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EUCLIDEAN LINEAR INVARIANCE AND UNIFORM LOCAL CONVEXITY

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Abstract

Let $S(\rho)$ be the family of holomorphic functions f defined on the unit disk $\mathbb{D}$, normalized by $f(0) = f'(0) - 1 = 0$ and univalent in every hyperbolic disk of radius ρ. Let $C(\rho)$ be the subfamily consisting of those functions which are convex univalent in every hyperbolic disk of radius ρ. For $\rho = \infty$ these become the classical families S and C of normalized univalent and convex functions, respectively. These families are linearly invariant in the sense of Pommerenke; a natural problem is to calculate the order of these linearly invariant families. More precisely, we give a geometrie proof that $C(\rho)$ is the universal linearly invariant family of all normalized locally schlicht functions of order at most $\coth(2\rho)$. This gives a purely geometric interpretation for the order of a linearly invariant family. In a related matter, we characterize those locally schlicht functions which map each hyperbolically k-convex subset of $\mathbb{D}$ onto a euclidean convex set. Finally, we give upper and lower bounds on the order of the linearly invariant family $S(\rho)$ and prove that this class is not equal to the universal linearly invariant family of any order.

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1. Introduction

Pommerenke ([14], [15]) initiated the study of linearly invariant families of locally schlicht (univalent) holomorphic functions defined in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. (We shall use the words "schlicht" and "univalent" interchangeably throughout this paper.) A family $\mathscr{F}$ of locally univalent func-

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tions holomorphic on the unit disk $\mathbb{D}$ is called *linearly invariant* if each function $f \in \mathscr{F}$ is normalized by $f(0) = f'(0) - 1 = 0$, and also, for each $f \in \mathscr{F}$ and any conformal automorphism T of $\mathbb{D}$ the Koebe transform of f,

$$f_T(z) = \frac{f(T(z)) - f(T(0))}{f'(T(0))T'(0)},$$

also belongs to $\mathscr{F}$. For $T(z) = (z+a)/(1+\overline{a}z)$ we write $f(z, a)$ in place of f_T. Explicitly,

$$f(z, a) = \frac{f\left(\frac{z+a}{1+\overline{a}z}\right) - f(a)}{(1-|a|^2)f'(a)}.$$

Let $\operatorname{Aut}(\mathbb{D})$ denote the group of conformal automorphisms of $\mathbb{D}$. A locally schlicht holomorphic function f on $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ is called *linearly invariant* if the linearly invariant family $\mathscr{F}(f) = \{f_T : T \in \operatorname{Aut}(\mathbb{D})\}$ of all Koebe transformations of f has finite linearly invariant order, which is defined by

$$\alpha(f) = \sup\left\{\left|(1-|z|^2)\frac{f''(z)}{2f'(z)} - \overline{z}\right| : z \in \mathbb{D}\right\} = \sup\left\{\left|\frac{f_T''(0)}{2}\right| : T \in \operatorname{Aut}(\mathbb{D})\right\}.$$

The order of a linearly invariant function satisfies $\alpha(f) \geq 1$ and equality holds if and only if f is a normalized convex univalent function [14]. The order of a linearly invariant family $\mathscr{F}$ is defined by

$$\alpha(\mathscr{F}) = \sup\{\alpha(f) : f \in \mathscr{F}\}.$$

For $\alpha \geq 1$, let $\mathscr{F}(\alpha)$ be the linearly invariant family consisting of all linearly invariant functions with linearly invariant order less than or equal to α. For a linearly invariant family $\mathscr{F}$ one often wishes to determine its order and to obtain growth, distortion and covering theorems for the family. Pommerenke [14] derived a number of sharp growth, distortion and covering theorems for the family $\mathscr{F}(\alpha)$. He also showed that $\mathscr{F}(\alpha)$ is a compact, normal family.

Linear invariance is closely related to the concepts of uniform local univalence and uniform local convexity. These latter two notions are defined relative to hyperbolic geometry on $\mathbb{D}$. The density for the hyperbolic metric is $\lambda_{\mathbb{D}}(z) = 1/(1-|z|^2)$. The hyperbolic distance function on $\mathbb{D}$ induced by this metric is

$$d_h(a, b) = \operatorname{arctanh}\left|\frac{a-b}{1-a\overline{b}}\right|.$$

The hyperbolic disk in $\mathbb{D}$ with hyperbolic center $a \in \mathbb{D}$ and hyperbolic radius ρ, $0 < \rho \leq \infty$, is defined by $D_h(a, \rho) = \{z \in \mathbb{D} : d_h(a, z) < \rho\}$. For an analytic function f in $\mathbb{D}$ we let $\rho(z, f)$ be the hyperbolic radius of the largest hyperbolic disk in $\mathbb{D}$ centered at z in which f is univalent. Note that $\rho(z, f)$ can be zero or infinite. Define $\rho(f) = \inf\{\rho(z, f) : z \in \mathbb{D}\}$.

