

A CHARACTERIZATION OF LATTICE-ORDERED GROUPS BY THEIR CONVEX L -SUBGROUPS

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1. Introduction

In this paper we show that whether or not a group admits a lattice-order often depends upon whether or not it possesses a set of subgroups that satisfy certain algebraic conditions. Using these techniques we are able to determine large classes of groups that can be lattice-ordered.

There are many places in the literature where a class of groups that can be totally ordered has been characterized as those groups with a chain of subgroups that satisfy certain restrictions — for example see Iwasawa [6] and Neumann [8]. Malcev [7], Podderyugin [9] and Rieger [10] have derived conditions on a chain of subgroups that are necessary and sufficient for the ordering of a group (an account of part of their work can be found in Fuchs [5]). In our theory we recover the results of Podderyugin and Rieger.

In section 2 we recall some of the definitions and the theory of lattice-ordered groups (“ l -groups”) that we need. Most of these results are from [4]. We then prove the necessity of the conditions that are used in section 3 to characterize l -groups. Also we derive some new results. For example, Theorem 2.1 has proven to be extremely useful in this paper and also in other contexts.

In section 3 we determine all those groups that admit a lattice-order which is finite valued. We also determine those groups that admit a normal lattice-order. In fact, the techniques that we use can only describe normal lattice-orders, but since this includes all representable l -groups as well as all finite valued l -groups, this class is quite large.

The results in section 3 all have content if we restrict our attention to abelian groups. In section 4 we show that our theory fits in nicely with the representation theory for abelian l -groups that is developed in [2].

Throughout this paper we assume that the reader is familiar with the results in [4].

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NOTATION. If A and B are subgroups of a group G , then $N(A)$ will denote the normalizer of A in G , $[A, B]$ will denote the subgroup of G that is generated by all commutators $-a-b+a+b$ where $a \in A$ and $b \in B$, $A \triangleleft B$ will denote that A is a normal subgroup of B , and $A \setminus B$ will denote the set of all elements in A but not in B . We shall denote the null set by $\square$. We shall write l -group (o -group) for lattice-ordered group (totally ordered group). If x belongs to an l -group, G , then $G(x)$ will denote the subgroup $\{y \in G \mid |y| < n|x| \text{ for some } n > 0\}$. We shall denote the fact that $a, b \in G$ are not comparable by $a \parallel b$ and also that subsets X and Y are not comparable with respect to the inclusion relation by $X \parallel Y$.

2. Some properties of l -groups

Throughout this section G will denote an l -group. We first recall some definitions and results from [4] that we need. A subgroup C of G is called a *convex l -subgroup* if C is a sublattice of G such that $0 < g < c \in C$ implies $g \in C$. The set of all convex l -subgroups of G is a complete distributive sublattice of the lattice of all subgroups of G . A convex l -subgroup C is called *prime (regular)* if the convex l -subgroups of G that contain it form a chain (if it is maximal without containing some element $g \in G$). Theorem 3.2 in [4] establishes six equivalent definitions of a prime convex l -subgroup. In particular, each regular convex l -subgroup is prime. If C is a regular convex l -subgroup of G , then C is a proper subgroup of the intersection C^* of all convex l -subgroups that properly contain C . In fact, this property is equivalent to regularity. Also if C is a prime or a regular convex l -subgroup of G , then so is $-g+C+g$ for all $g \in G$.

Let C be a regular convex l -subgroup of G that is covered by the convex l -subgroup C^* and suppose that $C \triangleleft C^*$.

$$(a) \quad [[N(C), N(C)], C^*] \subseteq C.$$

PROOF. Since C^* is the unique convex l -subgroup that covers C it follows that $N(C) \subseteq N(C^*)$. Thus each $a \in N(C)$ induces an o -automorphism $\hat{a}$ of C^*/C

$$(C+x)\hat{a} = C-a+x+a.$$

But C^*/C is o -isomorphic to a naturally ordered additive group of real numbers. Thus the o -automorphisms of C^*/C are essentially multiplications by positive real numbers, and hence $\hat{a}\hat{b} = \hat{b}\hat{a}$ for all $a, b \in N(C)$, which is equivalent to property (a).

(b) If $x \in C^* \setminus C$ and $a_1, \dots, a_n \in N(C)$, then

$$(-a_1+x+a_1)+\dots+(-a_n+x+a_n) \notin C.$$

PROOF. We may assume that $C+x > C$ and thus it follows that $C+(-a_i+x+a_i) > C$ for $i = 1, \dots, n$, and hence

$$C+(-a_1+x+a_1)+\dots+(-a_n+x+a_n) > C.$$

Note that (a) and (b) imply that C^*/C is an $N(C)$ -group such that the elements of $N(C)$ commute as operators and the total order on C^*/C is an $N(C)$ -order.

Let $\Gamma_1 = \Gamma_1(G)$ be an index set for the set of all regular convex L -subgroups G_γ of G . We shall frequently identify this set with Γ_1 . Define $\alpha \leq \beta$ in Γ_1 if $G_\alpha \subseteq G_\beta$. Then Γ_1 is a *root system* (that is, Γ_1 is a partially ordered set and for each $\gamma \in \Gamma_1$, $\{a \in \Gamma_1 \mid a \geq \gamma\}$ is a chain). Let G^γ be the convex L -subgroup of G that covers G_γ ($\gamma \in \Gamma_1$). If $g \in G^\gamma \setminus G_\gamma$, then γ is said to be a *value* of g , and if γ is the only value of g in Γ_1 , then both γ and g are called *special*. It follows from Theorems 3.5 and 3.6 in [4] that if γ is special, then $G_\gamma \triangleleft G^\gamma$. A subset Δ of Γ_1 is called *plenary* if

(i) $\cap \{G_\gamma \mid \gamma \in \Delta\} = 0$, and

(ii) if $g \in G^\gamma$ ($\gamma \in \Delta$), then there exists a value α of g in Δ that exceeds γ .

If in addition, for each $\alpha \in \Delta$ and $g \in G$

(iii) $G_\alpha \triangleleft G^\alpha$, and

(iv) $-g+G_\alpha+g \in \{G_\gamma \mid \gamma \in \Delta\}$,

then we shall call Δ a *normal plenary subset* of Γ_1 . Note that Γ_1 satisfies (i), (ii) and (iv) but that it need not satisfy (iii). We shall call the lattice-order of G *normal* if Γ_1 contains a normal plenary subset. Let Δ be the set of all special elements of Γ_1 . Clearly for each $\gamma \in \Delta$, (iii) and (iv) are satisfied. Thus Δ is a normal plenary subset of Γ_1 provided that it is a plenary subset.

