ARTICLES

PARETO-OPTIMAL PROFIT-SHARING

BY MARTINA VANDEBROEK

Instituut voor Actuariële Wetenschappen, K.U.Leuven, Belgium

ABSTRACT

The Pareto-optimal design for profit-sharing is derived under general assumptions as to the utility function of both the insured and the insurer. This generalizes the result of Jones and Gerber and explains commonly used dividend formulas in terms of risk aversion.

KEYWORDS

Profit-sharing; Pareto-optimality; optimal control theory.

1. INTRODUCTION

Experience rating in group life insurance arose because of the threat of self-insurance on the part of the "good risks." This antiselection was — and still is — prevented by offering to the policyholder a profit-sharing plan in which it is stipulated that the insurer will refund some part of the profit he makes on that particular policy. This profit-sharing usage is also commonly used in most non-life insurances and in reinsurance where the greater uncertainty about the total claim distribution is reflected in higher safety loadings and therefore in higher potential profits.

The repayment is defined by a dividend formula which expresses the refund in terms of the claim experience. Several formulas are conceivable, but there are two designs that are frequently used and have been studied by BERNHARDT and ENDRES (1979), DRUDE and NIEDERHAUSEN (1973–1974), JONES and GERBER (1974), SCHMUTZ (1985), STRICKLER (1982) and ZOPPI (1982):

(1) $$W = (\alpha P' - S)_+, \qquad \alpha \leqslant 1$$

(2) $$W = \beta(P' - S)_+, \qquad \beta \leqslant 1$$

where W = the refund

$P' = P(1+\theta)$ with P the net premium and θ the safety loading

S = the total claim amount

and $(x)_+ = \max(0, x)$.

In (1), the insurer refunds that part of the profit that exceeds some predetermined constant $(1-\alpha)P'$, whereas in (2) the profit $(P' - S)_+$ is divided according to some proportional rule.

Further restrictions on the parameters are obtained by stating that the expected result of the insurer must satisfy a given solvency requirement. The parameter values actually used are mostly a matter of bargaining.

The choice between the two designs is harder to tackle in a theoretical context. One of the possible ways to deal with competing preferences is using utility functions and looking for a Pareto-optimal solution (i.e. a solution such that it can not be improved for one of the two competitors without harming the other).

JONES and GERBER (1974) proved that the Pareto-optimal solution responds to formula (1), in case the insured is risk-neutral and the insurer has a concave utility function. The Pareto-optimal solution for the general case, which allows both the insured and the insurer to have a concave utility function, will be derived in this paper with the aid of optimal control theory.

As was pointed out by the referees, this result can also be obtained as a special case of the theorem of Borch. This approach will be demonstrated in the Appendix.

2. NOTATIONS AND DEFINITIONS

Denote the utility function of the policyholder by $u(x)$ and assume that the conditions $u'(x) > 0$ and $u''(x) \leqslant 0$ are satisfied. Analogously, let $v(x)$ be the utility function of the insurer, with $v'(x) > 0$ and $v''(x) \leqslant 0$. Thus, both the insured and the insurer are supposed to be risk-averse or risk-neutral. Their risk-aversion can be measured by

$$(3) \qquad R_u(x) = -\frac{u''(x)}{u'(x)} \qquad \text{and} \qquad R_v(x) = -\frac{v''(x)}{v'(x)}.$$

Let S denote the total claim amount and $X = (P' - S)_+$ the profit in the considered period. Note that the premium P' is the risk premium supplemented by a safety loading but without any loading for administration costs. The refund that corresponds to gain x will be represented by $W(x)$.

Denote by $f_X(x)$ and $f_S(s)$ the probability density functions of X and S. There exists a close relationship between these two functions:

$$f_X(0) = \int_{P'}^{\infty} f_S(s)\, ds \qquad \text{and} \qquad f_X(x) = f_S(P' - x) \qquad \text{for } x > 0.$$

With these notations, and denoting by w_u the initial capital of the policyholder and by w_v the capital of the insurer, it is possible to express the conditions for a Pareto-optimal solution in a more formal way.

