MONTEL ALGEBRAS ON THE PLANE

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1. Introduction. The results of Rudin in [**7**] show that under certain conditions, the maximum modulus principle characterizes the algebra $A(G)$ of functions analytic on an open subset G of the plane C (see below). In [**2**], Birtel obtained a characterization of $A(C)$ in terms of the *Liouville* theorem; he proved that every singly generated F-algebra of continuous functions on C which contains no non-constant bounded functions is isomorphic to $A(C)$ in the compact-open topology. In this paper we show that the *Montel* property of the topological algebra $A(G)$ also characterizes it. In particular, any Montel algebra A of continuous complex-valued functions on G which contains the polynomials and has continuous homomorphism space $M(A)$ homeomorphic to G is precisely $A(G)$.

An example is given to show that this is not true if we do not require $M(A) = G$. For each $n \geqq 1$, a subalgebra of continuous complex-valued functions on G is constructed which contains the polynomials and is isomorphic to $P(G^n)$, the closure of polynomials in n variables in the topology of uniform convergence on compact subsets of the open set G^n in C^n. For polynomially convex open sets G, the algebras so constructed are Montel but cannot be isomorphic to $A(G)$ unless $n = 1$. In case $G = C$, the algebras obtained provide an answer to a question asked in [**3**]: Do there exist subalgebras of continuous functions on the plane which properly contain $A(C)$ but contain no non-constant bounded functions?

2. Preliminaries. We shall use the result of Rudin mentioned above in the following form. Define a *uniform algebra* on a topological space X to be an algebra of continuous complex-valued functions on X which contains the constants and is closed under uniform convergence on compact subsets of X. By a *maximum modulus algebra* on X we shall mean a uniform algebra A on X having the property that for every compact subset K of X, the Šilov boundary of the restriction algebra $A|K$ is contained in the topological boundary of K. Rudin's result can be formulated as follows: if A is a maximum modulus algebra on an open subset G of C, if A contains the polynomials, and if $M(A) = G$, then $A = A(G)$.

Let A be a uniform algebra on X and K a compact subset of X. The A-convex hull of K, denoted $\text{hull}_A K$, is the set $\{x \in M(A): |\hat{a}(x)| \leqq ||a||_K, a \in A\}$, where $\hat{a}(x) = x(a)$ defines the Gelfand transform $\hat{a}$ of a. For compact subsets K of X, $\text{hull}_A K$ is compact and the algebra A_K obtained as the uniform

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closure of the restriction algebra $A|K$ has non-zero continuous homomorphism space $M(A_K) = \text{hull}_A K$ [**6**].

If X is a σ-compact locally compact space, then any uniform algebra on X is an F-algebra; by σ-compactness, the topology of uniform convergence on compact subsets is metrizable, and it is complete since X is a k-space [**5**]. Moreover, there exist compact subsets K_n of X such that $K_{n+1} \supset K_n$, $X = \bigcup_{n=1}^{\infty} K_n$, and every compact subset of X is contained in some K_n. Such a sequence $\{K_n\}_{n=1}^{\infty}$ is called a *hemi-compact covering* of X. If X has a hemi-compact covering $\{K_n\}_{n=1}^{\infty}$ and A is a uniform algebra on X, then $\{\text{hull}_A K_n\}_{n=1}^{\infty}$ is a hemi-compact covering of $M(A)$ [**5**]. It follows that when X is σ-compact and locally compact, the algebra $\hat{A}$ of Gelfand transforms of elements in A is a uniform algebra on $M(A)$ and $\hat{A}$ is algebraically and topologically isomorphic to A. If $M(A) = X$ (topologically), we shall identify the isomorphic algebras A and $\hat{A}$ and say that A is a uniform algebra on $M(A)$.

If K is a compact subset of C^n, then $\text{hull}_{P(C^n)} K$ is also a compact set in C^n, denoted by $\hat{K}$. Call a subset X of C^n polynomially convex provided $\hat{K} \subset X$ whenever K is a compact subset of X. For arbitrary X, define $\hat{X}$ to be the intersection of all polynomially convex sets containing X.