G is said to be *representable* if there exists an L -isomorphism of G into a cardinal sum of o -groups. Byrd [1] has shown that G is representable if and only if for each $\gamma \in \Gamma_1$ and $g \in G$, $-g+G_\gamma+g$ and G_γ are comparable, and if G is representable, then Γ_1 is normal.

We shall call a subset $\{g_\lambda \mid \lambda \in \Delta\}$ of G *disjoint* if each $g_\lambda > 0$ and $g_\lambda \wedge g_{\lambda'} = 0$ for $\lambda \neq \lambda'$. A convex L -subgroup C of G is called *$\mathcal{L}$ -closed* if $\bigvee g_\lambda \in C$ for each disjoint subset $\{g_\lambda \mid \lambda \in \Delta\}$ of C for which $\bigvee g_\lambda$ exists.

LEMMA 2.1. *If $\gamma \in \Gamma_1$ is special, then G_γ is $\mathcal{L}$ -closed.*

PROOF. Suppose (by way of contradiction) that $0 < g = \bigvee g_\lambda \notin G_\gamma$, where $\{g_\lambda \mid \lambda \in \Delta\}$ is a disjoint subset of G_γ . Let α be the value of g such that $\alpha \geq \gamma$, and pick a special element $0 < y \in G^\gamma \setminus G_\gamma$. Let $x = y \wedge g$. Then $x \in G^\gamma \setminus G_\gamma$ is special, and

$$x = g \wedge x = (\bigvee g_\lambda) \wedge x = \bigvee (g_\lambda \wedge x)$$

where each $g_\lambda \wedge x$ belongs to $G_\gamma \cap G(x)$. But $G(x)$ is a lexicographic extension

of $G_\gamma \cap G(x)$ ([4] Theorem 3.6) and hence for each positive integer n , $x > n(g_\lambda \wedge x)$ and it follows that

$$x \geq \bigvee n(g_\lambda \wedge x) = n \bigvee (g_\lambda \wedge x) = nx$$

for all $n > 0$ which is impossible.

LEMMA 2.2. *If $g = \bigvee a_\alpha = \bigvee b_\beta$, where $\{a_\alpha \mid \alpha \in A\}$ and $\{b_\beta \mid \beta \in B\}$ are disjoint sets of special elements, then $\{a_\alpha \mid \alpha \in A\} = \{b_\beta \mid \beta \in B\}$.*

PROOF. Let δ be the value of b_β . It is easy to show that $\bigvee_{\lambda \neq \beta} b_\lambda$ exists and is disjoint from b_β , and hence $g = b_\beta + \bigvee_{\lambda \neq \beta} b_\lambda$. Thus since G_δ is prime, $\bigvee_{\lambda \neq \beta} b_\lambda \in G_\delta$ and hence $g \in G^\delta \setminus G_\delta$. Now by Lemma 2.1, there exists $a_\alpha \notin G_\delta$ and since $\{a_\alpha \mid \alpha \in A\}$ is a disjoint set this a_α is unique. If $\gamma \leq \delta$, then $a_\alpha, b_\beta \notin G_\gamma$, and hence $\bigvee_{\mu \neq \alpha} a_\mu$ and $\bigvee_{\lambda \neq \beta} b_\lambda$ must belong to G_γ . Thus $G_\gamma + a_\alpha = G_\gamma + g = G_\gamma + b_\beta$. If $\gamma \not\leq \delta$, then $a_\alpha, b_\beta \in G_\gamma$. Therefore $G_\gamma + a_\alpha = G_\gamma + b_\beta$ for all $\gamma \in \Gamma_1$ and hence $a_\alpha = b_\beta$.

COROLLARY. *If $b_\beta \in G^\delta \setminus G_\delta$, then $g \in G^\delta \setminus G_\delta$.*

THEOREM 2.1. *Let Δ be the set of special elements in Γ_1 . Then the following are equivalent.*

- (a) *Each $0 < g \in G$ is the join of disjoint special elements in G .*
- (b) *Δ is a plenary subset of Γ_1 .*

If this is the case, then Δ is the set of all regular $\mathcal{L}$ -closed convex l -subgroups of G and Δ is a normal plenary subset of Γ_1 . Moreover, the representation $g = \bigvee g_\lambda$ of g as the join of disjoint special elements is unique and the values of g in Δ are precisely the values of the g_λ .

PROOF. Consider $0 < g \in G$ and let Δ_g be the set of all values of g in Δ .

(a) $\rightarrow$ (b). $g = \bigvee g_\lambda$, where $\{g_\lambda \mid \lambda \in \Lambda\}$ is a disjoint set of special elements. If δ is the value of g_λ , then by the above corollary $\delta \in \Delta_g$. Thus $\bigcap \{G_\gamma \mid \gamma \in \Delta\} = O$. If $g \notin G_\alpha$ ($\alpha \in \Delta$), then by Lemma 2.1 there exists $g_\lambda \notin G_\alpha$ and so the value of g_λ is $\geq \alpha$ and it is also a value of g . Thus Δ is a plenary subset of Γ_1 .

(b) $\rightarrow$ (a). Let $\delta \in \Delta_g$ and pick $0 < x \in G^\delta \setminus G_\delta$ so that x is special and $G_\delta + x > G_\delta + g$. If $\delta \not\geq \alpha \in \Delta$, then $x \in G_\alpha$ and hence $G_\alpha + x \wedge g = G_\alpha + x \wedge G_\alpha + g = G_\alpha$. If $\delta \geq \alpha \in \Delta$, then since the right cosets of G_α are totally ordered and $G_\delta + x > G_\delta + g$ it follows that $G_\alpha + x > G_\alpha + g$. Thus $G_\alpha + x \wedge g = G_\alpha + x \wedge G_\alpha + g = G_\alpha + g$. If we set $g(\delta) = x \wedge g$, then we have shown that

$$G_\alpha + g(\delta) = \begin{cases} G_\alpha + g & \text{for all } \delta \geq \alpha \in \Delta \\ G_\alpha & \text{for all } \delta \not\geq \alpha \in \Delta. \end{cases}$$

Clearly δ is the only value of $g(\delta)$ in Δ and hence ([4] Proposition 3.11) δ is the only value of $g(\delta)$ in Γ_1 . Thus $g(\delta)$ is special. For each $\delta \in \Delta_g$ define

a $g(\delta)$ as above. Then the $g(\delta)$ are special and pairwise disjoint ([4] Proposition 3.10). In order to complete the proof it suffices to show that $g = \bigvee g(\delta)$ ($\delta \in \Delta_g$). Clearly g exceeds each $g(\delta)$. Suppose that $h \in G$ and $h \geq g(\delta)$ for all $\delta \in \Delta_g$ and let α be a value of $g-h$ in Δ . If $g \in G_\alpha$, then $G_\alpha + g - h = G_\alpha - h < G_\alpha$ since $h \geq 0$, and if $g \notin G_\alpha$, then $\alpha \leq \delta \in \Delta_g$ and hence $G_\alpha \neq G_\alpha + g - h = G_\alpha + g(\delta) - h < G_\alpha$. Thus $g-h \leq 0$ and hence $g = \bigvee g(\delta)$. Therefore (a) and (b) are equivalent.

