sh}, F_{kh}$, is zero then the conclusions of the lemma imply the four matrices are PSS.

We proceed to the proof of the theorem. First observe that it is sufficient to show that up to pre and post composition with T by elements of G_m and G_n respectively, if $1 \leqq m < n$ we arrive at a contradiction and if $m = n$ we get T is the identity map on $M_n(R)$, $\text{id}_{M_n(R)}$, or $T = E(A_1, \ldots, A_n)$ for some PSS orthogonal matrices $A_1, \ldots, A_n$ with the latter occurring only for $n = 2, 4$, or 8.

We induct on n. The case $n = 1$ is trivial. Hence we assume $n > 1$ and divide the rest of the proof into three parts.

Part I. If $1 \leqq m < n$, then $L(n, m) = \emptyset$.

Part II. If $m = n$ and there exist (i, j), $1 \leqq i, j \leqq n$, such that F_{ij} is not a scalar multiple of an orthogonal matrix, then $T = \text{id}_{M_n(R)}$.

Part III. If $m = n$ and F_{ij} is a scalar multiple of an orthogonal matrix for each (i, j), $i, j = 1, \ldots, n$, then $n = 2, 4$, or 8 and $T = E(A_1, \ldots, A_n)$ for some PSS orthogonal matrices $A_1, \ldots, A_n$.

Proof of Part I. The matrices $F_{11}, \ldots, F_{nn}$ are PSS by Lemma 1. Lemma 3 implies we can assume

$$F_{ii} = \bigoplus_{j=1}^{k} a(i,j)B(i,j) \quad i = 1, \ldots, n$$

where $B(i, j) \in O_{s_j}(R) \cup \{0\}$, $\sum_{j=1}^{k} s_j = m$ and if $a(i, j)B(i, j) = 0$ then $a(i, j) = 0$ and $B(i, j) = 0$. Furthermore for each fixed j, $j = 1, \ldots, k$, we have $B(1, j), \ldots, B(n, j)$ are PSS. We claim that at least one of the F_{ii} is the zero matrix. To see this, let t denote the total number of non-zero $a(i, j)$, $i = 1, \ldots, n$, $j = 1, \ldots, k$. Then

$$t \leqq \sum_{j=1}^{k} \nu(s_j) \leqq \sum_{j=1}^{k} s_j = m < n$$

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where the first inequality follows from the definition of ν and the second inequality follows from Lemma 4. Now if each F_{ii} has at least one non-zero direct summand we would have $t \geqq n$. Hence our claim is valid. We can assume $F_{11} = 0$.

If $X \in O_{n-1}(R)$, then $T(0 + X) = T(1 + X) \in O_m(R)$. Hence T induces a map $\hat{T} \in L(n-1, m)$. By induction, $m = n - 1$ and we can assume $\hat{T} = \mathrm{id}_{M_{n-1}(R)}$ or $\hat{T} = E(A_1, \ldots, A_{n-1})$ for some PSS orthogonal matrices $A_1, \ldots, A_{n-1}$.

(i) If $\hat{T} = \mathrm{id}_{M_{n-1}(R)}$, then from Lemmas 1 and 6 and the fact that $F_{11} = 0$ we get $\{F_{1n}, F_{ii}\} = 0$ for $i = 1, \ldots, n$. Since $F_{ii} = E_{i-1\,i-1}$, $i = 2, \ldots, n$, we have $F_{1n} = 0$. Similarly $F_{n1} = 0$. Now let the matrix $A \in O_n(R)$ be defined by

$$A = E_{n1} + E_{1n} + \sum_{i=2}^{n-1} E_{ii}.$$

Then $T(A) = 0 + I_{n-2} \notin O_{n-1}(R)$ which is a contradiction.

(ii) If $\hat{T} = E(A_1, \ldots, A_{n-1})$, then by definition $F_{2i} = A_{i-1}$, $i = 2, \ldots, n$, and $F_{ij} = 0$ for $i = 3, \ldots, n$ and $j = 2, \ldots, n$. We set $A_n = F_{12} + F_{21}$. Then

$$A_n = T\left(E_{12} + E_{21} + \sum_{i=3}^{n} E_{ii}\right).$$

Hence $A_n \in O_{n-1}(R)$. By Lemmas 1 and 6, $A_1, \ldots, A_{n-1}, F_{12}, F_{21}$ are PSS. Hence $A_1, \ldots, A_n$ are PSS. But this contradicts the fact that $\nu(n-1) < n$. This completes the proof of Part I.

