Glasgow Math. J. **43A** (2001) 9–21.

BÄCKLUND TRANSFORMATIONS FROM PAINLEVÉ ANALYSIS

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(Received 15 February, 2000)

Abstract. Since its elaboration in 1983 by Weiss, Tabor and Carnevale, the method to explicitly build the Bäcklund transformation of a partial differential equation (PDE) from singularity analysis only has been improved in several complementary directions, and at the present time it succeeds for practically all PDEs in $1+1$-dimensions. The current state of the art is presented, and the emphasis is put on understanding the method. There are two important stages: first, the definition (identified with a Darboux transformation) of a resummation variable to make the Laurent series a finite one as requested by the definition of the word integrability; second, the link (identified with a linearizing formula to be taken from the classification of Painlevé and Gambier) between this resummation variable and the Lax pair to be found.

1991 *Mathematics Subject Classification*. 35Q58.

1. Introduction: Bäcklund transformation, Lax pair, Darboux transformation. This paper is a short version, restricted to integrable partial differential equations (PDEs), of a summer school course on the same subject also dealing with non-integrable PDEs [**7**]. Many examples, not repeated here, can be found there.

Given a *nonlinear* PDE (boldface means multicomponent)

$$\mathbf{E}(\mathbf{u}, \mathbf{x}) = 0, \tag{1}$$

which has some good reasons to be integrable (see definition below), such as passing the Painlevé test [**6**,**20**], the problem which we address is to find explicitly the "macroscopic" quantities which materialize this integrability. More precisely, we want to find the Bäcklund transformation (BT) if it exists, since this is a generator of exact solutions, and we want to do it by singularity analysis *only*.

We need to recall some basic definitions. For simplicity, but this is not a restriction, let us give them for a PDE defined as a single scalar equation for one dependent variable u and two independent variables (x, t).

Definition 1. (Refs. [**1**], [**11**] vol. III chap. XII, [**19**]) A *Bäcklund transformation* (*BT*) between two given PDEs

$$E_1(u, x, t) = 0, \;\; E_2(U, X, T) = 0 \tag{2}$$

https://doi.org/10.1017/S0017089501000027 Published online by Cambridge University Press

is a pair of relations

$$F_j(u, x, t, U, X, T) = 0, \ j = 1, 2 \tag{3}$$

with some transformation between (x, t) and (X, T), in which F_j depends on the derivatives of $u(x, t)$ and $U(X, T)$, such that the elimination of u (resp. U) between (F_1, F_2) implies $E_2(U, X, T) = 0$ (resp. $E_1(u, x, t) = 0$). The BT is called the *auto-BT* or the *hetero-BT* according as the two PDEs are the same or not.

Example 1. The sine-Gordon equation (we identify sine-Gordon and sinh-Gordon since an affine transformation on u does not change the integrability nor the singularity structure)

$$\text{sine-Gordon} \ : E(u) \equiv u_{xt} + 2a \sinh u = 0 \tag{4}$$

admits the auto-BT

$$(u + U)_x + 4\lambda \sinh \frac{u - U}{2} = 0, \tag{5}$$

$$(u - U)_t - \frac{2a}{\lambda} \sinh \frac{u + U}{2} = 0, \tag{6}$$

in which λ is an arbitrary complex constant, called the *Bäcklund parameter*.

The importance of the BT is such that it is often taken as a definition of *integrability*.

Definition 2. A PDE in N independent variables is *integrable* if at least one of the following properties holds.

1. It is linearizable.
2. For $N > 1$, it possesses an auto-BT which, if $N = 2$, depends on an arbitrary complex constant, the Bäcklund parameter.
3. It possesses a hetero-BT to another integrable PDE.

Definition 3. Given a PDE, a *Lax pair* is a system of two linear differential operators

$$\text{Lax pair}: \ L_1(U, \lambda), \ L_2(U, \lambda), \tag{7}$$

depending on a solution U of the PDE and, in the $1+1$-dimensional case, on an arbitrary constant λ, called the *spectral parameter*, with the property that the vanishing of the commutator $[L_1, L_2]$ is equivalent to the vanishing of the PDE $E(U) = 0$.