Now suppose that (a) and (b) hold and that γ is a value of $0 < g \in G$ in $\Gamma_1 \setminus \Delta$. $g = \bigvee g_\lambda$, where $\{g_\lambda \mid \lambda \in \Lambda\}$ is a disjoint set of special elements. If $g_\lambda \notin G_\gamma$, then $\bigvee_{\lambda' \neq \lambda} g_{\lambda'} \in G_\gamma$ and thus γ is a value of g_λ , but this means that $\gamma \in \Delta$, a contradiction. Therefore $\{g_\lambda \mid \lambda \in \Lambda\} \subseteq G_\gamma$ and $\bigvee g_\lambda \notin G_\gamma$, and hence G_γ is not $\mathcal{L}$ -closed. Thus by Lemma 2.1 Δ is the set of all $\mathcal{L}$ -closed regular convex L -subgroups of G . The rest of the theorem follows from our previous results.

An element $0 < g \in G$ is said to be *irreducible* if

$$g = a + b \text{ and } a \wedge b = 0 \text{ imply } a = 0 \text{ or } b = 0.$$

If $0 < g \in G$ is special with value γ and $g = a + b$, where $a \wedge b = 0$, then one but not both of a and b must belong to G_γ . For if $a, b \notin G_\gamma$, then $0 = a \wedge b \notin G_\gamma$ ([4] Theorem 3.2). If $a \in G_\gamma$, then clearly γ is the only value of $b-a$ and $G_\gamma + b - a = G_\gamma + b > G_\gamma$. Thus $b > a$ and hence $0 = b \wedge a = a$. Therefore each special element of G is irreducible.

COROLLARY. *If the set Δ of all special elements of Γ_1 is a plenary subset of Γ_1 , then $0 < g \in G$ is special if and only if it is irreducible.*

PROOF. If g is not special, then by our theorem $g = \bigvee g_\lambda$, where $\{g_\lambda \mid \lambda \in \Lambda\}$ is a disjoint set of special elements. Therefore $g = g_\alpha + \bigvee_{\lambda \neq \alpha} g_\lambda$, and hence g is reducible.

3. Characterizations of lattice-ordered groups

Throughout this section we shall assume that G is a group with a set $\{G_\delta \mid \delta \in \Delta\}$ of proper subgroups such that

$$(1) \quad \bigcap \{G_\delta \mid \delta \in \Delta\} = 0$$

and such that for each $\delta \in \Delta$ the following properties are satisfied.

(2) $\{G_\alpha \mid \alpha \in \Delta \text{ and } G_\alpha \supset G_\delta\}$ is a chain whose intersection G^δ properly contains G_δ .

(3) If $g \in G \setminus G^\delta$, then there exists an $\alpha \in \Delta$ such that $G_\alpha \supseteq G^\delta$ and $g \in G^\alpha \setminus G_\alpha$.

(4) If $(-a_1 + x + a_1) + \cdots + (-a_n + x + a_n) \in G_\delta$, where $x \in G^\delta$ and $a_1, \dots, a_n \in N(G)_\delta$, then $x \in G_\delta$.

(5) $-g + G_\delta + g \in \{G_\delta \mid \delta \in \Delta\}$ for all $g \in G$.

$$(6) \quad [G^\delta, G^\delta] \subseteq G_\delta \text{ and } [[N(G_\delta), N(G_\delta)], G^\delta] \subseteq G_\delta.$$

It follows from the material in the last section that if H is an l -group, then each normal plenary subset Δ of $\Gamma_1(H)$ satisfies (1) through (6).

If $O \neq g \in G$, then by (1) and (3), $g \in G^\alpha \setminus G_\alpha$ for some $\alpha \in \Delta$ and hence by (4), $ng \in G^\alpha \setminus G_\alpha$ for each positive integer n . Thus G is torsion free, and also if $ng \in G_\delta$, then $g \in G_\delta$, and hence the G_δ and the G^δ are pure subgroups of G .

(a) *If $-g + G_\delta + g = G_\delta$, then $-g + G^\delta + g = G^\delta$, and hence $N(G_\delta) \subseteq N(G^\delta)$.*

$$\begin{aligned} \text{PROOF. } -g + G^\delta + g &= -g + \cap \{G_\alpha \mid G_\alpha \supset G_\delta\} + g \\ &= \cap \{-g + G_\alpha + g \mid G_\alpha \supset G_\delta\} = \cap \{G_\beta \mid G_\beta \supset -g + G_\delta + g\} \\ &= \cap \{G_\beta \mid G_\beta \supset G_\delta\} = G^\delta. \end{aligned}$$

In particular, each element $a \in N(G_\delta)$ induces an automorphism $\hat{a}$ on the group G^δ/G_δ

$$(G_\delta + x)\hat{a} = G_\delta - a + x + a.$$

(b) *Each G^δ/G_δ is an abelian $N(G_\delta)$ -group and the elements of $N(G_\delta)$ commute as operators. Also G^δ/G_δ is $N(G_\delta)$ -torsion-free.*

PROOF. By the first part of (6), G^δ/G_δ is an abelian group, and it is an immediate consequence of the second half of (6) that the elements in $N(G_\delta)$ commute as operators on G^δ/G_δ (see for example [5] p. 51). The content of (4) is that G^δ/G_δ is $N(G_\delta)$ -torsion-free.

Thus by a theorem of Podderiyugin [9] any partial $N(G_\delta)$ -order of G^δ/G_δ can be extended to a total $N(G_\delta)$ -order. In particular, the trivial partial order can be so extended and hence there exists a total $N(G_\delta)$ -order for G^δ/G_δ . Now define α and β in Δ to be equivalent if $G_\alpha = -g + G_\beta + g$ for some $g \in G$, and in each equivalence class pick a group G^δ/G_δ and give it a total $N(G_\delta)$ -order $\mathcal{P}_\delta$. Next define $X = -g + Y + g \in (-g + G^\delta + g)/(-g + G_\delta + g)$ to be positive if $Y \in \mathcal{P}_\delta$. Using the fact that $\mathcal{P}_\delta$ is an $N(G_\delta)$ -order it follows that this definition is independent of the particular choice of g , and that this is a total $N(-g + G_\delta + g)$ -order of $(-g + G^\delta + g)/(-g + G_\delta + g)$.