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A function f is called *uniformly locally univalent* (*in the hyperbolic sense*) in $\mathbb{D}$ provided $\rho(f) > 0$. For $0 < \rho \leq \infty$, let $S(\rho)$ be the family of all locally univalent holomorphic functions f in $\mathbb{D}$ normalized by $f(0) = 0$, $f'(0) = 1$ and satisfying $\rho(f) \geq \rho$. Then $S(\rho)$ is a linearly invariant family since a conformal automorphism of $\mathbb{D}$ maps a hyperbolic disk onto another hyperbolic disk with the same hyperbolic radius and $S = S(\infty)$ is the usual class of normalized univalent functions. For a locally schlicht holomorphic function f we define

$$\rho_c(f) = \sup\{\rho\colon f \text{ is univalent in } D_h(a, \rho) \text{ and } f(D_h(a, \rho)) \text{ is convex for all } a \in \mathbb{D}\}.$$

Clearly, $\rho_c(f) \leq \rho(f)$. A locally univalent holomorphic function f is called *uniformly locally convex* (*in the hyperbolic sense*) provided $\rho_c(f) > 0$. Clearly, every convex function is uniformly locally convex. Let $C(\rho)$, $0 < \rho \leq \infty$, be the class of normalized uniformly locally convex functions f such that $\rho_c(f) \geq \rho$. Then $C(\rho)$ is a linearly invariant family and $C(\infty)$ is the usual class C of normalized convex univalent functions.

It is not difficult to verify that the three families $\bigcup\{\mathscr{F}(\alpha)\colon \alpha \geq 1\}$, $\bigcup\{S(\rho)\colon \rho > 0\}$, and $\bigcup\{C(\rho)\colon \rho > 0\}$ are all equal. In words, a function f is linearly invariant if and only if it is uniformly locally univalent, or if and only if it is uniformly locally convex. In [7], we presented two extensions of the notion of linear invariance to general planar regions, one involving the hyperbolic metric and the other the quasihyperbolic metric. We related these concepts of linear invariance to uniform local univalence relative to each of these metrics. In this paper we will focus our attention on the special case of the unit disk.

Pommerenke [14] proved that $C(\infty) = \mathscr{F}(1)$ and $\mathscr{F}(\coth(2\rho)) \subset C(\rho)$. Harmelin [7] gave the inclusion $C(\rho) \subset \mathscr{F}(\coth(\rho))$. In Section 4 of this paper, we investigate the relationship between the families $C(\rho)$ and $\mathscr{F}(\alpha)$. We show that $C(\rho) = \mathscr{F}(\coth(2\rho))$. More precisely, we show that $\rho_c(f) = \rho$ if and only if $\alpha(f) = \coth(2\rho)$. So the order of the linearly invariant family $C(\rho)$ is $\coth(2\rho)$. This gives a geometric characterization of linearly invariant functions and provides a geometric interpretation for the order $\alpha(f)$ of a linearly invariant function.

It is then natural to inquire about the relationship between the families $S(\rho)$ and $\mathscr{F}(\alpha)$. On one hand, Pommerenke [14] proved $\mathscr{F}(\alpha) \subset S(\operatorname{arctanh}(s)/2)$, where s is defined by

$$\int_0^s \frac{\sqrt{1 - t^2/\alpha^2}}{1 - t^2}\, dt = \frac{\pi}{\alpha}.$$

On the other hand, it is easy to see that $S(\rho) \subset \mathscr{F}(2\coth(\rho))$. This inclusion,

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together with Hurwitz's Theorem, shows that $S(\rho)$ is a compact normal family. In Section 5, we will establish a stronger inclusion. Also, unlike the family $C(\rho)$, we show that $S(\rho)$ is not equal to $\mathscr{F}(\alpha(\rho))$, where $\alpha(\rho)$ is the order of the linearly invariant family $S(\rho)$.