A Lax pair can be represented in several equivalent ways, such as the *Lax representation*, the *zero-curvature representation*, the *projective Riccati representation*, the *scalar representation*, the *Sato representation*, etc. From the singularity point of view, the Riccati representation is the most suitable, as will be seen.

EXAMPLE 2. The sine-Gordon equation (4) admits the zero-curvature representation

$$(\partial_x - L)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0,\ \ L = \begin{pmatrix}U_x/2 & \lambda\\ \lambda & -U_x/2\end{pmatrix}, \tag{8}$$

$$(\partial_t - M)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0,\ \ M = -(a/2)\lambda^{-1}\begin{pmatrix}0 & e^U\\ e^{-U} & 0\end{pmatrix}, \tag{9}$$

equivalent to the Riccati representation, with $y = \psi_1/\psi_2$,

$$y_x = \lambda + U_x y - \lambda y^2, \tag{10}$$

$$y_t = -\frac{a}{2}\lambda^{-1}e^U + \frac{a}{2}\lambda^{-1}e^{-U}y^2. \tag{11}$$

The working definition of the Darboux transformation (DT) given below is very simplified (this is an involution) as compared to the one of Darboux [**10**], but it is sufficient for our purpose.

DEFINITION 4. Given a PDE, a *Darboux transformation* is a transformation between two solutions (u, U) of the PDE

$$\mathrm{DT}:\ u = \sum_f \mathcal{D}_f \operatorname{Log}\tau_f + U \tag{12}$$

linking their difference to a finite number of linear differential operators, denoted $\mathcal{D}_f$ (f like family), acting on the logarithm of functions τ_f.

In the definition (12), it is important to note that, despite the notation, each function τ_f is in fact the ratio of the "tau-function" of u by that of U.

Lax pairs, Bäcklund and Darboux transformations are not independent. In order to exhibit their interrelation, one needs some additional information, namely the link

$$\forall f\ :\ \mathcal{D}_f \operatorname{Log}\tau_f = F_f(\psi), \tag{13}$$

which most often is the identity $\tau = \psi$, between the functions τ_f and the function ψ in the definition of a scalar Lax pair.

EXAMPLE 3. The (integrable) sine-Gordon equation (4) admits the Darboux transformation

$$u = U - 2(\operatorname{Log}\tau_1 - \operatorname{Log}\tau_2), \tag{14}$$

in which (τ_1, τ_2) is a solution (ψ_1, ψ_2) of the system (8)–(9).

Then its BT (5)–(6) is the result of the elimination [**4**] of τ_1/τ_2 between the DT (14) and the Riccati form of the Lax pair (10)–(11), with the correspondence $\tau_f = \psi_f, f = 1, 2$. This elimination reduces to the substitution $y = e^{-(u-U)/2}$ in the

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Riccati system (10)–(11), and this is one of the advantages of the Riccati representation. Therefore the Bäcklund parameter and the spectral parameter are identical notions.

Thus, roughly speaking, the BT is equivalent to the couple (DT, Lax pair).

2. The ODE situation. For the six ordinary differential equations (ODE) (P1)–(P6) which bear his name, Painlevé proved the Painlevé property (PP) by showing [**26**,**27**] the existence of one (case of (P1)) or two ((P2)–(P6)) function(s) $\tau = \tau_1, \tau_2$, called *tau-functions*, linked to the general solution u by logarithmic derivatives

$$(\text{P1}) : u = \mathcal{D}_1 \operatorname{Log} \tau, \tag{15}$$

$$(\text{P}n),\ \ n = 2, \dots, 6 : u = \mathcal{D}_n(\operatorname{Log} \tau_1 - \operatorname{Log} \tau_2), \tag{16}$$