DEFINITION 1. The dividend formula $W(x)$ is Pareto-optimal if, for every other refund formula $\tilde{W}(x)$ for which

$$E[u(w_u - P' + \tilde{W}(X))] \geqslant E[u(w_u - P' + W(X))] \qquad \text{and}$$
$$E[v(w_v + P' - S - \tilde{W}(X))] \geqslant E[v(w_v + P' - S - W(X))],$$

both $\geqslant$ signs can only be equalities.

It is easy to see that this is equivalent to the following definition.

DEFINITION 2. The solution to the following optimization problem is Pareto-optimal: maximize $E[u(w_u - P' + W(X))]$ subject to the constraint $E[v(w_v + P' - S - W(X))] \geqslant c$ (c an arbitrary constant), over the feasible set of refund formulas.

One natural constraint on the set of all possible designs for $W(x)$ is

$$0 \leqslant W(x) \leqslant x$$

as the insured will not pay a surplus in case of bad experience and the insurer will not pay more than he gains in order to stay solvable.

Using the notation introduced above, the problem can be stated as follows:

$$\underset{W}{\text{Max}} \int_0^{P'} u(w_u - P' + W(P' - s)) f_S(s) ds + \int_{P'}^{\infty} u(w_u - P') f_S(s) ds$$

subject to

$$0 \leqslant W(x) \leqslant x$$

$$\int_0^{P'} v(w_v + P' - s - W(P' - s)) f_S(s) ds + \int_{P'}^{\infty} v(w_v + P' - s) f_S(s) ds \geqslant c$$

where c must be smaller than or equal to $E[v(w_v + P' - S)]$ in order to get a non-empty feasible set of refund formulas.

An equivalent formulation of this problem is:

$$\underset{W}{\text{Max}} \int_0^{P'} u(w_u - P' + W(x)) f_X(x) dx \tag{4}$$

subject to

$$0 \leqslant W(x) \leqslant x \tag{5}$$

$$\int_0^{P'} v(w_v + x - W(x)) f_X(x) dx \geqslant k \tag{6}$$

and

$$k \leqslant \int_0^{P'} v(w_v + P' - s) f_S(s) ds \tag{7}$$

where

$$k = c - \int_{P'}^{\infty} v(w_v + P' - s) f_S(s) ds.$$

In the following section we will derive the solution to this problem, where the maximum is sought over the family of all piecewise continuous functions on $[0, P']$.

3. THE PARETO-OPTIMAL DESIGN

THEOREM. *Depending on k, the solution to the problem (4) under the constraints (5), (6) and (7) takes one of two possible forms in which the k^*, x_1^* and x_2^* will be defined in the proof.*

If $k < k^*$

$$W(x) = x \qquad x \leqslant x_1^* \tag{8}$$
$$0 \leqslant W(x) \leqslant x \qquad x > x_1^*$$

where $W(x)$ *is determined by the differential equation*

$$W'(x) = \frac{R_v(w_v + x - W(x))}{R_u(w_u - P' + W(x)) + R_v(w_v + x - W(x))} \tag{9}$$

with the boundary condition $W(x_1^*) = x_1^*$.

If $k \geqslant k^*$

$$W(x) = 0 \qquad x \leqslant x_2^* \tag{10}$$
$$0 \leqslant W(x) \leqslant x \qquad x > x_2^*$$

where $W(x)$ *is also determined by the differential equation (9) but with the boundary condition* $W(x_2^*) = 0$.

PROOF. The problem can easily be solved via optimal control theory, see e.g. KAMIEN and SCHWARTZ (1981), if we rewrite the constraint (6) as:

$$z'(x) = v(w_v + x - W(x))f_X(x) \qquad \text{with} \qquad z(0) = 0 \qquad \text{and} \qquad z(P') \geqslant k.$$

The Lagrangian for this problem is:

$$L = u(w_u - P' + W(x))f_X(x) + \lambda(x)v(w_v + x - W(x))f_X(x) \tag{11}$$
$$+ \beta_1(x)W(x) + \beta_2(x)(x - W(x))$$

where $\lambda(x)$ is a continuous function and $\beta_1(x)$ and $\beta_2(x)$ are piecewise continuous functions, such that for all $x \in [0, P']$ where the β_i are continuous, the following conditions are satisfied:

$$\lambda'(x) = -\frac{\delta L}{\delta z} \tag{12}$$

$$\lambda(P') \geqslant 0 \qquad \text{and} \qquad \lambda(P') = 0 \qquad \text{if} \qquad z(P') > k \tag{13}$$

$$\beta_1(x)W(x) = 0, \qquad \beta_1(x) \geqslant 0 \tag{14}$$

$$\beta_2(x)(x - W(x)) = 0, \qquad \beta_2(x) \geqslant 0. \tag{15}$$

As all the concavity requirements are satisfied, the optimal $W(x)$ is then found by maximizing L.