One interesting aspect of the class $S(\rho)$ is that there exist both a necessary and a sufficient condition for $f \in S(\rho)$ in terms of the Schwarzian derivative S_f of f, where

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 .$$

The sufficient condition is that if f is holomorphic and locally univalent in $\mathbb{D}$ with

$$(1-|z|^2)^2|S_f(z)| \le 2(1+(\pi/2\rho)^2),$$

where $\rho > 0$, then $f \in S(\rho)$ ([1], [9]). This result is sharp for all $\rho > 0$. On the other hand, the following elementary necessary condition for $f \in S(\rho)$,

$$(1-|z|^2)^2|S_f(z)| \le 6/\tanh^2(\rho),$$

has been noted a number of times ([6], [9], [12], [18] and [20]). For finite ρ, the known upper bound is not sharp. Also, in Section 5, we use our inclusion result to improve this known upper bound on the Schwarzian derivative.

We want to thank the referee for carefully reading the paper and offering useful suggestions that helped clarify the paper.

2. Examples

We now investigate two examples of classes of functions that will be needed in the remainder of the paper.

EXAMPLE 1. First, we consider a function k_α in $\mathscr{F}(\alpha)$ which is known to be extremal for a number of problems. We determine those values of α for which this function belongs to either $C(\rho)$ or $S(\rho)$. For $\alpha \ge 1$ set

$$k_\alpha(z) = \frac{\left(\frac{1+z}{1-z}\right)^\alpha - 1}{2\alpha}.$$

It is not difficult to show that the order of the function k_α is α. Note that k_2 is the Koebe function $z/(1-z)^2$ and $k_1(z) = z/(1-z)$.

First, we determine when k_α belongs to $S(\rho)$. The density of the hyperbolic metric on the right half-plane $\mathbb{H} = \{z\colon \operatorname{Re} z > 0\}$ is $\lambda_{\mathbb{H}}(z) = 1/2\operatorname{Re}(z)$ and the hyperbolic distance is

$$d_h(a, b) = \operatorname{arctanh}\left|\frac{a-b}{a+\bar{b}}\right| .$$

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The hyperbolic disk in $\mathbb{H}$ with center a and radius ρ is $D_h(a, \rho) = \{z: d_h(a, z) < \rho\}$. If $R = \tanh(\rho)$, then in euclidean terms

$$D_h(a, \rho) = \left\{z: \left|z - i\operatorname{Im} a - \frac{1+R^2}{1-R^2}\operatorname{Re} a\right| < \frac{2R}{1-R^2}\operatorname{Re} a\right\},$$

which is a euclidean disk. Now, for each $a \in \mathbb{H}$,

$$\begin{aligned}\sup\{\arg(z_1/z_2)&: z_1, z_2 \in D_h(a, \rho)\}\\ &= 2\arcsin\{2R\operatorname{Re} a[(1+R^2)^2(\operatorname{Re} a)^2 + (1-R^2)^2(\operatorname{Im} a)^2]^{-1/2}\}\\ &\le 2\arcsin\{2R/(1+R^2)\} = 4\arctan R.\end{aligned}$$

For $\alpha \ge 1$, notice that

$$\left(\frac{1+z}{1-z}\right)^\alpha = \exp\left\{\alpha\log\left|\frac{1+z}{1-z}\right| + i\alpha\arg\frac{1+z}{1-z}\right\}$$

and that $(1+z)/(1-z)$ is a conformal map from $\mathbb{D}$ onto $\mathbb{H}$ that maps hyperbolic disks in $\mathbb{D}$ with hyperbolic radius ρ onto hyperbolic disks in $\mathbb{H}$ with hyperbolic radius ρ. We see that $((1+z)/(1-z))^\alpha$ is univalent in every hyperbolic disk in $\mathbb{D}$ with hyperbolic radius ρ if and only if $4\alpha\arctan R \le 2\pi$. Thus for $R = \tanh(\rho)$, $k_\alpha \in S(\rho)$ if and only if $4\alpha\arctan R \le 2\pi$. In particular, $k_\alpha \in S(\rho)$ for $\alpha = \pi/(2\arctan R)$.