where the operators $\mathcal{D}_n$ are linear: these operators are given by

$$\mathcal{D}_1 = -\partial_x^2,\ \ \mathcal{D}_2 = \mathcal{D}_4 = \pm\partial_x,\ \ \mathcal{D}_3 = \pm e^{-x}\partial_x, \tag{17}$$

$$\mathcal{D}_5 = \pm x e^{-x}(2\alpha)^{-1/2}\partial_x,\ \ \mathcal{D}_6 = \pm x(x-1)e^{-x}(2\alpha)^{-1/2}\partial_x. \tag{18}$$

These functions τ_1, τ_2 satisfy third order nonlinear ODEs and they have the same kind of singularities as solutions of *linear* ODEs, namely they have no movable singularities at all; they are entire functions for (P1)–(P5), and their only singularities for (P6) are the three fixed critical points (∞, 0, 1).

The important point is that the linear operator $\mathcal{D}$ contains the full information. In some sense, the tau-function τ realizes a (global) resummation of the (local) Laurent series

$$u = \sum_{j=0}^{+\infty} u_j (x - x_0)^{j+p}, \tag{19}$$

with $p = -1$ for (P1), $p = -2$ for the others.

ODEs cannot possess an auto-BT, since the number of independent arbitrary coefficients in a solution cannot exceed the order of the ODE. Nevertheless, some of the features above can be transposed to the PDE situation.

3. A degeneracy: linearizable equations. The Burgers equation in its potential form,

$$b v_t + v_x^2/a + v_{xx} + G(t) = 0, \tag{20}$$

under the transformation of Forsyth [**15**, p. 106]

$$v = a \operatorname{Log} \tau,\ \ \tau = \psi, \tag{21}$$

is linearized into the heat equation

https://doi.org/10.1017/S0017089501000027 Published online by Cambridge University Press

$$b\psi_t + \psi_{xx} + G(t)\psi = 0. \tag{22}$$

This can be considered as a degenerate Darboux transformation (12), in which U is identically zero and ψ satisfies a single linear equation, not a pair of linear equations, and so this fits the general scheme.

4. The local information: Laurent series. Let (1) be a PDE which passes the Painlevé test. One considers (which does not mean that one computes) all the Laurent series that can (maybe after suitable perturbations not described here) represent the *general* solution,

$$u = \sum_{j=0}^{+\infty} u_j \chi^{j+p}, \ -p \in \mathcal{N}, \ E = \sum_{j=0}^{+\infty} E_j \chi^{j+q}, -q \in \mathcal{N}^*. \tag{23}$$

The expansion variable χ is for instance $\chi = \varphi(x, t) - \varphi_0$, in which $\varphi(x, t) - \varphi_0 = 0$ is the singular manifold [**31**].

Definition 5. One calls the *singular part operator* of the family $f = (u_0, p)$ the linear differential operator $\mathcal{D}$ which has the property that

$$u - \mathcal{D}\,\mathrm{Log}(\varphi(x, t) - \varphi_0) \tag{24}$$

is regular when $\varphi \to \varphi_0$.

The only piece of information which one retains from the Laurent series is its singular part operator $\mathcal{D}$. Examples of $\mathcal{D}$ have been given in (15)–(16) and (21).

5. The global assumption: resummation. The full Laurent series is of no help, for this is not global information. Since this is the only piece of information directly available, let us represent, and this is the idea of Weiss, Tabor and Carnevale (WTC) [**31**,**30**], an unknown exact solution u as the sum of a singular part, built from the finite principal part of the Laurent series (i.e. the finite number of terms with negative powers), and of a regular part made up of one term denoted by U. This assumption is identical to that of a Darboux transformation (12), in which nothing would be specified about U.

This method is widely known as the *singular manifold method* or *truncation method* because it selects the beginning of the Laurent series and discards ("truncates") the remaining infinite part.