We will assume in the sequel that $f_X(x) > 0$, because the values of x where $f_X(x) = 0$ are of no interest to this problem. It follows from (11) and (12) that $\lambda(x)$ is a constant function of x because the Lagrangian does not depend on z. We will denote this constant by λ.

We must distinguish two cases.

(a) If constraint (6) is not binding, then it follows from (13) that $\lambda = 0$. The condition to maximize L with respect to W is then:

(16) $$u'(w_u - P' + W(x))f_X(x) + \beta_1(x) - \beta_2(x) = 0.$$

From (14) and (15) it is immediate that $W(x) = 0$ can never be optimal in this case and that $W(x) = x$ will be the optimal solution. This is to be expected: if the restriction that the insurer puts on his expected utility is not binding, the utility of the policyholder will be maximized by refunding as much as possible.

(b) Now consider the more realistic case that (6) is binding, then λ is uniquely determined by the equation

(17) $$\int_0^{P'} v(w_v + x - W(x))f_X(x)dx = k$$

where the optimal solution $W(x)$ is expressed in terms of λ. Then $W(x) = x$ will be optimal if

(18) $$H_1(x) = u'(w_u - P' + x) - \lambda v'(w_v) \geqslant 0$$

and $W(x) = 0$ is optimal if

(19) $$H_2(x) = u'(w_u - P') - \lambda v'(w_v + x) \leqslant 0.$$

Because H_1 is a continuous decreasing function in x and H_2 is a continuous increasing function in x, and since $H_1(0) = H_2(0)$, (18) and (19) can not occur simultaneously and one of these conditions has to be satisfied up to some x. So either

$W(x) = x$ for $x \leqslant x_1^*$ where x_1^* is the solution of

(20) $$H_1(x) = 0$$

or

$W(x) = 0$ for $x \leqslant x_2^*$ where x_2^* is the solution of

(21) $$H_2(x) = 0.$$

If $0 < W(x) < x$ then the solution is determined by

(22) $$u'(w_u - P' + W(x)) - \lambda v'(w_v + x - W(x)) = 0.$$

Differentiating (22) with respect to x, the following equation is obtained

(23) $$u''(w_u - P' + W(x))W'(x) - \lambda v''(w_v + x - W(x))(1 - W'(x)) = 0.$$

By solving (22) for λ and inserting this expression for λ in (23), (9) is obtained.

As $W(x)$ is a decreasing function of λ and of k, λ is increasing with k. So we can translate the conditions for the different solutions in terms of k. Denote by k^* the k-bound belonging to the special case

(24) $$H_1(0) = H_2(0) = 0 \qquad \text{or} \qquad \lambda = \frac{u'(w_u - P')}{v'(w_v)}.$$

Then the conditions (18) and (19) are equivalent with $k < k^*$ and $k \geqslant k^*$, which proves the theorem.

REMARK. It may not be clear how this solution can be computed, because of the unknown λ. As the relationship between k and λ is rather complex, the easiest way to obtain the solution is as follows: express x_1^* or x_2^* and the solution of the differential equation (9) in terms of λ. Insert this solution in constraint (6) and by trial and error the value of λ belonging to the given k-bound can be found.

4. SPECIAL CASES

In the special case where the insured is risk-neutral, $R_u(x) = 0$, the solution is given by

$$W'(x) = 1 \qquad \text{or} \qquad W(x) = x + \text{constant for } x \geqslant x^*.$$

From $W(x) \leqslant x$ it follows that the constant has to be negative.

If $k \leqslant k^*$ the constant must be zero, whereas for $k > k^*$ the boundary condition is $W(x_2^*) = 0$ and thus the constant must equal $-x_2^*$. So in any case the first design (1) is optimal, which is the result that was found by JONES and GERBER (1974).

