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PAIRINGS OF HOMOTOPY SETS OVER AND UNDER B

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ABSTRACT. We define pairings and copairings in the category of topological spaces over and under B. We prove a relation between pairings of homotopy sets induced by a pairing and a copairing of topological spaces over and under B. We obtain some properties of the axes of pairings and the homotopy set of the axes of pairings in the category of topological spaces over and under B. We also prove the dual results.

Introduction. In the previous papers [11, 12], we obtained fundamental properties of pairings and copairings in the *category of equivariant topological spaces with base point*. The purpose of this paper is to study pairings and copairings in the *category of topological spaces over and under B* [4, 5]. We show that many results on pairings and copairings in the category of topological spaces with base point can be generalized to the category of topological spaces over and under B.

In §1, we review some definitions in the category $\mathbf{Top}_B^B$ of topological spaces over and under B. In this category $\mathbf{Top}_B^B$, there are fibrewise product $X \times_B Y$ and fibrewise wedge sum $X \vee_B Y$ for any spaces X and Y. These constructions will be used for the definition of pairings and copairings.

In §2, we consider pairings and copairings in $\mathbf{Top}_B^B$. We call a map $\mu: X \times_B Y \longrightarrow Z$ a pairing with axes $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ if

$$\mu | X \times_B \{*_B\} \simeq_B f \text{ and } \mu | \{*_B\} \times_B Y \simeq_B g,$$

where $*_B$ is the fibrewise base point and the symbol $\simeq_B$ means that the maps are fibrewise pointed homotopic. Dually, we define a copairing $\theta: A \longrightarrow H \vee_B R$ with coaxes $h: A \longrightarrow H$ and $r: A \longrightarrow R$. A pairing $\mu: X \times_B Y \longrightarrow Z$ defines a pairing of homotopy sets in $\mathbf{Top}_B^B$

$$\dot{+}_B : [W, X]_B^B \times [W, Y]_B^B \longrightarrow [W, Z]_B^B,$$

and a copairing $\theta: A \longrightarrow H \vee_B R$ defines another pairing of homotopy sets

$$\dot{+}_B : [H, Z]_B^B \times [R, Z]_B^B \longrightarrow [A, Z]_B^B.$$

The main result of this section is the following theorem.

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THEOREM 2.3. *Let* $\mu: X \times_B Y \longrightarrow Z$ *be a pairing with axes* $f: X \longrightarrow Z$ *and* $g: Y \longrightarrow Z$, *and* $\theta: A \longrightarrow H \vee_B R$ *a copairing with coaxes* $h: A \longrightarrow H$ *and* $r: A \longrightarrow R$ *in* $\mathbf{Top}^B_B$. *Let* $\alpha: H \longrightarrow X$, $\beta: R \longrightarrow X$, $\gamma: H \longrightarrow Y$ *and* $\delta: R \longrightarrow Y$ *be maps in* $\mathbf{Top}^B_B$. *Then the following relations hold in* $[A, Z]^B_B$.

(1) $(\alpha \dot{+}_B \beta) +_B (\gamma \dot{+}_B \delta) = (\alpha +_B \gamma) \dot{+}_B (\beta +_B \delta)$

(2) $h^*(\alpha) \dot{+}_B r^*(\delta) = f_*(\alpha) \dot{+}_B g_*(\delta)$

(3) $r^*(\beta) \dot{+}_B h^*(\gamma) = g_*(\gamma) \dot{+}_B f_*(\beta)$

This theorem corresponds to Theorem 2.7 of [12].

Let us write $f \perp_B g$ if there exists a pairing $\mu: X \times_B Y \longrightarrow Z$ with axes $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$. In §3, we firstly prove some formulae for the relation $f \perp_B g$ of two maps $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ in $\mathbf{Top}^B_B$. We also consider homotopy sets of the axes of pairings in $\mathbf{Top}^B_B$. Let $v: X \longrightarrow Z$ be a map in $\mathbf{Top}^B_B$. Then we define the homotopy set of the axes of pairings by

$$v^{\perp_B}(Y, Z) = \{[g]^B_B: Y \longrightarrow Z \mid v \perp_B g\}.$$

This set is a generalization of the groups introduced by Gottlieb [1, 2], Jiang [6], Varadarajan [13] and Woo and Kim [15]. These results are also closely related with the works by Hoo [3], Kim [7] and Lim [8, 9, 10]. One of the results in this section is the following one.