Proof of Part II. We first show that we can assume $F_{11} = E_{11}$. Since at least one of the F_{ij} is not a scalar multiple of an orthogonal matrix we can assume F_{11} is not. Using the polar decomposition theorem, we can assume

$$F_{11} = \bigoplus_{i=1}^{k} a_i I_{s_i}$$

where $\sum_{i=1}^{k} s_i = n$, $a_i \geqq 0$ and $a_i \neq a_j$ if $i \neq j$. We observe that $k \geqq 2$ and $0 < s_i < n$ for all $i = 1, \ldots, k$. Furthermore we note that $0 \leqq a_i \leqq 1$ and we can assume $a_1 \neq 1$. By Lemma 2, we now have

$$F_{ij} = \bigoplus_{t=1}^{k} B(i, j, t)$$

where $B(i, j, t)$ is $s_t \times s_t$, $i, j = 2, \ldots, n$ and $t = 1, \ldots, k$. Now if X is $n - 1 \times n - 1$ orthogonal then T maps $0 \oplus X$ into $Y_1 \oplus \ldots \oplus Y_k$ where Y_t is $s_t \times s_t$. From T we obtain a map $\hat{T} : M_{n-1}(R) \rightarrow M_{s_1}(R)$ such that $\hat{T}(X) = Y_1$. An easy exercise shows that $(1 - a_1^2)^{-1/2}\hat{T} \in L(n-1, s_1)$. We have by induction $n - 1 = s_1$. Hence $F_{11} = a_1 I_{n-1} \oplus a_2$ where $0 \leqq a_1 < 1$ and $0 \leqq a_2 \leqq 1$, and $F_{ii} = B_i \oplus b_i$ where $B_i \in M_{n-1}(R)$ and $b_i \in R$, $i = 2, \ldots, n$. Since $F_{11}, \ldots, F_{nn}$ are PSS, we have that $a_2, b_2, \ldots, b_n$ are PSS. Since $\nu(1) = 1$, at most one of them is non-zero. If $a_2 = 0$, then $a_1 \neq 0$. A similar

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argument as before shows $s_2 = n - 1$ which means $n = 2$, $F_{11} = a_1 \oplus 0$ and $F_{22} = b_1 \oplus b_2$. Now a_1, b_1 are PSS and $a_1 \neq 0$ means $b_1 = 0$ which in turn implies $a_1 = 1$, a contradiction. Hence $a_2 \neq 0$ and $b_2, \ldots, b_n$ are all zero. This means $a_2 = 1$. Now F_{11} has the form $a_1 I_{n-1} \oplus 1$. If $a_1 = 0$ then we are done. Hence assume $0 < a_1 < 1$. There exist an s, $2 \leqq s \leqq n$, such that $F_{ss} \neq 0$. F_{ss} has the form $B + 0$. Precomposing T with a map interchanging the (s, s) and $(1, 1)$ entries allows us to assume $F_{11} = B \oplus 0$. As before bring our new F_{11} to the form

$$F_{11} = \bigoplus_{i=1}^{k} c_i I_{s_i}$$

where $c_i \neq c_j$ if $i \neq j$ and $c_i \geqq 0$. We know zero must occur among the eigenvalues of F_{11}. Hence we can assume $c_2 = 0$. Applying previous arguments again, we get $s_2 = n - 1$. Hence $F_{11} = E_{11}$.

This means T maps matrices of the form $0 \oplus X$ into $0 \oplus Y$ where X, Y are in $M_{n-1}(R)$. Hence T induces a map $\hat{T} \in L(n-1, n-1)$. By induction we can assume $\hat{T} = \mathrm{id}_{M_{n-1}(R)}$ or $\hat{T} = E(A_1, \ldots, A_{n-1})$ for some PSS $n - 1 \times n - 1$ orthogonal $A_1, \ldots, A_{n-1}$.

We show that the latter case may be reduced to the former. If $\hat{T} = E(A_1, \ldots, A_{n-1})$, we can assume $A_1 = I_{n-1}$. Hence $F_{22} = 0 \oplus I_{n-1}$. Lemma 2 forces F_{ij} to have the form $b_{ij} \oplus O_{n-1}$, $i, j = 1, 3, \ldots n$. This means T induces a map in $L(n-1, 1)$. Using our induction hypothesis again we get that $n = 2$ which implies $\hat{T} = \mathrm{id}_{M_1(R)}$.

We now have $\hat{T}$ is the identity map on $M_{n-1}(R)$ which means $F_{ij} = E_{ij}$ for $i, j = 2, \ldots, n$. Now F_{12} and F_{21} must have the form

$$\begin{vmatrix} * & 0 \\ 0 & O_{n-2} \end{vmatrix} .$$

By Lemma 6, they satisfy the equations $\{F_{12}, F_{11}\} = \{F_{21}, F_{22}\}$ and $\{F_{12}, F_{22}\} = \{F_{21}, F_{11}\}$. Furthermore F_{12} and F_{21} are PSS. We use these facts to conclude that F_{12} and F_{21} must satisfy one of the following four possibilities:

(1) $F_{12} = E_{12}$, $F_{21} = E_{21}$
(2) $F_{12} = -E_{12}$, $F_{21} = -E_{21}$
(3) $F_{12} = E_{21}$, $F_{21} = E_{12}$
(4) $F_{12} = -E_{21}$, $F_{21} = -E_{12}$.