Next, we determine when $k_\alpha \in C(\rho)$. Note that $(z+a)/(1+\overline{a}z)$ is a conformal automorphism of $\mathbb{D}$ that maps $\{z: |z| < R = \tanh(\rho)\}$ conformally onto $D_h(a, \rho)$. Now, f belongs to $C(\rho)$ if and only if for every $a \in \mathbb{D}$, the function $f(z, a)$ is convex in $|z| < R = \tanh(\rho)$. It is easy to verify that for $a \in \mathbb{D}$,

$$1 + \frac{zk_\alpha''(z, a)}{k_\alpha'(z, a)} = \frac{1 + \alpha w(e^{i\theta} + 1) + e^{i\theta}w^2}{(1 + e^{i\theta}w)(1-w)},$$

where $w = z(1-\overline{a})/(1-a)$ and $e^{i\theta}w = z(1+\overline{a})/(1+a)$. For $|z| = R$, let $w = \mathrm{Re}^{i\varphi}$ and $e^{i\theta}w = \mathrm{Re}^{i\psi}$. Then

$$\begin{aligned}|(1+e^{i\theta}w)(1-w)|^2 \operatorname{Re}&\left\{1 + \frac{zk_\alpha''(z, a)}{k_\alpha'(z, a)}\right\}\\ &= (1-R^2)\{1 + R^2 + R(\alpha-1)\cos\varphi + R(\alpha+1)\cos\psi\},\end{aligned}$$

which is nonnegative for all φ and ψ if and only if $\alpha \le \coth(2\rho)$. In particular, $k_\alpha \in C(\rho)$ for $\alpha = \coth(2\rho)$.

EXAMPLE 2. For $\beta > 0$ set

$$h_\beta(z) = \frac{\exp(\beta z) - 1}{\beta} = z + \frac{\beta}{2}z^2 + \cdots.$$

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Then h_β is a normalized entire function which is univalent in every euclidean disk with radius π/β, but not in any larger disk.

Now we regard h_β as defined just on $\mathbb{D}$ and determine $\alpha(h_\beta)$ and $\rho(h_\beta)$. It is an elementary calculation to show that for $\beta \geq 1$

$$\alpha(h_\beta) = \frac{1}{2}\left(\beta + \frac{1}{\beta}\right).$$

Consequently, $h_\beta \in \mathscr{F}(\alpha)$ when $\beta = \alpha + \sqrt{\alpha^2 - 1}$. For a fixed $\rho > 0$, each hyperbolic disk $D_h(a, \rho)$ in $\mathbb{D}$ is a euclidean disk which attains its maximal euclidean radius $\tanh(\rho)$ at the origin. Therefore, for $\beta > \pi$, $\rho(h_\beta) = \operatorname{arctanh}(\pi/\beta)$ and $h_\beta \in S(\operatorname{arctanh}(\pi/\beta))$.

3. Euclidean invariant differential operators

We introduce invariant differential operators for holomorphic functions mapping the unit disk into the complex plane.

DEFINITION. Suppose f is holomorphic in $\mathbb{D}$. Set

$$D_1 f(z) = (1 - |z|^2) f'(z),$$

and

$$D_2 f(z) = (1 - |z|^2)^2 f''(z) - 2\bar{z}(1 - |z|^2) f'(z).$$

Note that $D_1 f(0) = f'(0)$ and $D_2 f(0) = f''(0)$. In terms of these differential operators the linearly invariant order of a locally schlicht function f is

$$\alpha(f) = \sup\left\{\frac{|D_2 f(z)|}{2|D_1 f(z)|} : z \in \mathbb{D}\right\}.$$

For $a \in \mathbb{D}$ set $T_a(z) = (z - a)/(1 - \bar{a}z)$. Then T_a is a conformal automorphism of $\mathbb{D}$ which sends a to 0, $T_{-a} = T_a^{-1}$ and $T_a'(0) = 1 - |a|^2$, $T_a'(a) = 1/(1 - |a|^2)$. Also, for $b \in \mathbb{C}$, set $S_b(z) = z - b$. Note that $S_{-b} = S_b^{-1}$. If $\tilde{f} = S_{f(a)} \circ f \circ T_{-a}$, then $\tilde{f}$ is a holomorphic function on $\mathbb{D}$ with $\tilde{f}(0) = 0$. It is straightforward to verify that $D_1 f(a) = \tilde{f}'(0)$ and $D_2 f(a) = \tilde{f}''(0)$. We next establish an invariance property of these differential operators relative to the group $\operatorname{Aut}(\mathbb{D})$ of conformal automorphisms of $\mathbb{D}$ and the subgroup $\operatorname{Euc}(\mathbb{C})$ of $\operatorname{Aut}(\mathbb{C})$ consisting of euclidean motions.