THEOREM 3.5. *If* A *is a co-grouplike space in* $\mathbf{Top}^B_B$, *then* $(1_X)^{\perp_B}(A, X)$ *is an abelian subgroup which is contained in the center of* $[A, X]^B_B$.

This theorem is a generalization of some results by Gottlieb, Hoo, Lim and Varadarajan (*cf.* Remark 3.6 in §3).

In §4, we prove the dual results of §3, namely, results on copairings in the category $\mathbf{Top}^B_B$.

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1. **Category over and under** B. Let **Top** be the *category of topological spaces*. We work in the category $\mathbf{Top}^B_B$ of the topological spaces *over and under* B. We find terminologies and results in Chapters 1, 3, 5 of James [4] and Chapter IV of James [5].

An object in $\mathbf{Top}^B_B$ is a pair of maps in **Top**

$$B \xrightarrow{u} X \xrightarrow{p} B$$

which satisfies $p \circ u = 1_B$, the *identity map*. Over each point $b \in B$, we regard $u(b)$ the *base point* of the subspace $p^{-1}(b)$ of X, which is called the *fibre* over b. We write $u(b) = *_b$ and $p^{-1}(b) = X_b$. Then $*_B = \{*_b \mid b \in B\}$ is the *fibrewise base point*.

A morphism

$$f: (B \xrightarrow{u} X \xrightarrow{p} B) \longrightarrow (B \xrightarrow{v} Y \xrightarrow{q} B)$$

in $\mathbf{Top}_B^B$ is a map $f: X \longrightarrow Y$ which makes the following diagram

$$\begin{array}{ccccc} B & \xrightarrow{u} & X & \xrightarrow{p} & B \\ \| & & \downarrow f & & \| \\ B & \xrightarrow{v} & Y & \xrightarrow{q} & B \end{array}$$

commutative. So a map $f: X \longrightarrow Y$ in $\mathbf{Top}_B^B$ is a *fibrewise pointed map*. A *fibrewise pointed homotopy relation* is denoted by $\simeq_B$ and the set of the homotopy classes in $\mathbf{Top}_B^B$ by $[X, Y]_B^B$.

Let $X \vee_B Y$ denote the *fibrewise wedge sum* and $X \times_B Y$ the *fibrewise product*. We have natural equivalences

$$[X, Y_1 \times_B Y_2]_B^B \cong [X, Y_1]_B^B \times [X, Y_2]_B^B$$
$$[X_1 \vee_B X_2, Y]_B^B \cong [X_1, Y]_B^B \times [X_2, Y]_B^B.$$

We regard $X \vee_B Y$ as a subspace of $X \times_B Y$ by the inclusion map

$$j_B: X \vee_B Y \subset X \times_B Y.$$

Moreover we use the following maps in $\mathbf{Top}_B^B$.

$T: X \times_B Y \longrightarrow Y \times_B X$ is the *fibrewise switching map*.

$\triangle_{X,B}: X \longrightarrow X \times_B X$ is the *fibrewise diagonal map*.

$\nabla_{X,B}: X \vee_B X \longrightarrow X$ is the *fibrewise folding map*.

$*_B: X \longrightarrow Y$ is the *fibrewise constant map* or *fibrewise base point*.

2. **Pairings and copairings over and under B.** We now define pairings and copairings in the category $\mathbf{Top}_B^B$.

We call a map $\mu: X \times_B Y \longrightarrow Z$ a *pairing* with *axes* $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ if the diagram

$$\begin{array}{ccc} X \times_B Y & \xrightarrow{\mu} & Z \\ j_B \uparrow & & \uparrow \nabla_{Z,B} \\ X \vee_B Y & \xrightarrow[f \vee_B g]{} & Z \vee_B Z \end{array}$$

is homotopy commutative in $\mathbf{Top}_B^B$. A pairing $\mu: X \times_B Y \longrightarrow Z$ with axes $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ defines a pairing of homotopy sets

$$\dot{+}_B : [W, X]_B^B \times [W, Y]_B^B \longrightarrow [W, Z]_B^B$$

by the formula

$$\alpha \dot{+}_B \beta = \mu \circ (\alpha \times_B \beta) \circ \triangle_{W,B}: W \longrightarrow Z$$

for any maps $\alpha: W \longrightarrow X$ and $\beta: W \longrightarrow Y$ in $\mathbf{Top}_B^B$.