When A is a uniform algebra on $M(A)$, this concept of polynomial convexity may be generalized to one of A-convexity (cf. Rickart [**6**]). A subset $Y \subset M(A)$ is said to be *A-convex* provided $\text{hull}_A K \subset Y$ whenever K is a compact subset of Y. For arbitrary $Y \subset M(A)$, define $\text{hull}_A Y$ to be the intersection of all A-convex subsets of $M(A)$ which contain Y. Since $M(A)$ is A-convex, $\text{hull}_A Y$ always exists, and is A-convex. If Y is σ-compact and locally compact, then Lemma 1 below shows that the non-zero continuous homomorphism space $M(A_Y)$ of the uniform algebra A_Y (defined as the closure of the restriction algebra $A|Y$ in the space $C(Y)$, in the topology of uniform convergence on compact subsets of Y) is $\text{hull}_A Y$.

Finally, suppose that A is a uniform algebra on X and $S \subset X$. If there is a neighbourhood U of S and an element $a \in A$ such that $a(x) = 1$ for $x \in S$ and $|a(x)| < 1$ for $x \in U - S$, then S is said to be a *local peak set* in X, and a is said to *peak locally at S within U*. If U can be taken to be the whole space X, then S is a *peak set* of A. We obtain our characterization of Montel algebras by showing that, in the cases under consideration, they can have no (non-trivial) local peak sets.

3. A characterization of $A(G)$.

A uniform algebra A on X is said to be *Montel* if every bounded subset (that is, every set of functions in A which is uniformly bounded on compact subsets of X) is relatively compact in A.

Note that the Montel property is preserved under topological isomorphisms.

PROPOSITION 1. *Let A be a uniform algebra on a σ-compact locally compact space X. If A is Montel, then every local peak set of A in $M(A)$ is open and closed in $M(A)$.*

Proof. Suppose that $f \in A$ peaks locally on S within U in $M(A)$. For every positive integer n, the set $U_n = \{x \in U: |f(x) - 1| < 1/n\}$ is a neighbourhood of S in $M(A)$, and $\{U_n\}_{n=1}^{\infty}$ is a fundamental sequence of neighbourhoods of S. Let $\{K_n\}_{n=1}^{\infty}$ be a hemi-compact covering of $M(A)$ by A-convex sets K_n. There is an integer n_0 such that $S \cap K_n \neq \emptyset$ for $n \geqq n_0$. Thus for $n \geqq n_0$, $S \cap K_n$ is a local peak set of $A|K_n$ in K_n; hence by a well-known result (see [**4**, p. 62]) it is known that $S \cap K_n$ is a peak set of A_{K_n} in K_n. It follows that there exist functions $f_n \in A$ such that $||f_n - 1||_{S \cap K_n} < 1/n$, $||f_n||_{K_n - U_n} < 1/n$, and $||f_n||_{K_n} < 2$, $n \geqq n_0$. However, $\{f_n\}_{n=n_0}^{\infty}$ is a bounded subset of A and therefore relatively compact. Let $\{f_{ni}\}_{i=1}^{\infty}$ be a subsequence converging uniformly on compact subsets of X to $f \in A$. Clearly $f(x) = 1$ for $x \in S$ and if $y \in M(A) - S$, then $f(y) = 0$. Since $f \in C(M(A))$, it must be that S is open and closed in $M(A)$.

COROLLARY 1. *Let A be a uniform algebra on a connected σ-compact, locally compact space X. If A is Montel, then A is a maximum modulus algebra on $M(A)$.*

Proof. Suppose that there is a compact subset K of $M(A)$ and a function $f \in A$ such that $\{x \in M(A): |f(x)| = ||f||_K\}$ does not meet the boundary of K in $M(A)$. If x is chosen to be any element of this set, then the function $g \in A$ defined by $g = ((f/f(x)) + 1)/2$ peaks in K on $\{y \in K: f(y) = f(x)\} = S$, which is in the interior of K. Thus S is a local peak set of A in $M(A)$, hence S is open and closed in $M(A)$, whence $S = M(A)$, which is impossible.