For the more realistic case, where the insurer is assumed to be risk-neutral and the insured is risk-averse, the optimal dividend formula is derived from $W'(x) = 0$, and thus $W(x) = \text{constant}$ for $x > x^*$. So the optimum takes the form of

$$\begin{aligned} W(x) &= x & x &\leqslant x_1^* \\ W(x) &= x_1^* & x &> x_1^* \end{aligned}$$

with the limiting case

$$W(x) = 0 \quad \text{for all } x.$$

Note that this case, which corresponds to the most intuitive ideas with respect to the utility functions of an insurer and an insured, has a Pareto-optimal solution that has not been considered before.

5. EXAMPLE

We assume that both the insured and the insurer have exponential utility functions:

$$u(x) = \frac{1 - \exp(-ax)}{a} \qquad \text{and} \qquad v(x) = \frac{1 - \exp(-bx)}{b}.$$

In this case the risk aversion coefficients are constant:

$$R_u(x) = a \qquad \text{and} \qquad R_v(x) = b.$$

It follows from the theorem that a Pareto-optimal refund formula is either of the type

$$W(x) = \begin{cases} x & x \leqslant x_1^* \\ x_1^* + \dfrac{b}{a+b}(x - x_1^*) & x > x_1^* \end{cases}$$

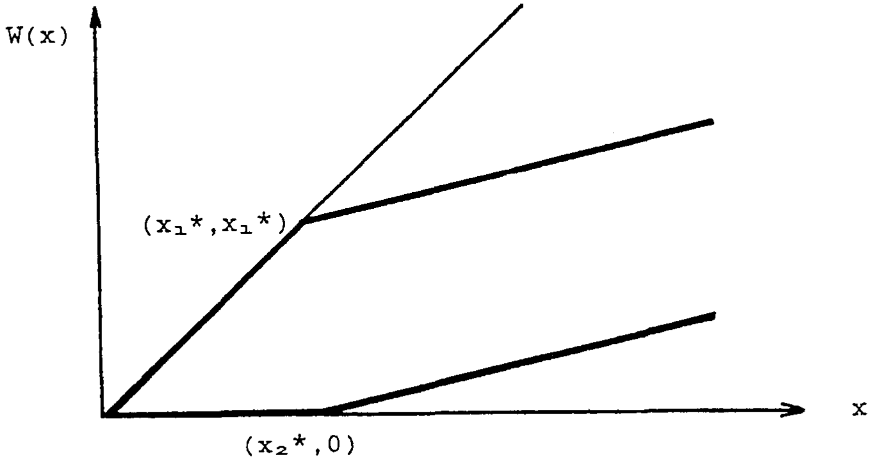

FIGURE 1.

or of the type

$$W(x) = \begin{cases} 0 & x \leqslant x_2^* \\ \dfrac{b}{a+b}(x - x_2^*) & x > x_2^*. \end{cases}$$

These results are illustrated in Figure 1.

ACKNOWLEDGEMENT

The author is grateful to A. Gisler and the referees for suggesting considerable improvements to the paper.

APPENDIX

We will demonstrate how the same result can be derived by applying Theorem 2 of BÜHLMANN and JEWELL (1979) which generalizes Borch's theorem to exchange functions that are subject to constraints.

Denote by $X_1(x)$ and $X_2(x)$ the amounts the policyholder and the insurer get in case the gain is x and there is no profit sharing. Analogously, denote by $Y_1(x)$ and $Y_2(x)$ the amounts after the profit sharing (cfr risk exchange). Remark that we have to consider only the cases where there is profit and thus can denote everything in terms of x. Table 1 gives an overview of the situation. Denote by $\tilde{u}(Y_1(x)) = u(w_u - P' + Y_1(x))$ and by $\tilde{v}(Y_2(x)) = v(w_v + Y_2(x))$.