This satisfies the relations

(2.1) $$\alpha \dot{+}_B *_B = f_*(\alpha) \text{ and } *_B \dot{+}_B \beta = g_*(\beta) \text{ in } [W, Z]_B^B.$$

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PROOF. If $\beta = *_B$, then we see

$$(\alpha \times_B *_B) \circ \triangle_{W,B} = j_B \circ i_{1B} \circ \alpha \colon W \longrightarrow X \times_B Y$$

for the inclusion map $i_{1B}\colon X \longrightarrow X \vee_B Y$, which is defined by $i_{1B}(x) = (x, *_b)$ when $p(x) = b$. Then we have

$$\alpha \dot{+}_B *_B = \mu \circ (\alpha \times_B *_B) \circ \triangle_{W,B} = \mu \circ j_B \circ i_{1B} \circ \alpha \simeq_B f \circ \alpha = f_*(\alpha).$$

By a similar argument, we have the case that $\alpha = *_B$. ∎

We call a map $\theta\colon A \longrightarrow H \vee_B R$ a *copairing* with *coaxes* $h\colon A \longrightarrow H$ and $r\colon A \longrightarrow R$ if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\theta} & H \vee_B R \\ \triangle_{A,B} \downarrow & & \downarrow j_B \\ A \times_B A & \xrightarrow[h \times_B r]{} & H \times_B R \end{array}$$

is homotopy commutative in $\mathbf{Top}_B^B$. A copairing $\theta\colon A \longrightarrow H \vee_B R$ with coaxes $h\colon A \longrightarrow H$ and $r\colon A \longrightarrow R$ defines a pairing of homotopy sets

$$\dot{+}_B \colon [H, Z]_B^B \times [R, Z]_B^B \longrightarrow [A, Z]_B^B$$

by the formula

$$\alpha \dot{+}_B \beta = \nabla_{Z,B} \circ (\alpha \vee_B \beta) \circ \theta \colon A \longrightarrow Z$$

for any maps $\alpha\colon H \longrightarrow Z$ and $\beta\colon R \longrightarrow Z$ in $\mathbf{Top}_B^B$.

This pairing satisfies relations

(2.2) $$\alpha \dot{+}_B *_B = h^*(\alpha) \text{ and } *_B \dot{+}_B \beta = r^*(\beta) \text{ in } [A, Z]_B^B.$$

PROOF. We prove only the first relation of (2.2), since the other case is proved similarly. If $\beta = *_B$, then we have

$$\nabla_{Z,B} \circ (\alpha \vee_B *_B) = \alpha \circ j_{1B} \colon H \vee_B R \longrightarrow Z$$

for the projection map $j_{1B}\colon H \vee_B R \longrightarrow H$, which is defined by $j_{1B}(x, *_b) = x$ for any element x of H and $j_{1B}(*_b, y) = *_b$ for any element y of R, where $p(x) = q(y) = b$. We remark that

$$j_{1B} \circ \theta = p'_{1B} \circ j_B \circ \theta \simeq_B p'_{1B} \circ (h \times_B r) \circ \triangle_{A,B} = h \circ p_{1B} \circ \triangle_{A,B} = h \circ 1_A = h,$$

where $j_B\colon H \vee_B R \longrightarrow H \times_B R$ is the inclusion map, and $p_{1B}\colon A \times_B A \longrightarrow A$ and $p'_{1B}\colon H \times_B R \longrightarrow H$ are projection maps to the first factor. Then we have

$$\alpha \dot{+}_B *_B = \nabla_{Z,B} \circ (\alpha \vee_B *_B) \circ \theta = \alpha \circ j_{1B} \circ \theta \simeq_B \alpha \circ h = h^*(\alpha).$$

THEOREM 2.3. *Let $\mu: X \times_B Y \longrightarrow Z$ be a pairing with axes $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$, and $\theta: A \longrightarrow H \vee_B R$ a copairing with coaxes $h: A \longrightarrow H$ and $r: A \longrightarrow R$ in* $\mathbf{Top}_B^B$. *Let $\alpha: H \longrightarrow X$, $\beta: R \longrightarrow X$, $\gamma: H \longrightarrow Y$ and $\delta: R \longrightarrow Y$ be maps in* $\mathbf{Top}_B^B$. *Then the following relations hold in* $[A, Z]_B^B$.