Define $\alpha < \beta$ in Δ if $G^\alpha \subseteq G_\beta$ or equivalently if $G_\alpha \subset G_\beta$. If $g \in G^\delta \setminus G_\delta$, then we shall say that δ is a *value* of g . Note that each $O \neq g \in G$ has at least one value and that the set of all values of g is a trivially ordered subset of Δ which we shall denote by Δ_g . Define $O \neq g \in G$ to be *strictly positive* (notation $g > O$) if $G_\delta + g > G_\delta$ in G^δ/G_δ for all $\delta \in \Delta_g$.

PROPOSITION 3.1. *G is a semiclosed po-group and the G_δ and the G^δ are pure convex subgroups of G . We shall call a partial order of G that is defined in this way a Δ -order.*

PROOF. Let $P = \{g \in G \mid g > O\}$, and consider $a, b \in P$ and $\alpha \in \Delta_a$. Then $\alpha \in \Delta_{-a}$ and $G_\alpha - a < G_\alpha$, and so $P \cap -P = \square$. If $a+b = O$, then $\alpha \in \Delta_b$ and $G_\alpha + a = G_\alpha - b < G_\alpha$, which contradicts the fact that $a \in P$. Thus $a+b \neq O$ and hence there exists a value γ of $a+b$. Clearly $\gamma \in \Delta_a \cup \Delta_b$ and $a, b \in G_\gamma$. Thus $G_\gamma + a+b = G_\gamma + a + G_\gamma + b > G_\gamma$ and hence $P+P \subseteq P$. If $-g+a+g \in G_\beta \setminus G_\beta$, then $a \in (g+G_\beta-g) \setminus (g+G_\beta-g)$ and hence $(g+G_\beta-g)+a$ is positive. Thus $G_\beta - g + a + g = -g + (g+G_\beta-g) + a + g$ is positive and hence $-g+a+g \in P$. Therefore G is a po -group.

If $ng \in P$, where $g \in G$ and $n > 0$, and $\gamma \in \Delta_g$, then $\gamma \in \Delta_{ng}$ and $n(G_\gamma + g) = G_\gamma + ng > G_\gamma$ in the o -group G_γ/G_γ . Thus $G_\gamma + g > G_\gamma$ and it follows that $g \in P$ and thus the po is semiclosed.

If $0 < x < g \in G_\delta$ and $x \notin G_\delta$, then there exists $\delta \leq \beta \in \Delta_x$ and since $g \in G_\beta$, $\beta \in \Delta_{x-g}$. Thus $G_\beta < G_\beta + x - g < G_\beta$, which is impossible. Therefore $x \in G_\delta$ and hence G_δ is convex. Moreover, since G^δ is the intersection of convex subgroups it must also be convex.

We have not yet made use of our hypothesis that Δ is a root system, but we shall make repeated use of it in determining when a Δ -order is a lattice-order. Note that if Δ is a chain, then each Δ -order of G is a total order. Thus a group H can be totally ordered if and only if it contains a chain of subgroups $\{H_\delta \mid \delta \in \Delta\}$ that satisfy (1) through (6). This is entirely similar to the results of Podderugin [9] and Rieger [10] about o -groups.

We shall say that $\delta \in \Delta_g$ is a *positive (negative) value of g* if $G_\delta + g > G_\delta$ ($G_\delta + g < G_\delta$). For each subset Π of Δ let Π' be the *ideal* determined by Π ,

$$\Pi' = \{\delta \in \Delta \mid \delta \leq \gamma \text{ for some } \gamma \in \Pi\}.$$

PROPOSITION 3.2. *If Π is the set of positive values of $g \in G$ in a Δ -order of G and if $h \in \cap G_\gamma$ ($\gamma \in \Delta \setminus \Pi'$), and $h-g \in \cap G_\gamma$ ($\gamma \in \Pi'$) then $h = g \vee O$.*

PROOF. We first show that $\Pi = \Delta_h$. If $\delta \in \Pi$, then $G_\delta + h = G_\delta + g > G_\delta$ and hence $h \in G^\delta \setminus G_\delta$. Conversely suppose that $\delta \in \Delta_h$. If $\delta \in \Delta \setminus \Pi'$, then $h \in G_\delta$ and if $\delta \in \Pi' \setminus \Pi$, then $\delta < \gamma \in \Pi \subseteq \Delta_h$, both of which are impossible. Thus $\delta \in \Pi$ and hence $\Pi = \Delta_h$. In particular, h has only positive values and hence $h \geq O$. In order to prove that $h \geq g$ we must show that each value γ of $h-g$ is positive. If $\gamma \in \Pi'$, then $h-g \in G_\gamma$, a contradiction. If $\gamma \in \Delta \setminus \Pi'$, then $h \in G_\gamma$ and $G_\gamma + h-g = G_\gamma - g > G_\gamma$ (because γ must be a negative value of g). Therefore $h \geq g$ and O .

Finally suppose that $c \in G$ and $h \neq c \geq g$ and O , and consider $\gamma \in \Delta_{c-h}$. If $h \in G_\gamma$, then $G_\gamma + c-h = G_\gamma + c > G_\gamma$. If $h \notin G_\gamma$, then there exists $\gamma \leq \beta \in \Delta_h = \Pi$ and hence $\gamma \in \Pi'$. Thus $G_\gamma + h-c = G_\gamma + g-c$ and hence $G_\gamma + c-h = G_\gamma + c-g > G_\gamma$ because $c \geq g$. Thus all the values of $c-h$ are positive and hence $c \geq h$. Therefore $h = g \vee O$.

In more detail the hypothesis of this proposition is

- (i) $\Delta_h \subseteq \Delta_g$,
- (ii) if γ is a negative value of g , then $h \in G_\alpha$ for all $\gamma \geq \alpha \in \Delta$, and
- (iii) if γ is a positive value of g , then $h - g \in G_\alpha$ for all $\gamma \geq \alpha \in \Delta$.

Proposition 3.9 in [4] is the corresponding result for l -groups.

THEOREM 3.1. *A Δ -order of G is a lattice-order and the G_δ are prime convex l -subgroups if and only if*

(7) *for each $g \in G$, there exists $h \in \cap G_\gamma$ ($\gamma \in \Delta \setminus \Pi'$) such that $h - g \in \cap G_\gamma$ ($\gamma \in \Pi'$), where Π is the set of positive values of g .*

If this is the case, then $h = g \vee 0$.