Applying the result of Rudin in the form stated above, we obtain the following result.

COROLLARY 2. *Let A be a uniform algebra on an open subset G of C and suppose that A contains the polynomials and $M(A) = G$. Then A is Montel if and only if $A = A(G)$.*

4. Montel algebras of non-analytic functions. In this section we show that if G is a polynomially convex open connected subset of C and $n \geqq 1$, there is a uniform algebra A on G which is algebraically and topologically isomorphic to the algebra of all analytic functions on an open subset of C^n, in the compact-open topology. Since the Montel property is preserved under isomorphisms, the algebra is Montel. However, if $n > 1$, then $A \neq A(G)$ since the continuous homomorphism space of A is an open subset of C^n while that of $A(G)$ is G (cf. [**4**, p. 58]).

In the construction, the following standard fact is used.

LEMMA. *If K is a compact connected subset of C and ϵ is any positive real number, then there exists a simple closed curve J such that K is contained in the relatively compact component of $C - J$ and every point of J is at a distance less than ϵ from some point of K;* cf. [**8**, p. 35].

PROPOSITION 2. *If G is an open connected subset of C and $n \geqq 1$, then $P(G^n)$ is algebraically and topologically isomorphic to a subalgebra of $C(G)$.*

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Proof. G is σ-compact and locally compact, thus there exists a hemi-compact covering $\{K_j\}_{j=1}^{\infty}$ of G. Since G is connected, locally connected, and locally compact, every compact subset of G is contained in a compact connected subset, thus we may assume that $\{K_j\}_{j=1}^{\infty}$ is chosen so that each K_j is connected.

Choose a sequence of simple closed curves J_j in G as follows. K_1 is a compact connected subset of the open set G in C, thus there exists a simple closed curve J_1 in G such that the relatively compact component $i(J_1)$ of $C - J_1$ contains K_1. Applying the lemma now to the compact connected set J_1, a simple closed curve J_2 in G may be chosen so that the closure $c(J_1)$ of $i(J_1)$ in C lies in $i(J_2)$ and $c(J_2) - i(J_1) \subset G$. Suppose by way of induction that J_j have been chosen, $1 \leqq j \leqq 2k$, such that

(1) $J_j \subset G$, $1 \leqq j \leqq 2k$,

(2) $K_i \cup c(J_{2i-2}) \subset i(J_{2i-1}) \subset c(J_{2i-1}) \subset i(J_{2i})$, $1 \leqq i \leqq k$, and

(3) $c(J_{2i}) - i(J_{2i-1}) \subset G$, $1 \leqq i \leqq k$.

Now $K_{k+1} \cup J_{2k}$ is a compact subset of G, hence is contained in a compact connected subset L_{k+1}. By the lemma, there exists a simple closed curve J_{2k+1} in G such that $L_{k+1} \subset i(J_{2k+1})$ and another curve J_{2k+2} such that $J_{2k+1} \subset i(J_{2k+2})$ and $c(J_{2k+2}) - i(J_{2k+1}) \subset G$. Thus

(1) $J_{2k+1}, J_{2k+2} \subset G$,

(2) $K_{k+1} \cup c(J_{2k}) \subset i(J_{2k+1}) \subset c(J_{2k+1}) \subset i(J_{2k+2})$,

(3) $c(J_{2k+2}) - i(J_{2k+1}) \subset G$,

and by induction, (1), (2), and (3) hold for all positive integers. Define $R_j = c(J_{2j}) - i(J_{2j-1})$, $j \geqq 1$. Note that $\bigcup_{j=1}^{\infty} R_j$ is closed in G.