(1) $(\alpha \dot{+}_B \beta) \underset{\cdot}{+}_B (\gamma \dot{+}_B \delta) = (\alpha \underset{\cdot}{+}_B \gamma) \dot{+}_B (\beta \underset{\cdot}{+}_B \delta)$

(2) $h^*(\alpha) \underset{\cdot}{+}_B r^*(\delta) = f_*(\alpha) \dot{+}_B g_*(\delta)$

(3) $r^*(\beta) \underset{\cdot}{+}_B h^*(\gamma) = g_*(\gamma) \dot{+}_B f_*(\beta)$

PROOF. (1) Over each point $b \in B$, the maps on both sides of the equation coincide (*cf.* proof of Theorem 2.7 of Oda [12]). So we have the equation.

(2) is obtained by setting $\beta = \gamma = *$ in (1) and using (2.1) and (2.2).

(3) is obtained by setting $\alpha = \delta = *$ in (1) and using (2.1) and (2.2). ∎

As immediate consequences of Theorem 2.3, we have the following results.

COROLLARY 2.4. *Let $\mu: X \times_B Y \longrightarrow Z$ be a pairing with axes $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ and $\theta: A \longrightarrow H \vee_B R$ a copairing with coaxes $h: A \longrightarrow H$ and $r: A \longrightarrow R$ in* $\mathbf{Top}_B^B$.

(1) If $X = Y$ and $f \simeq_B g$, then

$$h^*(\alpha) \underset{\cdot}{+}_B r^*(\beta) = r^*(\beta) \underset{\cdot}{+}_B h^*(\alpha)$$

in $[A, Z]_B^B$ for any maps $\alpha: H \longrightarrow X$ and $\beta: R \longrightarrow X$ in $\mathbf{Top}_B^B$.

(2) If $H = R$ and $h \simeq_B r$, then

$$f_*(\beta) \dot{+}_B g_*(\delta) = g_*(\delta) \dot{+}_B f_*(\beta)$$

in $[A, Z]_B^B$ for any maps $\beta: H \longrightarrow X$ and $\delta: H \longrightarrow Y$ in $\mathbf{Top}_B^B$.

3. **Axes of pairings over and under B.** We write $f \perp_B g$ when there exists a pairing $\mu: X \times_B Y \longrightarrow Z$ with axes $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$ in $\mathbf{Top}_B^B$.

Dually we write $h \top_B r$ when there exists a copairing $\theta: A \longrightarrow H \vee_B R$ with coaxes $h: A \longrightarrow H$ and $r: A \longrightarrow R$ in $\mathbf{Top}_B^B$.

We call a map $g: Y \longrightarrow X$ a *cyclic map* in $\mathbf{Top}_B^B$ when $1_X \perp_B g$.

We call a map $r: A \longrightarrow R$ a *cocyclic map* in $\mathbf{Top}_B^B$ when $1_A \top_B r$.

We call a space X a *Hopf space* in $\mathbf{Top}_B^B$ when $1_X \perp_B 1_X$.

We call a space A a *co-Hopf space* in $\mathbf{Top}_B^B$ when $1_A \top_B 1_A$.

If A is a co-Hopf space in $\mathbf{Top}_B^B$ or if X is a Hopf space in $\mathbf{Top}_B^B$, then the homotopy set $[A, X]_B^B$ has a binary operation $\dot{+}_B$ or $\underset{\cdot}{+}_B$ respectively. If A is a co-Hopf space and X is a Hopf space, then the two binary operations $\dot{+}_B$ and $\underset{\cdot}{+}_B$ in $[A, X]_B^B$ coincide and, moreover, these binary operations are abelian and associative (Set $h \simeq_B r \simeq_B 1_A$ and $f \simeq_B g \simeq_B 1_X$ in Theorem 2.3).