PROOF. First suppose that (7) is satisfied. Then by Proposition 3.2, $h = g \vee 0$ exists for each $g \in G$ and so G is an l -group. Also if $g \in G_\delta$, then no value of $g \vee 0$ exceeds δ and so $g \vee 0 \in G_\delta$. Therefore G_δ is an l -subgroup of G and hence by Proposition 3.1, G_δ is a convex l -subgroup of G . Suppose (by way of contradiction) that G_δ is not prime. Then there exists $a, b \in G^+ \setminus G_\delta$ such that $a \wedge b = 0$. Let $\alpha(\beta)$ be the value of $a(b)$ that is $\geq \delta$ and without loss of generality assume that $\alpha \geq \beta$. If $\alpha = \beta$, then $G_\alpha + a$ and $G_\alpha + b$ are strictly positive in the o -group G^α/G_α and hence $G_\alpha = G_\alpha + a \wedge b = G_\alpha + a \wedge G_\alpha + b = \min [G_\alpha + a, G_\alpha + b] > G_\alpha$, a contradiction. If $\alpha > \beta$, then α is a positive value of $a - b$ and hence by Proposition 3.2, $G_\beta + a - b = G_\beta + (a - b) \vee 0 > G_\beta$ in the po set of right cosets of G_β . Thus $G_\beta + a > G_\beta + b$ and hence $G_\beta = G_\beta + a \wedge b = G_\beta + a \wedge G_\beta + b = G_\beta + b > G_\beta$, a contradiction. Therefore each G_δ is a prime convex l -subgroup of G .

Conversely suppose that G is an l -group and the G_δ are prime convex l -subgroups of G . Then it follows from the proof of Proposition 3.9 in [4] that (7) is satisfied.

COROLLARY. *If for each $g \in G$ and each subset Π of Δ_g there exists $h \in \cap G_\gamma$ ($\gamma \in \Delta \setminus \Pi'$) such that $h - g \in \cap G_\gamma$ ($\gamma \in \Pi'$), then every Δ -order of G is a lattice order for which the G_δ are prime convex l -subgroups.*

Theorem 3.6 provides a converse to this corollary. It is easy to show that a wreath product G of one torsion free abelian group by another has a natural set of subgroups $\{G_\delta \mid \delta \in \Delta\}$ that satisfy (1) through (6) and also satisfy the hypothesis of this corollary. Thus each Δ -order of G is a lattice-order. However, since G contains elements a and b such that $-a + b + a = -b$, it is clear that G does not admit a total order.

Suppose that we have a Δ -order for G that satisfies (7) and let Δ be the set of all regular convex l -subgroups M of G such that $M \supseteq G_\delta$ for some $\delta \in \Delta$. Then it is easy to verify that Δ is a normal plenary subset of $\Gamma_1(G)$, and hence the Δ -order is a normal lattice order. We have proven the following theorem.

THEOREM 3.2. *A group H admits a normal lattice-order if and only if there exists a set $\{H_\delta \mid \delta \in \Delta\}$ of proper subgroups of H that satisfy (1) through (7).*

This result is not too satisfactory because whether or not (7) is satisfied depends upon the particular choice of the total order for the G^δ/G_δ . For the remainder of this section we derive conditions that insure that all the Δ -orderings of G satisfy (7). Note that the Corollary to Theorem 3.1 provides one such condition.

An element $\delta \in \Delta$ will be called *special* if there exist $g \in G^\delta \setminus G_\delta$ whose only value is δ . In this case g will be called Δ -special. Note that the hypothesis in the Corollary to Theorem 3.1 forces each $\delta \in \Delta$ to be special.

LEMMA 3.1. *If a and b are Δ -special with values $\alpha \parallel \beta$ respectively, then $a+b = b+a$ and $\Delta_{a+b} = \{\alpha, \beta\}$.*

PROOF. Let $C = \cap G_\gamma$ ($\gamma \leq \alpha$) and $D = \cap G_\gamma$ ($\gamma \leq \beta$). Then since b has only the value β it follows that $b \in C$, and similarly $a \in D$. Since $G_\alpha \triangleleft G^\alpha$ it follows that an inner automorphism of G by an element in G^α must induce a permutation on $\{\gamma \in \Delta \mid \gamma \leq \alpha\}$. Therefore $C \triangleleft G^\alpha$ and similarly $D \triangleleft G^\beta$. $D+g = D+a+b = D+b$, where $g = a+b$, and hence $a+b = g = b+d$, where $d \in D$.

If $\gamma \leq \beta$, then $G_\gamma + a = G_\gamma = G_\gamma + d$. If $\delta \not\leq \beta$, then $b \in G_\delta$. If $a \in G_\delta$, then $d = -b+a+b \in G_\delta$ and hence $G_\delta + a = G_\delta = G_\delta + d$. If $a \notin G_\delta$, then $\delta \leq \alpha$ and hence $C \subseteq G_\delta$. But then $C+a = C+a+b = C+b+d = C+d$ and hence $G_\delta + a = G_\delta + d$. Therefore $G_\delta + a = G_\delta + d$ for all $\delta \in \Delta$ and hence $a = d$.

Since $a \in G_\beta$, it follows that $\beta \in \Delta_a$ and similarly $\alpha \in \Delta_b$. Consider $\gamma \in \Delta_a$. If $a, b \notin G_\gamma$, then $\alpha, \beta \geq \gamma$ and $\alpha \parallel \beta$, but this contradicts the fact that Δ is a root system. Thus exactly one of a and b belongs to G_γ and it follows that $\gamma = \alpha$ or $\gamma = \beta$.

THEOREM 3.3. *If Δ_g is finite for each $g \in G$, then the following are equivalent.*

(8) *Each $0 \neq g \in G$ has a representation $g = g_1 + \cdots + g_n$, where g_i is Δ -special with value δ_i and $\delta_i \parallel \delta_j$ if $i \neq j$.*

(b) *Each Δ -order of G is a lattice-order for which the G_δ are prime convex L -subgroups.*

(c) *There exists a Δ -order of G that is a lattice-order and such that the G_δ are prime convex L -subgroups.*

PROOF. (8) $\rightarrow$ (b). Select a Δ -order for G and consider $0 \neq g \in G$. Then $g = g_1 + \cdots + g_n$, where g_i is Δ -special with value δ_i and $\delta_i \parallel \delta_j$ if $i \neq j$. By Lemma 3.1 $\Delta_g = \delta_1, \dots, \delta_n$ and $g_i + g_j = g_j + g_i$. Thus we may

assume that $\Pi = \delta_1, \dots, \delta_k$ ($k \leq n$) is the set of positive values of g . To complete the proof it suffices to show that $h = g_1 + \dots + g_k \in \cap G_\gamma$ ($\gamma \in \Delta \setminus \Pi'$), and $h - g \in \cap G_\gamma$ ($\gamma \in \Pi'$). But this follows at once from the fact that $g_1, \dots, g_k \in \cap G_\gamma$ ($\gamma \in \Delta \setminus \Pi'$) and $g_{k+1}, \dots, g_n \in \cap G_\gamma$ ($\gamma \in \Pi'$).