Since its introduction by WTC [**31**], it has been improved in many directions [**22**,**12**,**17**,**24**,**8**,**28**,**25**], and we present below the current status of the method.

For PDEs, the analogue of (15)–(16), with an additional rhs U, is now the Darboux transformation (12), and the scalar(s) ψ to which the scalar(s) τ are linked by (13) are assumed to satisfy a linear system, the Lax pair.

More precisely, the (infinite) Laurent series (23) in the variable $\varphi - \varphi_0$ is resummed as the sum of two terms

$$u = \mathcal{D}\,\mathrm{Log}\,\tau + \text{regular part}. \tag{25}$$

The first term is a finite Laurent series in τ, in which $\mathcal{D}$ is the singular part operator defined in Section 4, and τ is any variable fulfilling the requirement that τ^{-1} be linear in $(\varphi - \varphi_0)^{-1}$, so as to capture all the singularities of u when $\varphi \to \varphi_0$ without altering the structure of singularities. The second term, temporarily called "regular part" for this reason, is yet unspecified. The sum of these two terms is therefore a finite Laurent series and the variable τ is a *resummation variable* which has made the former infinite series in $\varphi - \varphi_0$ a finite one. One then requires the identification of this resummation (25) with the definition of a Darboux transformation (12). This involves two features. The first feature is to uncover a link (13) between τ and a scalar component ψ of a Lax pair. The second feature is to prove that the left over "regular part" is indeed a second solution to the PDE under study.

EXAMPLE 4. For the KdV and modified KdV equations, the assumptions are respectively

$$u = U + \partial_x^2 \operatorname{Log} \tau, \tag{26}$$

$$u = U + (\partial_x \operatorname{Log} \tau_1 - \partial_x \operatorname{Log} \tau_2), \tag{27}$$

in which nothing is assumed about U.

To make the assumption a constructive one, the needed additional input is as follows.

1. The *order of the Lax pair* and its definition containing coefficients to be found.
2. A *link* between the set of functions τ_f and the component(s) ψ_i of the Lax pair. This link will not be guessed arbitrarily but *obtained* from the classifications of ODEs performed by the Painlevé school (Painlevé, Gambier, Chazy, Bureau, Cosgrove, ...).

6. The possible links between τ and ψ. When the number of independent variables is two, one can eliminate ∂_t between the two PDEs defining the unknown BT, and thus obtain an ODE, e.g. (5). This nonlinear ODE, with coefficients depending on U and, in the $1+1$-dimensional case, on an arbitrary constant λ, has the property [**25**] of being linearizable, since it results from the Lax pair, a linear system, and the Darboux transformation by an elimination process [**4**].

This very strong property restricts the admissible choices (13) to a finite number of possibilities, summarized in Table 1.

This provides much information. If the scattering order is at most three, the unknown so called x-part of the BT is either a Riccati equation (for shortness, the symbol $'$ means here ∂_x)

Table 1. This gives the nonlinear ODEs of first degree and first or second order that have the Painlevé property and are linearizable. Their name is in the last column. For the order of the nonlinear ODE given in column 2, column 3 indicates the number of homographically inequivalent ODEs with the PP, column 1 the order of the associated linear equation.

Linear order	Nonlinear order	with PP	with PP and linearizable
2	1	1	Riccati
3	2	50	Gambier nos. 5 and 25

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$$y' = a_2 y^2 + a_1 y + a_0, \text{ (Riccati)} \tag{28}$$

or a fifth Gambier equation [**16**]

$$y'' + 3yy' + y^3 + ry + q = 0, \text{ (G5)} \tag{29}$$

or a twenty-fifth Gambier equation.

$$Y'' - 3Y'^2/(4Y) + 3YY'/2 + Y^3/4 - (Y' + Y^2)\frac{Q'}{2Q} - RY - Q = 0. \text{ (G25)}$$