Although the homotopy set $[A, X]$ has a binary operation with unit when A is a co-Hopf space or when X is a Hopf space, it is not necessarily a monoid in a usual sense (since we do not assume associativity for co-Hopf structure of A or Hopf structure of X

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in general). So we give the following definition for Proposition 3.1 and Theorems 3.5 and 4.6: If S is a set with a binary operation $\bullet$, then we call the subset

$$\{z \in S \mid z \bullet x = x \bullet z \text{ for all } x \in S\} \subset S$$

the *center* of S.

The following results are special cases of Corollary 2.4 and say that cyclic maps and cocyclic maps are contained in the center of homotopy sets (*cf.* Remark 3.6).

PROPOSITION 3.1. (1) *Let X be a Hopf space in* $\mathbf{Top}_B^B$. *If $r: A \longrightarrow R$ is a cocyclic map, then the image of*

$$r^*: [R, X]_B^B \longrightarrow [A, X]_B^B$$

is contained in the center of $[A, X]_B^B$.

(2) *Let A be a co-Hopf space in* $\mathbf{Top}_B^B$. *If $g: Y \longrightarrow X$ is a cyclic map, then the image of*

$$g_*: [A, Y]_B^B \longrightarrow [A, X]_B^B$$

is contained in the center of $[A, X]_B^B$.

PROOF. (1) Set $f \simeq_B g \simeq_B 1_X$ and $h \simeq_B 1_A$ in Corollary 2.4(1), then we have the result.

(2) Set $h \simeq_B r \simeq_B 1_A$ and $f \simeq_B 1_X$ in Corollary 2.4(2), then we have the result. ∎

Now, we study some properties of the relation $f \perp_B g$ for maps $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$.

THEOREM 3.2. *If $f \perp_B g$ for maps $f: X \longrightarrow Z$ and $g: Y \longrightarrow Z$, then the following results hold.*

(1) $(f \circ f') \perp_B (g \circ g')$ for any maps $f': X' \longrightarrow X$ and $g': Y' \longrightarrow Y$.

(2) $(w \circ f) \perp_B (w \circ g)$ for any map $w: Z \longrightarrow W$.

PROOF. Let $\mu: X \times_B Y \longrightarrow Z$ be a pairing for $f \perp_B g$. Then $\mu \circ (f' \times_B g')$ and $w \circ \mu$ are those for $(f \circ f') \perp_B (g \circ g')$ and $(w \circ f) \perp_B (w \circ g)$ respectively.

PROPOSITION 3.3. *Let $f_1: X_1 \longrightarrow Z_1$, $f_2: X_2 \longrightarrow Z_2$, $g_1: Y_1 \longrightarrow Z_1$ and $g_2: Y_2 \longrightarrow Z_2$ be maps in* $\mathbf{Top}_B^B$. *Then $f_1 \perp_B g_1$ and $f_2 \perp_B g_2$ implies $(f_1 \times_B f_2) \perp_B (g_1 \times_B g_2)$.*

PROOF. Let $\mu_1: X_1 \times_B Y_1 \longrightarrow Z_1$ and $\mu_2: X_2 \times_B Y_2 \longrightarrow Z_2$ be pairings for $f_1 \perp_B g_1$ and $f_2 \perp_B g_2$ respectively. Then the composition of maps

$$(\mu_1 \times_B \mu_2) \circ (1_{X_1} \times_B T \times_B 1_{Y_2}): (X_1 \times_B X_2) \times_B (Y_1 \times_B Y_2) \longrightarrow Z_1 \times_B Z_2$$

is a pairing for $(f_1 \times_B f_2) \perp_B (g_1 \times_B g_2)$ where $T: X_2 \times_B Y_1 \longrightarrow Y_1 \times_B X_2$ is the switching map.

THEOREM 3.4. *Let $f: X \longrightarrow Z$, $v: V \longrightarrow Z$, $g: Y \longrightarrow V$ and $w: W \longrightarrow V$ be maps and $\theta: A \longrightarrow X \vee_B Y$ a copairing in* $\mathbf{Top}_B^B$. *Then $f \perp_B v$ and $g \perp_B w$ implies $\{f \dotplus_B (v \circ g)\} \perp_B (v \circ w)$.*