(c) $\rightarrow$ (8). Select a Δ -order for G that is a lattice-order and such that each G_δ is a prime convex l -subgroup of G . Let $\mathcal{A}$ be the set of regular convex l -subgroups M of G such that $M \supseteq G_\delta$ for some $\delta \in \Delta$. Then $\mathcal{A}$ is a plenary subset of $\Gamma_1(G)$ and each $g \in G$ has at most a finite number of values in $\mathcal{A}$. (8) is now an immediate consequence of Theorem 3.7 in [4].

REMARK. Suppose that G satisfies (8) and that we have selected a Δ -order for G . Then from the theory in [4], Theorem 2.1 and Theorem 3.3 we have the following results. The representation of g given in (8) is unique and $g \vee 0$ is just the sum of the positive g_i . $\Gamma_1(G) = \mathcal{A}$ and the lattice $\mathcal{M}$ of all convex l -subgroups of G is freely generated by $\mathcal{A}$.

It is easy to show by an example that even if G is abelian and generated by its Δ -special elements, (8) need not be satisfied.

THEOREM 3.4. *There exists a lattice-ordering of a group H such that each element has at most a finite number of values if and only if there exists a set $\{H_\delta \mid \delta \in \Delta\}$ of proper subgroups of H that satisfy properties (1) through (6) and (8). If this is the case, then each Δ -order of H is a lattice-order.*

PROOF. Suppose there exists a lattice-order for H such that each element has at most a finite number of values. Then each element in $\Gamma_1(H)$ is special ([4] Theorem 3.9) and hence $H_\gamma \triangleleft H^\gamma$ for all $\gamma \in \Gamma_1(H)$. Thus $\{H_\gamma \mid \gamma \in \Gamma_1(H)\}$ satisfies (1) through (6), and (8) follows from Theorem 2.1 or from Theorem 3.7 in [4]. The converse is an immediate consequence of Theorem 3.3.

THEOREM 3.5. *If there exists a Δ -order of G that is a lattice-order and for which the G_δ are prime convex l -subgroups, then the following are equivalent.*

- (a) G is representable (as a subdirect sum of a cardinal sum of 0-groups).
- (b) For each $\delta \in \Delta$ and $g \in G$, $-g + G_\delta + g \subseteq G_\delta$ or $\supseteq G_\delta$.

PROOF. Byrd [1] has shown that if M is a prime convex l -subgroup of a representable l -group H , then $-h + M + h$ is comparable with M for all $h \in H$, and hence it follows that (a) implies (b). Conversely suppose that (b) is satisfied. Let $N_\delta = \cap \{-g + G_\delta + g \mid g \in G\}$. Then $N_\delta \triangleleft G$ and since N_δ is the intersection of a chain of prime convex l -subgroups, N_δ is a prime convex l -subgroup ([4] Theorem 3.2). Thus G/N_δ is an o -group. The natural homomorphism of G into the large cardinal sum of the G/N_δ is a representation of G .

THEOREM 3.6. *If for each $\delta \in \Delta$ and $g \in G$, $-g + G_\delta + g \subseteq G_\delta$ or $\supseteq G_\delta$, then the following are equivalent.*

(a) Each Δ -order of G is a lattice-order for which the G_δ are prime convex L -subgroups.

(b) If $g \in G_\gamma \setminus G_\gamma$ for all $\gamma \in \Pi \subseteq \Delta$, then there exists $h \in \cap G_\gamma (\gamma \in \Delta \setminus \Pi')$ such that $h - g \in \cap G_\gamma (\gamma \in \Pi')$.

(c) If Q_1 and Q_2 are dual ideals of Δ such that $Q_1 \cup Q_2 = \Delta$, then

$$\bigcap_{Q_1 \cap Q_2} G_\delta = \bigcap_{Q_1} G_\delta \oplus \bigcap_{Q_2} G_\delta.$$

PROOF. (a) $\rightarrow$ (b). If α and β are values of g , then $G_\alpha \parallel G_\beta$ and hence G_α and G_β are not conjugate. Thus we can choose a Δ -order for G so that $G_\gamma + g > G_\gamma$ for all $\gamma \in \Pi \subseteq \Delta$, and $G_\gamma + g < G_\gamma$ for all other values γ of g . Then by Theorem 3.1, $h = g \vee 0$ satisfies (b).

(b) $\rightarrow$ (c). Since $Q_1 \cap Q_2 \subseteq Q_1$ and Q_2

$$\bigcap_{Q_1 \cap Q_2} G_\delta \supseteq \bigcap_{Q_1} G_\delta \text{ and } \bigcap_{Q_2} G_\delta$$

and since

$$\cap \{G_\delta \mid \delta \in \Delta = Q_1 \cup Q_2\} = 0$$

we have

$$\left(\bigcap_{Q_1} G_\delta\right) \cap \left(\bigcap_{Q_2} G_\delta\right) = 0.$$

In order to show that $\cap G_\delta (\delta \in Q_1)$ is normal in $\cap G_\delta (\delta \in Q_1 \cap Q_2)$ it suffices to show that if $g \in \cap G_\delta (\delta \in Q_1 \cap Q_2)$, then the inner automorphism of G determined by g induces a mapping of Q_1 into itself. If there exists $\delta \in Q_1$ such that $-g + G_\delta + g = G_\beta$, where $\beta \notin Q_1$, then $\beta \in Q_2$ and $\delta > \beta$. Thus $\delta \in Q_1 \cap Q_2$ and hence $g \in G_\delta$, but then $-g + G_\delta + g = G_\delta$, a contradiction. Thus in general

$$\bigcap_{Q_1 \cap Q_2} G_\delta \supseteq \bigcap_{Q_1} G_\delta \oplus \bigcap_{Q_2} G_\delta.$$

Consider $g \in \cap G_\delta (\delta \in Q_1 \cap Q_2)$ and let $\Pi_1(\Pi_2)$ be the set of all values of g in $Q_1(Q_2)$, and let Π'_1 and Π'_2 be the corresponding ideals in Δ . Since $Q_1 \cup Q_2 = \Delta$, $\Pi_1 \cup \Pi_2$ is the set of all values of g .

$$\Pi'_1 \subseteq Q_1 \subseteq \Delta \setminus \Pi'_2 \text{ and } \Pi'_2 \subseteq Q_2 \subseteq \Delta \setminus \Pi'_1.$$

For if $\alpha \in \Pi'_1$, then $\alpha \leq \beta \in \Pi_1 \subseteq Q_1$, and if $\alpha \notin Q_1$, then $\alpha \in Q_2$ and hence $\beta \in Q_1 \cap Q_2$. But then $g \in G_\beta$ which contradicts the fact that β is a value of g . If $\gamma \in Q_1 \cap \Pi'_2$, then $\gamma \leq \delta \in \Pi_2 \subseteq Q_2$ and hence $\delta \in Q_1 \cap Q_2$, which is impossible.