The coefficients in the equations above are arbitrary functions of (x, t), and these equations are to be considered *modulo* the group of homographic transformations. The classes (G5) and (G25), inequivalent under the homographic group, are equivalent under the birational group, with the explicit transformation between G5(y; q, r) and G25(Y; Q, R) given by

$$Y = \frac{Q}{2z' + z^2 - (Q'/Q)z - R}, \quad z = y + \frac{Q'}{2Q}, \quad 2y = \frac{Y'}{Y} + Y - \frac{Q'}{Q}, \tag{30}$$

$$r = -R + \frac{Q''}{Q} - \frac{5Q'^2}{4Q^2}, \quad q = -\frac{Q}{2} - \frac{R'}{2} + \frac{Q'''}{2Q} - \frac{7Q'Q''}{4Q^2} + \frac{5Q'^3}{4Q^3}. \tag{31}$$

The linearizing transformations are

$$\text{(Riccati)} \quad y = -a_2^{-1}\frac{\tau'}{\tau} = -a_2^{-1}\frac{\psi'}{\psi}, \quad \psi'' - (a_1 + (a_2'/a_2))\psi' + a_0 a_2 \psi = 0, \tag{32}$$

$$\text{(G5)} \quad y = \frac{\tau'}{\tau} = \frac{\psi'}{\psi}, \quad \psi''' + r\psi' + q\psi = 0, \tag{33}$$

$$\text{(G25)} \quad \begin{cases} Y = \dfrac{\tau'}{\tau} = \dfrac{Q}{2z' + z^2 - (Q'/Q)z - R}, \quad z = \dfrac{\psi'}{\psi}, \\ \psi''' - \dfrac{3Q'}{2Q}\psi'' - \left(R + \dfrac{Q''}{Q} - \dfrac{Q'^2}{Q^2}\right)\psi' - \left(\dfrac{R'}{2} + \dfrac{Q}{2} - \dfrac{Q'R}{2Q}\right)\psi = 0. \end{cases} \tag{34}$$

The formulae above define the *a priori* link between τ and ψ.

These two and only two possibilities for a third order Lax pair were rediscovered in 1980 in the context of scattering theory by Caudrey [**2**] and Kaup [**18**].

7. The route from the PDE to the Bäcklund transformation. Again, the full details of this algorithm are given in [**7**], with complements (on nonlinear superposition formulae) in [**21**].

Zeroth step. Find the singular part operator $\mathcal{D}$.

First step. Assume a Darboux transformation defined as

$$u = U + \mathcal{D}(\text{Log}\,\tau_1 - \text{Log}\,\tau_2), \quad E(u) = 0, \tag{35}$$

with u a solution of the PDE under consideration, U an unspecified field, τ_f the "entire" function (or more precisely ratio of entire functions) attached to each

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family f. For one-family PDEs, one denotes $\tau_1 = \tau$, $\tau_2 = 1$, so that the DT assumption (35) becomes

$$u = U + \mathcal{D} \operatorname{Log} \tau, \;\; E(u) = 0. \tag{36}$$

Second step. Choose the order two, then three, then ..., for the unknown Lax pair, and define one or two (as many as the number of families) *scalars* ψ_f from the component(s) of its wave vector (e.g. the scalar wave vector if the PDE has one family and the pair is defined in scalar form). Such Lax pairs and scalars are for instance, in the scalar representation,

$$L_1\psi \equiv \psi_{xx} + \frac{S}{2}\psi = 0, \tag{37}$$

$$L_2\psi \equiv \psi_t + C\psi_x - \frac{C_x}{2}\psi = 0, \tag{38}$$

$$2[L_1, L_2] \equiv X = S_t + C_{xxx} + CS_x + 2C_xS = 0, \tag{39}$$

or

$$L_1\psi \equiv \psi_{xxx} - a\psi_x - b\psi = 0, \tag{40}$$

$$L_2\psi \equiv \psi_t - c\psi_{xx} - d\psi_x - e\psi = 0, \tag{41}$$

$$[L_1, L_2] \equiv X_0 + X_1\partial_x + X_2\partial_x^2 = 0. \tag{42}$$

Third step. Select (in a finite list) a link F

$$\forall f: \;\; \mathcal{D} \operatorname{Log} \tau_f = F(\psi_f), \tag{43}$$

the same for each family f, between the functions τ_f and the scalars ψ_f defined from the Lax pair. The most frequent choice (Riccati, (G5), see Section 6) is