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PROOF. Let $\mu_1: X \times_B V \longrightarrow Z$ and $\mu_2: Y \times_B W \longrightarrow V$ be pairings for $f \perp_B v$ and $g \perp_B w$ respectively. Then the composition of maps

$$\mu_1 \circ (1_X \times_B \mu_2) \circ (j_B \times_B 1_W) \circ (\theta \times_B 1_W): A \times_B W \longrightarrow Z$$

is a pairing for $\{f \dotplus_B (v \circ g)\} \perp_B (v \circ w)$. ∎

For a fixed map $v: X \longrightarrow Z$ in $\mathbf{Top}_B^B$, we define

$$v^{\perp_B}(Y, Z) = \{[g]_B^B: Y \longrightarrow Z \mid v \perp_B g\}$$

where $[g]_B^B: Y \longrightarrow Z$ is the homotopy class of $g: Y \longrightarrow Z$ in $\mathbf{Top}_B^B$.

We call a space A a *co-grouplike space* [4, 5, 14] in $\mathbf{Top}_B^B$ when A is a homotopy associative co-Hopf space in $\mathbf{Top}_B^B$ with an inverse $\nu: A \longrightarrow A$, namely, $1_A \dotplus_B \nu \simeq_B *_B \simeq_B \nu \dotplus_B 1_A$.

THEOREM 3.5. *If A is a co-grouplike space in $\mathbf{Top}_B^B$, then $(1_X)^{\perp_B}(A, X)$ is an abelian subgroup which is contained in the center of $[A, X]_B^B$.*

PROOF. The subset $(1_X)^{\perp_B}(A, X)$ is closed under the operation $\dotplus_B$ induced by the co-Hopf structure of A, as we see by setting $v = w = 1_X$, $X = Y = A$ and $f, g: A \longrightarrow X$ in Theorem 3.4. It is contained in the center of $[A, X]_B^B$ by Proposition 3.1(2) (Set $A = Y$).

If $[\alpha]_B^B \in (1_X)^{\perp_B}(A, X)$, then $\alpha \perp_B 1_X$. It follows that $(\alpha \circ \nu) \perp_B 1_X$ for the inverse $\nu: A \longrightarrow A$ by Theorem 3.2(1) or $-[\alpha]_B^B = [\alpha \circ \nu]_B^B \in (1_X)^{\perp_B}(A, X)$. Thus $(1_X)^{\perp_B}(A, X)$ is a subgroup of $[A, X]_B^B$.

REMARK 3.6. There are following works in connection with Theorem 3.5; Theorem I.4 of Gottlieb [1], Theorem 2 of Hoo [3], Proposition 4.13 of Lim [8], Proposition 4.3 of Lim [9] and Theorem 1.5 of Varadarajan [13].

Dual results: Corollary 3.10 and Theorem 4.2 of Lim [10] (*cf.* Theorem 4.6).

PROPOSITION 3.7. (1) *If $v \perp_B g$ for maps $v: X \longrightarrow Z$ and $g: Y \longrightarrow Z$, then*

$$\mathrm{Im}(g_*: [A, Y]_B^B \longrightarrow [A, Z]_B^B) \subset v^{\perp_B}(A, Z)$$

for any space A in $\mathbf{Top}_B^B$.

(2) *$v^{\perp_B}(Y, Z) \subset (v \circ d)^{\perp_B}(Y, Z)$ for any maps $v: X \longrightarrow Z$ and $d: D \longrightarrow X$.*

PROOF. (1) Let $\alpha: A \longrightarrow Y$ be any map. Then $v \perp_B g$ implies $v \perp_B (g \circ \alpha)$ by Theorem 3.2(1). It follows that $g_*([\alpha]_B^B) = [g \circ \alpha]_B^B \in v^{\perp_B}(A, Z)$.

(2) is proved similarly.

THEOREM 3.8. *If Z is a Hopf space in $\mathbf{Top}_B^B$, then $v^{\perp_B}(Y, Z) = [Y, Z]_B^B$ for any map $v: X \longrightarrow Z$.*

PROOF. Since Z is a Hopf space, we have $1_Z \perp_B 1_Z$. If $\alpha: Y \longrightarrow Z$ is any map, we have $(1_Z \circ v) \perp_B (1_Z \circ \alpha)$ by Theorem 3.2(1) and hence $v \perp_B \alpha$. It follows that $[\alpha]_B^B \in v^{\perp_B}(Y, Z)$.

4. **Coaxes of copairings over and under** B. We now consider the duals of the results in the previous section.