For the unconstrained case, Borch's theorem provides us with the optimum as the solution of

$$k_1\tilde{u}'(Y_1) = k_2\tilde{v}'(Y_2)$$

https://doi.org/10.2143/AST.18.1.2014959 Published online by Cambridge University Press

TABLE 1

	$X_i(x)$	$Y_i(x)$	Constraints on $Y_i(x)$
Policyholder	0	$W(x)$	$0 \leqslant Y_1(x) \leqslant x$
Insurer	x	$x - W(x)$	$0 \leqslant Y_2(x) \leqslant x$

where k_1 and k_2 are positive constants. Differentiating this equation, we obtain

$$k_1 W'(x) u''(w_u - P' + W(x)) = k_2 (1 - W'(x)) v''(w_v + x - W(x)).$$

If we divide this equation by the former, we get

$$W'(x) = \frac{R_v(w_v + x - W(x))}{R_u(w_u - P' + W(x)) + R_v(w_v + x - W(x))}.$$

For the constrained case Theorem 2 of BÜHLMANN and JEWELL (1979), page 249, states that $W(x)$ is an optimal solution if and only if there exists a positive function $\Lambda(x)$ such that

$$\begin{aligned}
k_1 \tilde{u}'(Y_1(x)) &= \Lambda(x) && \text{if } 0 < Y_1(x) < x \\
k_2 \tilde{v}'(Y_2(x)) &= \Lambda(x) && \text{if } 0 < Y_2(x) < x \\
k_1 \tilde{u}'(Y_1(x)) &\leqslant \Lambda(x) && \text{if } Y_1(x) = 0 \\
k_2 \tilde{v}'(Y_2(x)) &\leqslant \Lambda(x) && \text{if } Y_2(x) = 0 \\
k_1 \tilde{u}'(Y_l(x)) &\geqslant \Lambda(x) && \text{if } Y_1(x) = x \\
k_2 \tilde{v}'(Y_2(x)) &\geqslant \Lambda(x) && \text{if } Y_2(x) = x.
\end{aligned}$$

Remark that some signs are reversed compared with their theorem because we are dealing with utility functions instead of disutility functions.

Let $\tilde{W}(x)$ be a solution of the differential equation in the unconstrained case and let

$$W(x) = \begin{cases} 0 & \text{if } \quad \tilde{W}(x) < 0 \\ \tilde{W}(x) & \text{if } 0 \leqslant \tilde{W}(x) \leqslant x \\ x & \text{if } x < \tilde{W}(x). \end{cases}$$

It is easy to see that $W(x)$ fulfills the above theorem with $\Lambda(x) = k_1 \tilde{u}'(\tilde{W}(x))$. Hence $W(x)$ is an optimal solution.

Note that $0 < \tilde{W}'(x) < 1$ and that therefore $\tilde{W}(x)$ will attain either the boundary $f(x) = x$ or else the boundary $f(x) = 0$ at some point. Thus the solutions are of the same type as described by our theorem.

REFERENCES

BARDOLA, J. (1981) Optimaler Risikoaustausch als Anwendung für den Versicherungsvertrag. *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* 41–67.

BERNHARDT, R. and ENDRES, J. (1979) Erfahrungstarifierung und getrennte Abrechnungsverbände in der Lebensversicherung. *Blätter der Deutschen Gesellschaft für Versicherungsmathematik* Band XIV, Heft 2, 200–229.

BOLNICK, H. J. (1974) Experience rating group life insurance. *Transactions of the Society of Actuaries* **XXVI**, 123–170.
BÜHLMANN, H. and JEWELL W. S. (1979) Optimal risk exchanges. *ASTIN Bulletin* **10**, 243–262.
DRUDE, G. and NIEDERHAUSEN, W. (1973–1974) Gewinnbeteiligung von Gruppen-Lebensversicherungen bei getrennter Gewinnabrechnung. *Blätter der Deutschen Gesellschaft für Versicherungsmathematik* Band XI, Heft 3, 445–467.
JONES, D. A. and GERBER, H. U. (1974) Dividend formulas in group insurance. *Transactions of the Society of Actuaries* **XXVI**, 77–86.
KAMIEN, M. I. and SCHWARTZ, N. L. (1981) *Dynamic Optimization: The Calculus of Variation and Optimal Control in Economics and Management*. North Holland, New York.
SCHMUTZ, R. (1985) La participation aux excédents aléatoires. *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* 147–155.
STRICKLER, P. (1982) Im welchem Ausmass kann in der Lebensvericherung der Versicherungsnehmer am Risikogewinn einzelner Gruppen oder Verträge beteiligt werden. *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* pp 239–260.
ZOPPI, G. (1982) Refund formula in Group Life Assurance. *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* 225–238.

MARTINA VANDEBROEK
Instituut voor Actuariële Wetenschappen, K.U.Leuven, Dekenstraat 2, 3000 Leuven, Belgium.