THEOREM 1. *Suppose f is holomorphic in $\mathbb{D}$. If $R \in \operatorname{Euc}(\mathbb{C})$ and $S \in \operatorname{Aut}(\mathbb{D})$, then*

$$|D_j(R \circ f \circ S)| = |D_j(f)| \circ S \qquad (j = 1, 2).$$

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PROOF. Fix $a \in \mathbb{D}$ and set $b = S(a)$, $g = R \circ f \circ S$. We establish this result only in the case $j = 1$; the case $j = 2$ is virtually identical. Then $D_1 f(b) = \tilde{f}'(0)$, where $\tilde{f} = S_{f(b)} \circ f \circ T_{-b}$ and $D_1 g(a) = \tilde{g}'(0)$, where $\tilde{g} = S_{g(a)} \circ g \circ T_{-a}$. Note that the functions $\tilde{f}$ and $\tilde{g}$ both fix the origin. It suffices to show that $|\tilde{f}'(0)| = |\tilde{g}'(0)|$. This is elementary because $\tilde{g} = \widetilde{R} \circ \tilde{f} \circ \widetilde{S}$, where $\widetilde{R} = S_{g(a)} \circ R \circ S_{-f(b)}$ is a rotation of $\mathbb{C}$ and $\widetilde{S} = T_b \circ S \circ T_{-a}$ is a rotation of $\mathbb{D}$.

There is a close connection between these invariant differential operators and hyperbolic curvature. The hyperbolic curvature of a path $\gamma: z = z(t)$ in $\mathbb{D}$ is

$$\kappa_h(z, \gamma) = (1 - |z|^2)\kappa_e(z, \gamma) + \operatorname{Im}\left\{\frac{2\overline{z(t)}z'(t)}{|z'(t)|}\right\},$$

where $\kappa_e(z, \gamma)$ denotes the euclidean curvature,

$$\kappa_e(z, \gamma) = \frac{1}{|z'(t)|}\operatorname{Im}\left\{\frac{z''(t)}{z'(t)}\right\}.$$

Note that $\kappa_e(0, \gamma) = \kappa_h(0, \gamma)$. In words, the hyperbolic curvature and euclidean curvature coincide at the origin. Recall that hyperbolic curvature is invariant under the group $\operatorname{Aut}(\mathbb{D})$, that is, $\kappa_h(S(z), S \circ \gamma) = \kappa_h(z, \gamma)$ for any $S \in \operatorname{Aut}(\mathbb{D})$. For more details, see [2], [5] and [10]. Similarly, euclidean curvature is invariant under the group $\operatorname{Euc}(\mathbb{C})$.

EXAMPLE 3. If γ is the positively oriented boundary of the hyperbolic disk $D_h(a, \rho)$, then $\kappa_h(z, \gamma) = 2\coth(2\rho)$.

The formula for the change of euclidean curvature under a locally schlicht holomorphic function f is [2]

$$\kappa_e(f(z), f \circ \gamma)|f'(z)| = \kappa_e(z, \gamma) + \operatorname{Im}\left\{\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right\}.$$

Now we derive an analogous formula for the change from hyperbolic curvature to euclidean curvature under a locally schlicht holomorphic function.

THEOREM 2. *Suppose f is holomorphic and locally schlicht in $\mathbb{D}$. Then*

$$\kappa_e(f(z), f \circ \gamma)|D_1 f(z)| = \kappa_h(z, \gamma) + \operatorname{Im}\left\{\frac{D_2 f(z)}{D_1 f(z)}\frac{z'(t)}{|z'(t)|}\right\}.$$

PROOF. Fix $a \in \mathbb{D}$. Set $\tilde{f} = S_{f(a)} \circ f \circ T_{-a}$ and $\tilde{\gamma} = T_a \circ \gamma$. If γ is parametrized by $z = z(t)$ and $z(t_0) = a$, then $\tilde{\gamma}$ is parametrized by $z = \tilde{z}(t) = T_a \circ z(t)$ and $\tilde{z}(t_0) = 0$. Also, note that the unit tangent