As we defined in §3, we write $h \top_B r$ if there exists a copairing $\theta: A \longrightarrow H \vee_B R$ with coaxes $h: A \longrightarrow H$ and $r: A \longrightarrow R$ in $\mathbf{Top}_B^B$.

THEOREM 4.1. *If* $h \top_B r$ *for maps* $h: A \longrightarrow H$ *and* $r: A \longrightarrow R$, *then the following results hold.*

(1) $(h' \circ h) \top_B (r' \circ r)$ *for any maps* $h': H \longrightarrow H'$ *and* $r': R \longrightarrow R'$.

(2) $(h \circ d) \top_B (r \circ d)$ *for any map* $d: D \longrightarrow A$.

PROOF. Let $\theta: A \longrightarrow H \vee_B R$ be a copairing for $h \top_B r$. Then the compositions of maps $(h' \vee_B r') \circ \theta: A \longrightarrow H' \vee_B R'$ and $\theta \circ d: D \longrightarrow H \vee_B R$ are copairings for $(h' \circ h) \top_B (r' \circ r)$ and $(h \circ d) \top_B (r \circ d)$ respectively.

PROPOSITION 4.2. *Let* $h_1: A_1 \longrightarrow H_1$, $h_2: A_2 \longrightarrow H_2$, $r_1: A_1 \longrightarrow R_1$ *and* $r_2: A_2 \longrightarrow R_2$ *be maps in* $\mathbf{Top}_B^B$. *Then* $h_1 \top_B r_1$ *and* $h_2 \top_B r_2$ *implies* $(h_1 \vee_B h_2) \top_B (r_1 \vee_B r_2)$.

PROOF. Let $\theta_1: A_1 \longrightarrow H_1 \vee_B R_1$ and $\theta_2: A_2 \longrightarrow H_2 \vee_B R_2$ be copairings for $h_1 \top_B r_1$ and $h_2 \top_B r_2$ respectively. Then the composition of maps

$$(1_{H_1} \vee_B T \vee_B 1_{R_2}) \circ (\theta_1 \vee_B \theta_2): A_1 \vee_B A_2 \longrightarrow (H_1 \vee_B H_2) \vee_B (R_1 \vee_B R_2)$$

is a copairing for $(h_1 \vee_B h_2) \top_B (r_1 \vee_B r_2)$, where $T: R_1 \vee_B H_2 \longrightarrow H_2 \vee_B R_1$ is the switching map.

THEOREM 4.3. *Let* $h: A \longrightarrow H$, $r: A \longrightarrow R$, $u: H \longrightarrow U$ *and* $d: H \longrightarrow D$ *be maps and* $\mu: D \times_B R \longrightarrow Z$ *a pairing in* $\mathbf{Top}_B^B$. *Then* $u \top_B d$ *and* $h \top_B r$ *implies*

$$(u \circ h) \top_B \{(d \circ h) \dotplus_B r\}.$$

PROOF. Let $\theta_1: A \longrightarrow H \vee_B R$ and $\theta_2: H \longrightarrow U \vee_B D$ be copairings for $h \top_B r$ and $u \top_B d$ respectively. Then the composition of maps

$$(1_U \vee_B \mu) \circ (1_U \vee_B j_B) \circ (\theta_2 \vee_B 1_R) \circ \theta_1: A \longrightarrow U \vee_B Z$$

is a copairing for $(u \circ h) \top_B \{(d \circ h) \dotplus_B r\}$. ∎

Let $u: A \longrightarrow U$ be a fixed map. We define

$$u^{\top_B}(A, R) = \{[r]_B^B: A \longrightarrow R \mid u \top_B r\}.$$

PROPOSITION 4.4. (1) *If* $u \top_B r$ *for maps* $u: A \longrightarrow U$ *and* $r: A \longrightarrow R$, *then*

$$\mathrm{Im}(r^*: [R, X]_B^B \longrightarrow [A, X]_B^B) \subset u^{\top_B}(A, X)$$

for any space X *in* $\mathbf{Top}_B^B$.

(2) $u^{\top_B}(A, X) \subset (w \circ u)^{\top_B}(A, X)$ *for any maps* $u: A \longrightarrow U$ *and* $w: U \longrightarrow W$.