Since $\{K_j\}_{j=1}^{\infty}$ is a hemi-compact covering of G, it follows from the definition of $\hat{G}$ that $\hat{G} = \bigcup_{j=1}^{\infty}\hat{K}_j$. Furthermore, $\hat{J}_j = (c(J_j))^{\wedge} = c(J_j)$ for all j, thus by (2) above, $\hat{G} = \bigcup_{j=1}^{\infty}\hat{K}_j \subset \bigcup_{j=1}^{\infty}(c(J_j))^{\wedge} = \bigcup_{j=1}^{\infty}c(J_j)$. Moreover, by (1), we have $\bigcup_{j=1}^{\infty}c(J_j) = \bigcup_{j=1}^{\infty}\hat{J}_j \subset \hat{G}$, hence

$$\hat{G} = \bigcup_{j=1}^{\infty} c(J_j). \tag{4}$$

Now let $\{T_j\}_{j=1}^{\infty}$ be a sequence of disjoint closed annuli $T_j = \{t \in C: r_j' \leqq |t| \leqq r_j\}$ whose outer radii r_j increase to infinity. Let I_j be the closed interval $I_j = \{t \in C: \arg(t) = 0 \text{ and } r_j' \leqq t \leqq r_j\}$. By the representation (4) of $\hat{G}$ and the fact that $c(J_j) \subset i(J_{j+1})$, $j \geqq 1$, there is a homeomorphism $\varphi: \hat{G} \to C$ such that $\varphi(R_j) = T_j$, $j \geqq 1$. To show the existence of φ, it is enough to note that if J' and J are simple closed curves with $J' \subset i(J)$ and if r' and r are real numbers with $r' < r$, then any onto homeomorphism

$$\varphi: c(J') \to \{t \in C: |t| \leqq r'\}$$

can be extended to a homeomorphism $\bar{\varphi}: c(J) \to \{t \in C: |t| \leqq r\}$.

For each positive integer j, take a space-filling continuous function

$$g_j: I_j \to C^{n-1} \quad \text{with } g_j(I_j) = D_{r_j}^{\,n-1},$$

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where $D_{r_j}{}^{n-1}$ is by definition the polydisc

$$\{s = (s_1, \ldots, s_{n-1}) \in C^{n-1}: |s_k| \leqq r_j, 1 \leqq k \leqq n-1\}.$$

For $1 \leqq k \leqq n-1$, let π_k denote the projection map to the kth coordinate and define functions h_k on $\bigcup_{j=1}^{\infty} R_j$ by

$$h_k(t) = \pi_k(g_j(|\varphi(t)|)), \qquad t \in R_j.$$

Then h_k is continuous on the closed subset $\bigcup_{j=1}^{\infty} R_j$ of the normal space G and by the Tietze extension theorem has a continuous extension to a function, which we shall also call h_k, on G to C. Let $f_k = \varphi^{-1} \circ h_k$, $1 \leqq k \leqq n-1$.

Define a map $F: G \to C^n$ by $F(t) = (f_1(t), \ldots, f_{n-1}(t), t)$ for $t \in G$. Clearly $F(G) \subset (\hat{G})^n$ by definition of the functions $f_1, \ldots, f_{n-1}$. On the other hand, if $s = (s_1, \ldots, s_n)$ is an element of $(\hat{G})^n$, then by the representation (4) there is a positive integer j such that $s_1, \ldots, s_n \in c(J_{2j-3})$, thus

$$\varphi(s_1), \ldots, \varphi(s_n) \in D_{r_{j-1}}{}^1 \subset D_{r_j}{}^1.$$

To see this, observe that the image under the homeomorphism φ of the connected set $i(J_{2j-2}) - R_{j-1} = i(J_{2j-3})$ must lie in a single component of $C - \varphi(R_{j-1})$. However, $\varphi(c(J_{2j-3}))$ is compact and equal to the closure of $\varphi(i(J_{2j-3}))$; therefore $\varphi(i(J_{2j-3}))$ must be contained in the bounded component of the complement of $\varphi(R_{j-1}) = T_{j-1}$. Thus $|\varphi(t)| \leqq r_{j-1}$ for all $t \in c(J_{2j-3})$. It now follows that there exists $r \in I_j$ such that $g_j(r) = (\varphi(s_1), \ldots, \varphi(s_{n-1}))$, and $|\varphi(s_n)| \leqq r_{j-1} < r$. Let p be any polynomial on C^n (in fact, any entire function on C^n). Then