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to γ at a and the unit tangent to $\tilde{\gamma}$ at 0 are equal since $T_a'(a) > 0$; in symbols, $z'(t_0)/|z'(t_0)| = \tilde{z}'(t_0)/|\tilde{z}'(t_0)|$. Because hyperbolic curvature is invariant under $\mathrm{Aut}(\mathbb{D})$, euclidean curvature is invariant under $\mathrm{Euc}(\mathbb{C})$ and $\tilde{f} \circ \tilde{\gamma} = S_{f(a)} \circ f \circ \gamma$, from Theorem 1 it suffices to show that

$$\kappa_e(\tilde{f}(0), \tilde{f} \circ \tilde{\gamma})|D_1\tilde{f}(0)| = \kappa_h(0, \tilde{\gamma}) + \operatorname{Im}\left\{\frac{D_2\tilde{f}(0)}{D_1\tilde{f}(0)} \frac{\tilde{z}'(t_0)}{|\tilde{z}'(t_0)|}\right\},$$

or equivalently,

$$\kappa_e(\tilde{f}(0), \tilde{f} \circ \tilde{\gamma})|\tilde{f}'(0)| = \kappa_e(0, \tilde{\gamma}) + \operatorname{Im}\left\{\frac{\tilde{f}''(0)}{\tilde{f}'(0)} \frac{\tilde{z}'(t_0)}{|\tilde{z}'(t_0)|}\right\}.$$

But this is just the formula for the change of euclidean curvature under a locally schlicht holomorphic function.

4. Uniform local convexity

Now, we relate linear invariance and uniform local convexity in a precise, quantitative manner.

THEOREM 3. *Suppose f is locally schlicht in $\mathbb{D}$. Then $\alpha(f) = \coth(2\rho_c(f))$. In particular, $C(\rho) = \mathcal{F}(\coth(2\rho))$.*

PROOF. First, we show that $\alpha(f) \le \coth(2\rho_c(f))$. Fix $z_0 \in \mathbb{D}$. It suffices to show that

$$\frac{|D_2 f(z_0)|}{2|D_1 f(z_0)|} \le \coth(2\rho_c(f)).$$

We need only consider the case in which $D_2 f(z_0) \ne 0$. Then there is a unique point $a \in \mathbb{D}$ such that $z_0 \in \partial D_h(a, \rho_c(f))$ and

$$\operatorname{Im}\left\{\frac{D_2 f(z_0)}{D_1 f(z_0)} \frac{z'(t_0)}{|z'(t_0)|}\right\} = -\left|\frac{D_2 f(z_0)}{D_1 f(z_0)}\right|.$$

Let $\gamma: z = z(t)$ be a parametrization of $\partial D_h(a, \rho_c(f))$ and $z(t_0) = z_0$. Then $f \circ \gamma$ is a euclidean convex curve, so $\kappa_e(f(z_0), f \circ \gamma) \ge 0$. Hence,

$$0 \le \kappa_e(f(z_0), f \circ \gamma)|D_1 f(z_0)| = \kappa_h(z_0, \gamma) + \operatorname{Im}\left\{\frac{D_2 f(z_0)}{D_1 f(z_0)} \frac{z'(t_0)}{|z'(t_0)|}\right\}$$
$$= \kappa_h(z_0, \gamma) - \left|\frac{D_2 f(z_0)}{D_1 f(z_0)}\right|,$$

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or,

$$\frac{|D_2 f(z_0)|}{2|D_1 f(z_0)|} \le \frac{1}{2}\kappa_h(z_0, \gamma) = \coth(2\rho_c(f)).$$

Here we have used Example 3.

Next, we establish $\coth(2\rho_c(f)) \le \alpha(f)$. Determine ρ from $\coth(2\rho) = \alpha(f) \ge 1$. For any a in $\mathbb{D}$ let $\gamma = \partial D_h(a, \rho)$. Then from Theorem 2 we see that for $z \in \gamma$,

$$\kappa_e(f(z), f\circ\gamma)|D_1 f(z)| = \kappa_h(z, \gamma) + \operatorname{Im}\left\{\frac{D_2 f(z)}{D_1 f(z)}\frac{z'(t)}{|z'(t)|}\right\}$$
$$\ge 2\coth(2\rho) - \left|\frac{D_2 f(z)}{D_1 f(z)}\right| \ge 2(\coth(2\rho) - \alpha(f)) = 0.$$

Thus, $\kappa_e(f(z), f\circ\gamma) \ge 0$, so $f(D_h(a, \rho))$ is a euclidean convex set. Since $a \in \mathbb{D}$ is arbitrary, this yields $\rho_c(f) \ge \rho$, which is equivalent to $\coth(2\rho_c(f)) \le \alpha(f)$.

Part of Theorem 3 follows from the work of Pommerenke [14]; we have given our geometric proof since the proof of both halves of the theorem are similar. From Theorem 3 and Pommerenke's results on $\mathscr{F}(\alpha)$ we can obtain a number of sharp growth, distortion and covering theorems as well as some coefficient estimates for functions in the class $C(\rho)$, the first subclass of $S(\rho)$ for which we can get some sharp results.