By (b) there exist elements h and k in G such that

$$\begin{aligned} h &\in \bigcap_{\Delta \setminus \Pi'_1} G_\gamma \subseteq \bigcap_Q G_\gamma \text{ and } h - g \in \bigcap_{\Pi'_1} G_\gamma \\ k &\in \bigcap_{\Delta \setminus \Pi'_2} G_\gamma \subseteq \bigcap_{Q_1} G_\gamma \text{ and } k - g \in \bigcap_{\Pi'_2} G_\gamma. \end{aligned}$$

To complete the proof it suffices to show that $g = k + h$. If $\gamma \in \Pi'_1$, then $k, h - g \in G_\gamma$ and if $\gamma \in \Pi'_2$, then $h, k - g \in G_\gamma$. If $\gamma \notin \Pi'_1 \cup \Pi'_2$, then $h, k \in G_\gamma$, and if $g \notin G_\gamma$, then there exists a value β of g such that $\gamma \leq \beta \in \Pi_1 \cup \Pi_2$ and so $\gamma \in \Pi'_1 \cup \Pi'_2$, a contradiction. Thus using the fact that $h + k = k + h$ we have that $h + k - g \in \cap G_\gamma = 0$ ($\gamma \in \Delta$) and hence $g = k + h$.

(c) $\rightarrow$ (a). We assume that G has a Δ -order and pick $0 \neq g \in G$. Let Π be the set of all positive values of g . By Theorem 3.1 it suffices to show that there exists $h \in \cap G_\gamma$ ($\gamma \in \Delta \setminus \Pi'$) such that $h - g \in \cap G_\gamma$ ($\gamma \in \Pi'$). Let

$$Q_1 = \{\delta \in \Delta \mid \delta \geq \gamma \text{ for some } \gamma \in \Pi'\} \supseteq \Pi' \text{ and } Q_2 = \Delta \setminus \Pi'.$$

Then Q_1 and Q_2 are dual ideal ideals of Δ and $Q_1 \cup Q_2 = \Delta$.

Suppose (by way of contradiction) that $g \notin G_\gamma$ for some $\gamma \in Q_1 \cap Q_2$. Then there exists a value β of g such that $\beta \geq \gamma$. Since $\gamma \in Q_2$, $\gamma \notin \Pi'$ and hence β is a negative value of g . Since $\gamma \in Q_1$, $\beta \geq \gamma \geq \delta \in \Pi'$ and hence β is a positive value of g , which is impossible. Therefore

$$g = k + h \in \bigcap_{Q_1 \cap Q_2} G_\gamma = \bigcap_{Q_1} G_\gamma \oplus \bigcap_{Q_2} G_\gamma,$$

where $k \in \cap G_\gamma$ ($\gamma \in Q_1$) and $h \in \cap G_\gamma$ ($\gamma \in Q_2 = \Delta \setminus \Pi'$). If $\gamma \in \Pi'$, then $k \in G_\gamma$ and hence $h - g \in \cap G_\gamma$ ($\gamma \in \Pi'$). This completes the proof of Theorem 3.6.

We can use Theorems 3.5 and 3.6 to characterize a class of groups that admit a representable lattice-order, but instead we shall only apply Theorem 3.6 to representable l -groups. Let H be a representable l -group and let Δ be a normal plenary subset of $\Gamma_1(H)$. Then $\{H_\gamma \mid \gamma \in \Delta\}$ satisfies properties (1) through (7) and $-h + H_\gamma + h$ is comparable with H_γ for all $\gamma \in \Delta$ and $h \in H$. Also since the given lattice-order is a Δ -order, we have the equivalence of (a), (b) and (c) in Theorem 3.6.

THEOREM 3.7. *If H is a representable l -group and Δ is a normal plenary subset of $\Gamma_1(H)$, then each of (a), (b) and (c) is equivalent to*

(d) *each $\delta \in \Delta$ is special and if $\bigvee a_\alpha$ exists, where $\{a_\alpha \mid \alpha \in A\}$ is a disjoint set of positive elements of H , then $\bigvee a_\beta$ exists for each set $\{a_\beta \mid \beta \in B\} \subseteq \{a_\alpha \mid \alpha \in A\}$.*

PROOF. (b) $\rightarrow$ (d). Consider $0 \leq h \in H^\delta \setminus H_\delta$, where $\delta \in \Delta$. By (b) there exists $k \in \cap H_\gamma$ ($\gamma \not\leq \delta$) such that $k - h \in \cap H_\gamma$ ($\gamma \leq \delta$). In particular δ is the only value of k , and hence each $\delta \in \Delta$ is special. By Theorem 2.1, $h = \bigvee h_\lambda$, where $\{h_\lambda \mid \lambda \in \Lambda\}$ is a set of disjoint special elements. Moreover, the values of h in Δ are precisely the values of the h_λ . Let Φ be a subset of Δ and let Π be the set of all the values of the h_λ for $\lambda \in \Phi$. Then by (b) there exists $k \in \cap H_\gamma$ ($\gamma \in \Delta \setminus \Pi'$) such that $k - h \in \cap H_\gamma$ ($\gamma \in \Pi'$). Now by Theorem 2.1, k has a unique representation as the join of disjoint special elements. Thus it follows that $k = \bigvee h_\lambda$ ($\lambda \in \Phi$), and hence it is clear that (d) is satisfied.

(d) $\rightarrow$ (a). By Theorem 2.1, each positive element in H has a unique representation as the join of a set of disjoint special elements. Consider $h \in H$.

$$h = h \vee 0 - (-h \vee 0) = \bigvee_A a_\alpha - \bigvee_B b_\beta$$

where $\{a_\alpha \mid \alpha \in A\}$ and $\{b_\beta \mid \beta \in B\}$ are sets of disjoint special elements and $h \vee 0 = \bigvee a_\alpha$ and $-h \vee 0 = \bigvee b_\beta$. (If $h \vee 0 = 0$, then we let $A = \square$ and $\bigvee_A a_\alpha = 0$.) Without loss of generality we may assume that α is the value of a_α in Δ for each $\alpha \in A$ and that β is the value of b_β in Δ for each $\beta \in B$. In particular, $A \cup B$ is the set of values of h in Δ .