$$\begin{aligned} (5) \qquad |p(s)| &\leqq \sup\{|p(s_1, \ldots, s_{n-1}, t)|: t \in \hat{G}, |\varphi(t)| \leqq r\} \\ &= \sup\{|p(s_1, \ldots, s_{n-1}, t)|: t \in \hat{G}, |\varphi(t)| = r\} \\ &= \sup\{|p(s_1, \ldots, s_{n-1}, t)|: t \in G, |\varphi(t)| = r\} \\ &\leqq \sup\{|p(F(t))|: t \in G, |\varphi(t)| = r\}. \end{aligned}$$

However, $\{F(t): t \in G \text{ and } |\varphi(t)| = r\}$ is a compact subset of $F(G)$, thus $s \in (F(G))^\wedge$. That φ is a homeomorphism is used to conclude that $\{t \in \hat{G}: |\varphi(t)| \leqq r\}$ is compact and that $|\varphi(t)| = r$ implies $t \in G$.

We have shown that $F(G) \subset (\hat{G})^n \subset F(G)^\wedge$ and thus $F(G) \subset (G^n)^\wedge \subset F(G)^\wedge$ since $(\hat{G})^n = (G^n)^\wedge$ is immediate.

If L is a compact subset of $F(G)$, then $\pi_n(L)$ is compact in G and $L = F(\pi_n(L))$. However, G, and hence $F(G)$, is hemi-compact; since $F(G)$ is also first countable, it is σ-compact and locally compact [**1**]. It follows from Lemma 1 below that $P(F(G)) = P((G^n)^\wedge) = P(G^n)$. Finally, we use Lemma 2 to conclude that $P(G^n)$ is algebraically and topologically isomorphic to the subalgebra A of $C(G)$ generated by the functions $f_1, \ldots, f_{n-1}$, and z.

LEMMA 1. *Suppose that A is a uniform algebra on $M(A)$ and Y is a σ-compact locally compact subset of $M(A)$. If $Y \subset X \subset \text{hull}_A Y$, then $A_X = A_Y$ and*

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$M(A_X) = \text{hull}_A\, Y$; *in particular, the restriction map* $f \to f|Y$ *is an algebraic and topological isomorphism.*

Proof. Let $\{K_n\}_{n=1}^{\infty}$ be a hemi-compact covering of Y. Then $\{\text{hull}_A\, K_n\}_{n=1}^{\infty}$ is a hemi-compact covering of $M(A_Y)$, thus

$$M(A_Y) = \bigcup_{n=1}^{\infty} \text{hull}_A\, K_n \subset \text{hull}_A\, Y.$$

However, $M(A_Y)$ is easily seen to be A-convex, whence $\text{hull}_A\, Y = M(A_Y)$ is a hemi-compact union of the $\text{hull}_A\, K_n$. Clearly, $f \to f|Y$ is an algebraic homomorphism of A_X into A_Y. If $g \in A_Y$, the Gelfand transform $\hat{g} \in \widehat{A_Y}$ is such that $\hat{g}|Y = g$. However, $\hat{g} \in A_{\text{hull}_A Y}$; for if L is a compact subset of $\text{hull}_A\, Y$ and $\epsilon > 0$, then taking a compact subset K of Y such that $L \subset \text{hull}_A\, K$ and an element $p \in A$ such that $||g - p||_K < \epsilon$, it follows that

$$||\hat{g} - p||_L \leqq ||\hat{g} - p||_{\text{hull}_A K} = ||g - p||_K < \epsilon,$$

or $\hat{g} \in A_{\text{hull}_A Y}$. Let $f = \hat{g}|X$. Then $f \in A_X$ and $f \to f|Y = g$, thus the homomorphism is onto. This also shows that the map is one-to-one. Thus $f \to f|Y$ is an algebraic isomorphism. The inequalities $||f|Y||_K \leqq ||f||_{\text{hull}_A K}$ and $||f||_{\text{hull}_A K} = ||f|Y||_K$ which hold for K and L as above show that the map is in fact topological, whence $A_X = A_Y$ and $M(A_X) = M(A_Y)$.