If (3) or (4) occurs, then we replace T by $T \circ tp$ and get the cases (1) or (2). Hence we only need to consider (1) and (2).

We first take care of the case $n = 2$. If (1) occurs then $T = \mathrm{id}_{M_2(R)}$ and we are done. If (2) occurs, we let U be the matrix $-1 \oplus 1$. Then $M(U, U) \circ T = \mathrm{id}_{M_2(R)}$.

We now assume $n \geqq 3$. It is clear F_{13} and F_{31} must satisfy one of the possibilities (1)–(4) with suitably changed subscripts. This yields the following eight cases:

1. $F_{12} = E_{12}$, $F_{21} = E_{21}$, $F_{13} = E_{13}$, $F_{31} = E_{31}$
2. $F_{12} = -E_{12}$, $F_{21} = -E_{21}$, $F_{13} = -E_{13}$, $F_{31} = -E_{31}$
3. $F_{12} = E_{12}$, $F_{21} = E_{21}$, $F_{13} = -E_{13}$, $F_{31} = -E_{31}$
4. $F_{12} = E_{12}$, $F_{21} = E_{21}$, $F_{13} = E_{31}$, $F_{31} = E_{13}$
5. $F_{12} = E_{12}$, $F_{21} = E_{21}$, $F_{13} = -E_{31}$, $F_{31} = -E_{13}$
6. $F_{12} = -E_{12}$, $F_{21} = -E_{21}$, $F_{13} = E_{13}$, $F_{31} = E_{31}$
7. $F_{12} = -E_{12}$, $F_{21} = -E_{21}$, $F_{13} = E_{31}$, $F_{31} = E_{13}$
8. $F_{12} = -E_{12}$, $F_{21} = -E_{21}$, $F_{13} = -E_{31}$, $F_{31} = -E_{13}$.

Let A denote the matrix

$$\begin{vmatrix} 0 & a & b \\ 0 & -b & a \\ 1 & 0 & 0 \end{vmatrix} \oplus I_{n-3}$$

where $ab \neq 0$ and $a^2 + b^2 = 1$. Then $A \in O_n(R)$. Hence $T(A) \in O_n(R)$. This fact eliminates the cases 3–8.

It is clear that if we apply similar arguments to F_{1j} and F_{j1} for $4 \leqq j \leqq n$ we would get the following two cases:

1. $F_{1i} = E_{1i}$, $F_{i1} = E_{i1}$, $i = 2, \ldots, n$
2. $F_{1i} = -E_{1i}$, $F_{i1} = -E_{i1}$, $i = 2, \ldots, n$.

If case 1 holds then $T = \mathrm{id}_{M_n(R)}$ and we are done. If case 2 holds, we let $U = -1 \oplus I_{n-1}$. Then $M(U, U) \circ T = \mathrm{id}_{M_n(R)}$. This completes the proof of Part II.

Proof of Part III. We now have $F_{ij} = a_{ij}T_{ij}$, $i, j = 1, \ldots, n$, where $a_{ij} \in R$, $0 \leqq |a_{ij}| \leqq 1$ and $T_{ij} \in O_n(R) \cup \{0\}$. We assume that if $F_{ij} = 0$ then $a_{ij} = 0$ and $T_{ij} = 0$. At least one of the matrices $T_{11}, \ldots, T_{nn}$ is non-zero. We can assume that $T_{ii} \neq 0$, $i = 1, \ldots, k$, and $T_{jj} = 0$, $j = k + 1, \ldots, n$. From this we get that $T_{ij} = 0$ for $i, j = k + 1, \ldots, n$. Several applications of Lemmas 1 and 6 show the matrices

(3) $\quad T_{11}, \ldots, T_{kk}; T_{kk+1}, \ldots, T_{kn}; T_{k+1k}, \ldots, T_{nk}$

are PSS. Furthermore, it is not the case that there exist s and t, $k + 1 \leqq s$, $t \leqq n$, such that $T_{ks} = T_{tk} = 0$. Since we know each of $T_{11}, \ldots, T_{kk}$ is non-zero, we have that there are at least n non-zero matrices in (3). Lemma 4 now implies $n = 2$, 4 or 8 and that we can assume

(4) $\quad T_{kk+1} = \ldots = T_{kn} = 0$

and $T_{jk} \neq 0$ for all $j = k + 1, \ldots, n$.