Next we relate linear invariance and uniform local convexity to hyperbolic k-convexity. A region Ω in $\mathbb{D}$ is *hyperbolically k-convex relative to* $\mathbb{D}$ if for any pair of distinct points $a, b \in \Omega$ there exist two shortest arcs of constant hyperbolic curvature k in Ω connecting a and b. Note that curves in $\mathbb{D}$ of constant hyperbolic curvature are circular arcs. This notion was introduced independently by Flinn and Osgood [2] and Mejia [8]. If Ω is a region in $\mathbb{D}$ such that $\partial\Omega$ is a closed Jordan curve of class C^2 and $\kappa_h(z, \partial\Omega) \ge k$ for all $z \in \partial\Omega$, then Ω is hyperbolically k-convex (see [2] and [8]).

EXAMPLE 4. $D_h(a, \rho)$ is hyperbolically k-convex if $2\coth(2\rho) \ge k$. This follows easily from Example 3 and the sufficient condition given above.

DEFINITION. Suppose f is locally schlicht in $\mathbb{D}$. Let $k(f) = \inf\{k: f$ maps each hyperbolically k-convex subset of $\mathbb{D}$ injectively onto a euclidean convex set$\}$.

EXAMPLE 5. Let $g(z) = z/(1-z)$. We show that $k(g) = 2$. The work of Heins [4] and Pommerenke [16] implies that $k(g) \le 2$ since every convex univalent function maps each hyperbolically 2-convex subset of $\mathbb{D}$ conformally onto a euclidean convex set; see also [2]. Next, we show that $k(g) \ge 2$. Fix $r > 1$. Let $\Omega = \mathbb{D} \cap D(r/2, r/2)$ and assume the relative boundary $\gamma = \mathbb{D} \cap \partial D(r/2, r/2)$ is parametrized by $\gamma: z = z(t)$ so that Ω lies to the

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left and $z(0) = 0$. Note that $z'(0) = -i$. Then γ has constant hyperbolic curvature $2/r$ and Ω is hyperbolically $(2/r)$-convex. Because

$$\kappa_e(0, g \circ \gamma) = \kappa_e(g(0), g \circ \gamma)|g'(0)| = \kappa_e(0, \gamma) + \operatorname{Im}\left\{\frac{g''(0)}{g'(0)} \frac{z'(0)}{|z'(0)|}\right\}$$
$$= \frac{2}{r} - 2 < 0,$$

it follows that $g(\Omega)$ is not euclidean convex. Hence, $k(g) \geq 2/r$. Since $r > 1$ is arbitrary, we obtain $k(g) \geq 2$.

LEMMA. *Suppose f is locally univalent in $\mathbb{D}$. Then $k(f) \geq 2$ with equality if and only if f is convex univalent in $\mathbb{D}$.*

PROOF. We first show that

$$2\coth(2\rho_c(f)) \leq \max\{k(f), 2\} = k'.$$

We may assume that $k(f) < \infty$ without loss of generality. For any $\varepsilon > 0$, determine $\rho = \rho(\varepsilon)$ from $2\coth(2\rho) = k' + \varepsilon$. Note that $D_h(a, \rho)$ is hyperbolically $(k' + \varepsilon)$-convex from Example 4. It follows that f is convex univalent in $D_h(a, \rho)$. As $a \in \mathbb{D}$ is arbitrary, $\rho_c(f) \geq \rho$, which is equivalent to $2\coth(2\rho_c(f)) \leq k' + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this establishes the inequality. Observe that if $k(f) \leq 2$, then $k' = 2$ and so $\rho_c(f) = \infty$. Thus, $k(f) \leq 2$ implies that f is convex univalent in $\mathbb{D}$.