LEMMA 2. *Let X be σ-compact and locally compact and let $f_1, \ldots, f_n$ be functions in $C(X)$. Suppose that the map $F\colon X \to C^n$ defined by $F(x) = (f_1(x), \ldots, f_n(x))$, $x \in X$, has the property that if L is a compact subset of $F(X)$, then there exists K compact in X such that $L \subset F(K)$. Then the uniform algebra A on X generated by $f_1, \ldots, f_n$ is algebraically and topologically isomorphic to $P(F(X))$.*

Proof. X is σ-compact and locally compact, and the property of F assumed in the hypothesis guarantees that $F(X)$ is also σ-compact and locally compact, since it is hemi-compact and first countable. Thus the uniform algebras A and $P(F(X))$ are F-algebras. Define a mapping φ on $P(F(X))$ by $\varphi(g) = g \circ F$. Note that the image under φ of a dense subset of $P(F(X))$ is dense in A. Furthermore, if K is a compact subset of X, then $F(K)$ is compact in $F(X)$ and if p is any polynomial on C^n, then $||p||_{F(K)} = ||p \circ F||_K = ||\varphi(p)||_K$. Thus φ is continuous and, since A is complete, into A. It is clear that φ is one-to-one. We show that φ is onto. Suppose that $f \in A$ and K is a compact subset of X, δ a real number with $\delta > 0$. Choose a polynomial $p_{(K,\delta)}$ such that

$$||p_{(K,\delta)} \circ F - f||_K < \delta.$$

If the indices (K, δ) are ordered by $(K_1, \delta_1) < (K_2, \delta_2)$ if and only if $K_1 \subseteq K_2$ and $\delta_2 \leqq \delta_1$, then $\{p_{(K,\delta)}\}$ may be shown to be a Cauchy net as follows. Let L be an arbitrary compact set in $F(X)$ and let $\epsilon > 0$. Choose a compact set $K_L \subset X$ such that $F(K_L) \supseteq L$. Then, if $(K_L, \frac{1}{2}\epsilon) < (K_i, \delta_i)$ $(i = 1, 2)$, we have $||p_{(K_1,\delta_1)} - p_{(K_2,\delta_2)}||_L < \epsilon$, thus $\{p_{(K,\delta)}\}$ is a Cauchy net. By completeness of $P(F(X))$, $\{p_{(L,\epsilon)}\}$ has a limit $g \in P(F(X))$. By continuity, $\varphi(g) = f$.

We have established that φ is a continuous algebraic isomorphism of an F-algebra onto another, and the interior mapping principle enables us to conclude that φ is topological.

For polynomially convex open sets G in C, $P(G^n) = A(G^n)$ by Runge's theorem, hence we have the following corollary to Proposition 2.

COROLLARY 3. *If G is a polynomially convex open connected subset of C, then for each positive integer n there is a uniform algebra A_n on G containing the polynomials and such that $A_n = A(G^n)$ (algebraically and topologically).*

By an earlier remark, each of the algebras A_n is Montel since $A(G^n)$ is, thus we have found infinitely many non-isomorphic Montel algebras A_n on G. Of course, for $n > 1$, $M(A_n) = M(A(G_n)) = G^n \neq G$, thus $A(G) \subset A_n$ $(A(G) \neq A_n)$.

In the case $G = C$, the algebras A_n constructed above contain no non-constant bounded functions. For suppose that $f \in A_n$ is bounded. By Lemmas 1 and 2, $f = g \circ F$, where g can be taken in the algebra $P(F(C)) = P(C^n) = A(C^n)$. However, g is bounded on $F(C)$; thus by (5), g is bounded on C^n. It follows that g, and hence f, is constant. We have therefore also found (infinitely many non-isomorphic) uniform algebras on C having no non-constant bounded functions and properly containing $A(C)$, answering a question about the existence of such algebras raised by Birtel and Lindberg [**3**].

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