Let Π be the set of positive values of h in a new Δ -order of H . Then $\Pi = A^* \cup B^*$, where $A^* \subseteq A$ and $B^* \subseteq B$. It follows by an easy computation that

$$k = \bigvee_{A^*} a_\alpha - \bigvee_{B^*} b_\beta$$

satisfies property (7). Thus the new Δ -order of H is a lattice-order and the H_δ are prime convex L -subgroups. Therefore (a) is satisfied.

In the proof that (d) implies (a) we did not use the fact that H is a representable L -group, hence we have the following corollary.

COROLLARY. *If H is an L -group with a normal plenary subset Δ of $\Gamma_1(H)$ that satisfies (d), then each Δ -order of H is a lattice-order for which the H_δ are prime convex L -subgroups.*

4. Lattice-orderings of an abelian group

If G is an abelian group, then conditions (5) and (6) used in the definition of a Δ -order of G are trivially satisfied and condition (4) simply states that each group $G^\delta \setminus G_\delta$ is torsion free. The latter is equivalent to the hypothesis that G is torsion free and each of the G_δ is a pure subgroup of G . Thus for the remainder of this section we shall assume that G is a torsion free abelian group with a set $\{G_\delta \mid \delta \in \Delta\}$ of proper pure subgroups of G that satisfies the conditions (1), (2) and (3) from section 3.

Using the fact that the set consisting only of the zero subgroup satisfies our hypothesis, Theorem 3.2 simply states that an abelian group admits a lattice-order if and only if it is torsion free, and Theorem 3.5 states that each lattice-ordered abelian group is representable. The other theorems all establish significant relationships between a lattice-order for G and the subgroup structure of G .

Let $V = V(\Delta, G^\delta/G_\delta)$ be the set of all Δ -vectors $v = (\cdots, v_\delta, \cdots)$, where $v_\delta \in G^\delta/G_\delta$, for which the support $S_v = \{\delta \in \Delta \mid v_\delta \neq G_\delta\}$ contains no infinite ascending chains. Choose a Δ -order for G and define $O \neq v \in V$

to be positive if $v_\gamma > G_\gamma$ for every maximal element γ of S_v (such a v_γ will be called a *maximal component* of v). Then V is an l -group ([2] Theorem 2.2).

THEOREM 4.1. *If G is divisible, then there exists an o -isomorphism μ of G into $V(\Delta, G^\delta/G_\delta)$ such that δ is a value of g if and only if $(g\mu)_\delta$ is a maximal component of $g\mu$ and if this is the case $(g\mu)_\delta = G_\delta + g$.*

PROOF. G is a Γ -group, with $\Gamma = \Delta$ (see [3] p. 3) and hence by the embedding theorem in [3], there exists an isomorphism μ of G into V with the above property. It follows from the definition of the Δ -order for G and the lattice order for V that for $O \neq g \in G$ the following are equivalent.

- (a) g is positive in G .
- (b) $G_\delta + g > G_\delta$ for each $\delta \in \Delta_g$.
- (c) All maximal components of $g\mu$ are positive.
- (d) $g\mu$ is positive in V .

Therefore μ is an o -isomorphism of G into V , and we shall call such an o -isomorphism a Δ -isomorphism.

If we remove from (2) the assumption that $\{G_\alpha \mid \alpha \in \Delta \text{ and } G_\alpha \supset G_\delta\}$ is a chain, then $V(\Delta, G^\delta/G_\delta)$ is a po -group and Theorem 4.1 together with its proof remains valid.

As is usual when dealing with abelian po -groups we can dispense with the divisibility assumption. For let $\tilde{G}$ be the d -closure of G (that is, the minimal divisible abelian group that contains G) and let $\tilde{G}^\delta$ ($\tilde{G}_\delta$) be the d -closure of G^δ (G_δ) in $\tilde{G}$. Then $\tilde{G}^+ = \{\tilde{g} \in \tilde{G} \mid n\tilde{g} \in G^+ \text{ for some } n > 0\}$ is the minimal semiclosed po of $\tilde{G}$ that contains the given Δ -order of G . The mapping τ of $G_\delta + g$ upon $\tilde{G}_\delta + g$ is an isomorphism of G^δ/G_δ into $\tilde{G}^\delta/\tilde{G}_\delta$ and $\tilde{G}^\delta/\tilde{G}_\delta$ is the d -closure of $(G^\delta/G_\delta)\tau$. Thus there is a unique extension of the total order of $(G^\delta/G_\delta)\tau$ to a total order of $\tilde{G}^\delta/\tilde{G}_\delta$ and it is easy to verify that

$$\tilde{G}^+ = \{\tilde{g} \in \tilde{G} \mid \tilde{G}_\gamma + \tilde{g} > \tilde{G}_\gamma \text{ for all } \gamma \in \Delta_{\tilde{g}}\}.$$

COROLLARY I. *There exists a Δ -isomorphism of G into $V(\Delta, \tilde{G}^\delta/\tilde{G}_\delta)$.*

COROLLARY II. *If G satisfies (7), then each Δ -isomorphism of G into $V(\Delta, \tilde{G}^\delta/\tilde{G}_\delta)$ is necessarily and l -isomorphism.*

PROOF. Let μ be a Δ -isomorphism of G into V and consider $g \in G$. It suffices to show that $(g \vee O)\mu = g\mu \vee O$. Let Π be the set of positive values of g in Δ and let $h = g \vee O$. If $\gamma \in \Delta \setminus \Pi$, then (by (7)) $h \in G_\gamma$ and hence $(h\mu)_\gamma = \tilde{G}_\gamma$. For if $(h\mu)_\gamma \neq \tilde{G}_\gamma$, then there exists a maximum component $(h\mu)_\beta$ of $h\mu$ such that $\beta \geq \gamma$ and hence $h \in G^\beta \setminus G_\beta$, a contradiction. If $\gamma \in \Pi$ then $h - g \in G_\gamma$ and so $(h\mu)_\gamma - (g\mu)_\gamma = ((h - g)\mu)_\gamma = \tilde{G}_\gamma$. Thus we have

$$(h\mu)_\gamma = \begin{cases} (g\mu)_\gamma & \text{if } \gamma \leq \beta, \\ & \text{where } (g\mu)_\beta \text{ is a maximal positive component of } g\mu \\ \bar{G}_\gamma & \text{otherwise,} \end{cases}$$

but this means that $(g \vee O)\mu = h\mu = g\mu \vee O$.

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