Suppose for some (i, j), $|a_{ij}| = 1$. Then we can assume $F_{11} = I$. This means $F_{ij} = 0$ for all $i, j = 2, \ldots, n$. We further know from (4) that $F_{1j} = 0$ for all $j = 2, \ldots, n$. Let σ be a permutation such that $\sigma(t) = 1$ for some t, $1 \leqq t \leqq n$. If $P(\sigma)$ is the corresponding permutation matrix then $T(P(\sigma)) = F_{t1}$. Hence $F_{t1} \in O_n(R)$. From (3) we know the matrices $F_{11}, \ldots, F_{n1}$ are PSS. Hence we have $T \circ tp = E(F_{11}, \ldots, F_{n1})$.

We now consider the case $0 \leqq |a_{ij}| < 1$ for all $i, j = 1, \ldots, n$. We show that we can reduce this to the case $F_{11} = I$. To this end we split the argument into (i) $n = 2$ and (ii) $n = 4, 8$.

(i) If $n = 2$, then $T_{22} \neq 0$. Hence we can assume $F_{11} = a_{11}I$ and $F_{22} = a_{22}J_2$ where $0 < a_{11}, a_{22} < 1$. Using Lemmas 1 and 6 and an easy computation, we get that both F_{12} and F_{21} are non-zero linear combinations of I and J_2. We compute further. Write

$$F_{12} = a_{12}\begin{vmatrix} c_1 & d_1 \\ -d_1 & c_1 \end{vmatrix} \text{ and } F_{21} = a_{21}\begin{vmatrix} c_2 & d_2 \\ -d_2 & c_2 \end{vmatrix}$$

where $a_{12} \neq 0$, $a_{21} \neq 0$ and $c_i^2 + d_i^2 = 1$. Using $\{F_{12}, F_{21}\} = 0$, we get that

(5) $c_1c_2 + d_1d_2 = 0.$

Using $\{F_{12}, F_{11}\} = \{F_{21}, F_{22}\}$ and $\{F_{12}, F_{22}\} = \{F_{21}, F_{11}\}$, we get

(6) $c_1 = \dfrac{a_{22}a_{21}}{a_{11}a_{12}} d_2$ and $c_2 = \dfrac{a_{22}a_{12}}{a_{11}a_{21}} d_1.$

Combining (5) and (6) yields $d_1 = 0$ or $d_2 = 0$. We can assume $d_1 = 0$ which implies $c_1 = \pm 1$, $c_2 = 0$ and $d_2 = \pm 1$. Hence $F_{12} = a_{12}I$ and $F_{21} = a_{21}J_2$. Equations (6) then imply $a_{11}a_{12} = a_{22}a_{21}$. This along with the fact that $a_{11}^2 + a_{22}^2 = a_{12}^2 + a_{21}^2$ allow us to conclude $a_{12}^2 = a_{22}^2$. Now let $U \in O_2(R)$ be of the form

$$U = \begin{vmatrix} a_{11} & a_{12} \\ & * \end{vmatrix}.$$

Then $T \circ M(I, U)(E_{11}) = I$.

(ii) We now do the case $n = 4, 8$. Again we may assume $F_{11} = a_{11}I$ and $F_{22} = a_{22}J_n$. The matrices $T_{11}, \ldots, T_{kk}, T_{k+1k}, \ldots, T_{nk}$ are PSS. We also have $T_{12}, T_{21}, T_{33}, \ldots, T_{kk}, T_{k+1k}, \ldots, T_{nk}$ are PSS. Hence by Lemma 5 we have that T_{12} and T_{21} are linear combinations of I and J_n. Furthermore the numbers $a_{11}, a_{12}, a_{21}, a_{22}$ are all non-zero. We are now in a similar situation as that of the case $n = 2$. Since I and J_n are just direct sums of copies of I_2 and J_2, the same argument used in $n = 2$ applies here also. Hence we have $F_{12} = a_{12}I$, $F_{21} = a_{21}J_n$ and $a_{12}^2 = a_{22}^2$. Similarly $F_{1i} = a_{1i}I$ and $a_{1i}^2 = a_{ii}^2$ for all $i = 1, \ldots, k$. Now let $U \in O_n(R)$ be of the form

$$U = \begin{vmatrix} a_{11} & \ldots & a_{1k} & 0 & \ldots & 0 \\ & & * & & & \end{vmatrix}.$$

Then recalling the fact that $\sum_{i=1}^{n} a_{ii}^2 = 1$ and $a_{jj} = 0$, $k + 1 \leqq j \leqq n$, we conclude $T \circ M(I, U)(E_{11}) = I$. This completes the proof of the theorem.

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