$$\forall f: \;\; \tau_f = \psi_f. \tag{44}$$

Fourth step. Enforce the condition $E(u) = 0$ modulo the pair of linear equations for ψ [**23**]. This is the "truncation" properly said, that is to say : with the assumptions (35) for a DT, (43) for a link between τ_f and ψ_f, (37)–(38) or (40)–(41) or other for the Lax pair in ψ, express $E(u)$ as a polynomial in the derivatives of ψ_f which is irreducible modulo the Lax pair. For the above pairs and a one-family PDE, this amounts to eliminating any derivative of ψ of order in (x, t) higher than or equal to $(2, 0)$ or $(0, 1)$ (second order case) or to $(3, 0)$ or $(0, 1)$ (third order), thus resulting in a polynomial of one variable ψ_x/ψ (second order) or two variables ψ_x/ψ, ψ_{xx}/ψ (third order)

$$E(u) = \sum_{j=0}^{-q} E_j(S, C, U)(\psi/\psi_x)^{j+q} \text{ (one-family PDE, second order)}, \tag{45}$$

$$E(u) = \sum_{k\geq 0}\sum_{l\geq 0} E_{k,l}(a, b, c, d, e, U)(\psi_x/\psi)^k(\psi_{xx}/\psi)^l$$
$$\text{(one-family PDE, third order).} \tag{46}$$

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This generates the set of *determining equations*

$$\forall j \quad E_j(S, C, U) = 0 \text{ (one-family PDE, second order)} \tag{47}$$

$$\forall k\ \forall l \quad E_{k,l}(a, b, c, d, e, U) = 0 \text{ (one-family PDE, third order)} \tag{48}$$

for the unknown coefficients (S, C) or (a, b, c, d, e) as functions of U, and one establishes the constraint(s) on U by eliminating (S, C) or (a, b, c, d, e).

If the only constraint on U is to satisfy some PDE, one is on the way to an auto-BT if the PDE for U is the same as the PDE for u, or to an hetero-BT between the two PDEs.

The second, third and fourth steps must be repeated until a success occurs. The process is successful if and only if all the following conditions are met:

1. U comes out with one constraint exactly, namely: to be a solution of some PDE,
2. (if an auto-BT is desired) the PDE satisfied by U is identical to (1),
3. the vanishing of the commutator $[L_1, L_2]$ is equivalent to the vanishing of the PDE satisfied by U,
4. in the $1+1$-dimensional case only and if the PDE satisfied by U is identical to (1), the coefficients depend on an arbitrary constant λ, the spectral or Bäcklund parameter.

At this stage, one has obtained the DT and the Lax pair.

Fifth step. Obtain the two equations for the BT by eliminating ψ_f [**4**] between the DT and the Lax pair.

8. The privilege of second-order Lax pairs. The general second-order scalar Lax pair reads, in the case of two independent variables (x, t),

$$L_1\psi \equiv \psi_{xx} - d\psi_x - a\psi = 0, \tag{49}$$

$$L_2\psi \equiv \psi_t - b\psi_x - c\psi = 0, \tag{50}$$

$$[L_1, L_2] \equiv X_0 + X_1\partial_x, \tag{51}$$

$$X_0 \equiv -a_t + a_x b + 2ab_x + c_{xx} - c_x d = 0, \tag{52}$$

$$X_1 \equiv -d_t + (b_x + 2c - bd)_x = 0. \tag{53}$$

For the inverse scattering method to apply, the coefficients (d, a) of the x-part (49) are also required to depend linearly on the field U of the PDE.