PROOF. (1) Let $\alpha: R \longrightarrow X$ be any map. Then $u \top_B r$ implies $u \top_B (\alpha \circ r)$ by Theorem 4.1(1) and hence $r^*([\alpha]_B^B) = [\alpha \circ r]_B^B \in u^{\top_B}(A, X)$.

(2) is proved similarly.

THEOREM 4.5. *If A is a co-Hopf space in $\mathbf{Top}_B^B$, then $u^{\top_B}(A, R) = [A, R]_B^B$ for any map $u: A \longrightarrow U$ and any space R in $\mathbf{Top}_B^B$.*

PROOF. Since A is a co-Hopf space, we have $1_A \top_B 1_A$. Then for any map $\alpha: A \longrightarrow R$, we have $(u \circ 1_A) \top_B (\alpha \circ 1_A)$ by Theorem 4.1(1) and hence $u \top_B \alpha$. It follows that $[\alpha]_B^B \in u^{\top_B}(A, R)$. ∎

We call a space Z a *grouplike space* [4, 5, 14] in $\mathbf{Top}_B^B$ when Z is a homotopy associative Hopf space in $\mathbf{Top}_B^B$ with an inverse $\nu: Z \longrightarrow Z$, namely, $1_Z \dot{+}_B \nu \simeq_B *_B \simeq_B \nu \dot{+}_B 1_Z$.

The following theorem is a generalization of Theorem 4.2 of Lim [10].

THEOREM 4.6. *Let Z be a grouplike space and A any space in $\mathbf{Top}_B^B$. Then $(1_A)^{\top_B}(A, Z)$ is an abelian subgroup which is contained in the center of $[A, Z]_B^B$.*

PROOF. The subset $(1_A)^{\top_B}(A, Z)$ is closed under the operation $\dot{+}_B$ induced by the Hopf structure of Z by Theorem 4.3 (Set $u = h = 1_A$ and $D = R = Z$).

Suppose that $[\alpha]_B^B \in (1_A)^{\top_B}(A, Z)$ or $1_A \top_B \alpha$. Then we have $1_A \top_B (\nu \circ \alpha)$ by Theorem 4.1(1). It follows that $-[\alpha]_B^B = [\nu \circ \alpha]_B^B \in (1_A)^{\top_B}(A, Z)$. Thus $(1_A)^{\top_B}(A, Z)$ has a group structure.

Moreover, any element $[\alpha]_B^B \in (1_A)^{\top_B}(A, Z)$ is contained in the center of $[A, Z]_B^B$ by Proposition 3.1(1) (Set $R = X = Z$).

REFERENCES

1. D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87**(1965), 840–856.
2. ———, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91**(1969), 729–756.
3. C. S. Hoo, *Cyclic maps from suspensions to suspensions*, Canad. J. Math. **24**(1972), 789–791.
4. I. M. James, *General topology and homotopy theory*, Springer-Verlag, New York, 1984.
5. ———, *Fibrewise topology*, Cambridge Tracts in Mathematics **91**, Cambridge Univ. Press, Cambridge, 1989.
6. B. J. Jiang, *Estimation of the Nielsen numbers*, Chinese Math. Acta **5**(1964), 330–339.
7. J.-R. Kim, *Localizations and generalized evaluation subgroups of homotopy groups*, J. Korean Math. Soc. **22**(1985), 9–18.
8. K. L. Lim, *On cyclic maps*, J. Austral. Math. Soc. (A) **32**(1982), 349–357.
9. ———, *On evaluation subgroups of generalized homotopy groups*, Canad. Math. Bull. **27**(1984), 78–86.
10. ———, *Cocyclic maps and coevaluation subgroups*, Canad. Math. Bull. **30**(1987), 63–71.
11. N. Oda, *The homotopy set of the axes of pairings*, Canad. J. Math., **42**(1990), 856–868.
12. ———, *Pairings and copairings in the category of topological spaces*, Publ. RIMS Kyoto Univ., **28**(1992), 83–97.
13. K. Varadarajan, *Generalized Gottlieb groups*, J. Indian Math. Soc. **33**(1969), 141–164.

14. G. W. Whitehead, *Elements of homotopy theory*, Graduate texts in Math. **61**, Springer-Verlag, 1978.
15. M. H. Woo and J.-R. Kim, *Certain subgroups of homotopy groups*, J. Korean Math. Soc. **21**(1984), 109–120.

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