The Lax pair (49)–(50) is identical to a linearized version of the Riccati system satisfied by the most general expansion variable Y [**22**,**28**] which realizes the resummation explained in Section 5, that is by [**5**]

$$Y^{-1} = B(\chi^{-1} + A) \quad (B \neq 0), \tag{54}$$

$$\chi_x = 1 + \frac{S}{2}\chi^2, \tag{55}$$

$$\chi_t = -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2, \tag{56}$$

and the correspondence is

$$Y = B^{-1}\frac{\psi}{\psi_x}, \;\; B \neq 0, \tag{57}$$

$$d = 2A, \;\; a = A_x - A^2 - S/2, \;\; b = -C, \;\; c = C_x/2 + AC + \int A_t dx. \tag{58}$$

In particular, when the coefficient d is zero or when, by a linear change $\psi \mapsto e^{\int (d)dx/2}\psi$, it can be set to zero without altering the linearity of a on U, the correspondence is as given in [**22**]; that is

$$\chi = \frac{\psi}{\psi_x}, \;\; B = 1, \;\; A = 0, \tag{59}$$

$$d = 0, \;\; a = -S/2, \;\; b = -C, \;\; c = C_x/2, \tag{60}$$

and the ψ in (59) satisfies (37)–(38).

Therefore second order Lax pairs are privileged in the singularity approach, in the sense that their coefficients can be identified with the elementary homographic invariants S, C of the invariant Painlevé analysis and, if appropriate, A, B. Conversely, and this has historically been the reason for some errors (which we shared!), at the stage of searching for the BT (as opposed to the stage of performing the Painlevé test), these homographic invariants S, C are useless when the Lax order is higher than two and they should not be considered.

As explained in Section 7, given a Lax pair, one should define from it either one or two scalars ψ_f. Consider the second order Lax pair defined by the gradient of Y. Then, for one-family PDEs, this unique scalar ψ is defined by (57). For two-family PDEs, the two scalars ψ_f are defined by

$$Y = \frac{\psi_1}{\psi_2}, \tag{61}$$

which leads to the zero-curvature representation of the Lax pair

$$(\partial_x - L)\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = 0, \;\; L = \begin{pmatrix} -A - B^{-1}B_x/2 & B^{-1} \\ B(A_x - A^2 - S/2) & A + B^{-1}B_x/2 \end{pmatrix}, \tag{62}$$

$$(\partial_t - M)\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = 0, \tag{63}$$

$$M = \begin{pmatrix} AC + C_x/2 - B^{-1}B_t/2 & -CB^{-1} \\ B((CS + C_{xx})/2 + A_t + CA^2 + C_x A) & -AC - C_x/2 + B^{-1}B_t/2 \end{pmatrix}.$$

The reason why the Riccati form is the most suitable characterization of the Lax pair is that it allows two linearizations [**24**,**28**]; namely (57) and (61), depending on whether the PDE has one family or two opposite families.

9. Second order truncations. For (P1) on one side, (P2)–(P6) on the other side, one must distinguish two types of PDEs: the one-family ones, and the two-family ones. Table 2 summarizes the characteristics of the truncation in each case.

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Table 2. Compared characteristics of the one-family and the two-family truncations for second order scattering problems.

Type of PDE	One-family	Two-family
$\sum \mathcal{D}\,\mathrm{Log}$	$\mathrm{Log}\,\tau$	$\mathrm{Log}\,\tau_1 - \mathrm{Log}\,\tau_2$
link $\tau(\psi)$	$\tau = \psi$	$\tau_1 = \psi_1, \tau_2 = \psi_2$
powers in u	χ^p to χ^0	Y^p to Y^{-p}
truncation variable	$\chi = \dfrac{\tau}{\tau_x}$	$Y^{-1} = \dfrac{\tau_1}{\tau_2}$
link with φ	$\tau = (\varphi - \varphi_0)\varphi_x^{-1/2}$	$Y^{-1} = B(\chi^{-1} + A)$

10. Higher order Lax pairs and truncations. Two nice features of the second order case are now meaningless and must not be used or considered.

1. The truncation in χ or Y to access the BT (this would mix Lax orders).
2. The singular manifold equation $F(S, C) = 0$ (for the same reason).

Given the order n of the unknown Lax pair, one then has to expand $E(u)$ as a polynomial in $n-1$ variables, typically $\psi_x/\psi, \psi_{xx}/\psi, \ldots$, irreducible modulo the satisfaction of the Lax pair by ψ, and to require its identical vanishing.

The main new feature is that the link $\tau = \psi$ is no longer the only possible one. As seen in Section 6, there exists one more possibility at third order. Thanks to this unusual link, the BT of the Kaup-Kupershmidt (KK) equation could finally be obtained from **[25]**, and this was a new result, to the credit of singularity analysis. The well known duality (this is an hetero-BT) between the Sawada-Kotera equation and the KK one reduces, at the ODE level, to the birational transformation between (G5) and (G25). There are two such ODE levels at which this birational transformation takes place: the one of the x-part of the BT of each PDE, relevant to this paper, and the one of the fourth order ODEs arising as reductions of the PDEs. In this second level, each fourth order ODE is equivalent **[13]** to a two-degree of freedom Hénon-Heiles Hamiltonian system, with the feature that the separating variables of the Hamiltonian system in the SK case are obvious ($q_1 + q_2$ and $q_1 - q_2$, just like the link $\tau = \psi$ is obvious), while in the KK case they are not at all obvious **[29]**, but *a posteriori* easily recoverable from the SK case by the birational transformation.

Note that the two-family PDEs create no problem **[14]**.

11. Selected examples. Table 3 gathers the main features of the truncation for various integrable PDEs, with either one or two families of movable singularities, and with either a second order or a third order Lax pair.

Table 3. Selected examples. For each PDE, the successive lines indicate the singular part(s), the link between τ and ψ, the assumption for a Lax pair which leads to the BT, the basis variables for defining the determining equations, the reference where the correct truncation was performed for the first time.

PDE	KdV	KK	p-mKdV, SG	Tzitzéica
u–U	$\partial_x^2\,\mathrm{Log}\,\tau$	$\partial_x\,\mathrm{Log}\,\tau_1$	$\mathrm{Log}\,\tau_1 - \mathrm{Log}\,\tau_2$	$\partial_x\partial_t\,\mathrm{Log}\,\tau$
$\tau(\psi)$	$\tau = \psi$	$\tau = \mathrm{G25}(\psi)$	$Y^{-1} = \dfrac{\tau_2}{\tau_1} = \dfrac{\psi_2}{\psi_1}$	$\tau = \psi$
Lax	$\mathrm{scalar}_2(\psi)$	$\mathrm{scalar}_3(\psi)$	$\mathrm{matrix}_2(\psi_1, \psi_2)$	$\mathrm{matrix}_3(\psi_x, \psi_t, \psi)$
Basis	$\dfrac{\psi_x}{\psi}$	$\dfrac{\psi_x}{\psi}, \dfrac{\psi_{xx}}{\psi}$	$\dfrac{\psi_1}{\psi_2}$	$\dfrac{\psi_x}{\psi}, \dfrac{\psi_t}{\psi}$
Ref.	**[31]**	**[25]**	**[28]**	**[9]**

https://doi.org/10.1017/S0017089501000027 Published online by Cambridge University Press

We do not know of any unsuccessful truncation of a $1 + 1$-dimensional PDE.

ACKNOWLEDGEMENTS. RC thanks the organizers for their financial support. MM acknowledges the financial support extended within the framework of the IUAP Contract No. P4/08 funded by the Belgian government and the support of CEA. RC and MM acknowledge the financial support of the Tournesol grant No. T99/040.

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