All that remains is to show that if f is convex univalent in $\mathbb{D}$, then $k(f) \geq 2$. Suppose f is convex univalent in $\mathbb{D}$; then $\alpha(f) = 1$ [14]. This implies that there exist points $a_n \in \mathbb{D}$ and $\theta_n \in \mathbb{R}$ such that if $T_n(z) = e^{i\theta_n}(z - a_n)/(1 - \overline{a_n}z)$, then $f''_{T_n}(0) \to 2$. Set $g_n = f_{T_n}$. Note that each g_n is a normalized convex univalent function and the family of all such functions is compact relative to local uniform convergence. By passing to a subsequence if necessary, we may assume that the sequence $\{g_n\}$ is locally uniformly convergent. The condition $g''_n(0) \to 2$ implies that the limit function must be the function $g(z) = z/(1 - z)$. Clearly, $k(f) = k(R \circ f \circ S)$ whenever $R \in \operatorname{Aut}(\mathbb{C})$ and $S \in \operatorname{Aut}(\mathbb{D})$, so $k(g_n) = k(f)$ for all n. Now we show $k(f) \geq 2$. Let $k > k(f)$ and suppose Ω is any hyperbolically k-convex subset of $\mathbb{D}$. Fix $a, b \in \Omega$. Let γ_1 and γ_2 be the two shortest arcs of hyperbolic curvature k joining a and b. Then $\gamma_j \subset \Omega$ $(j = 1, 2)$ since Ω is hyperbolically k-convex. Also, $\gamma = \gamma_1 \cup \gamma_2$ is a closed Jordan curve whose interior belongs to Ω and $f \circ \gamma$ is a euclidean convex curve. Hence, $g_n \circ \gamma$ is also a euclidean convex Jordan curve. Because $g_n \to g$ locally uniformly, it follows that $g \circ \gamma$ is still euclidean convex. This implies that the line segment $[g(a), g(b)]$ lies in $g(\Omega)$. Hence, $g(\Omega)$ is euclidean convex and

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g maps every hyperbolically k-convex set onto a euclidean convex set. Since $k > k(f)$ is arbitrary, it follows that $2 = k(g) \le k(f)$.

THEOREM 4. *Suppose f is locally schlicht in $\mathbb{D}$. Then $k(f) = 2\coth(2\rho_c(f))$.*

PROOF. First, we show that $2\coth(2\rho_c(f)) \le k(f)$. This is an immediate consequence of the inequality in the proof of the preceding lemma since we know that $k(f) \ge 2$.

Next, we demonstrate that $k(f) \le 2\coth(2\rho_c(f))$. We may suppose that $\rho_c(f) > 0$. Set $k = 2\coth(2\rho_c(f))$. Let Ω be any subset of $\mathbb{D}$ which is hyperbolically k-convex. Since $k \ge 2$, Ω is contained in a hyperbolic disk with radius $\rho_c(f)$ [8]. In particular, f is univalent on Ω. We shall show that $f(\Omega)$ is euclidean convex. This will give $k(f) \le k$, the desired result. Because f is injective on Ω, it suffices to show that for any a, b in Ω, the euclidean line segment joining $f(a)$ and $f(b)$ lies in $f(\Omega)$. Consider distinct points $a, b \in \Omega$. Then there exist two closed hyperbolic disks Δ_1 and Δ_2 of hyperbolic radii $\rho_c(f)$ such that $a, b \in \partial\Delta_j$ $(j = 1, 2)$ and $\Delta_1 \cap \Delta_2 \subset \Omega$. Now, $f(\Delta_j)$ is euclidean convex, so the euclidean line segment $\Gamma = [f(a), f(b)]$ is contained in $f(\Delta_j)$ $(j = 1, 2)$. Hence, there is a path γ_j in Δ_j from a to b such that $f \circ \gamma_j = \Gamma$ $(j = 1, 2)$. Since f is univalent on both Δ_1 and Δ_2, and γ_1, γ_2 are both paths from a to b, the condition $f \circ \gamma_1 = f \circ \gamma_2$ implies that $\gamma_1 = \gamma_2$. Let $\gamma_1 = \gamma_2 = \gamma$. Then γ is contained in $\Delta_1 \cap \Delta_2$, so $\Gamma = f \circ \gamma$ lies in $f(\Delta_1 \cap \Delta_2) \subset f(\Omega)$. This shows that $f(\Omega)$ is euclidean convex. Hence, $k(f) \le k$.

THEOREM 5. *Let $\Omega \subset \mathbb{D}$ be a simply connected region. Then Ω is hyperbolically k-convex if and only if $f(\Omega)$ is convex for every f with $\alpha(f) \le k/2$.*

PROOF. If Ω is hyperbolically k-convex and $\alpha(f) \le k/2$, it follows from Theorems 3 and 4 that $k(f) \le k$. Thus $f(\Omega)$ is convex.

On the other hand, assume $f(\Omega)$ is convex for every f with $\alpha(f) \le k/2$. Let $g: \mathbb{D} \to \Omega$ be a conformal mapping; then $f(g(z))$ is convex for every f with $\alpha(f) \le k/2$. This implies that for every r, $0 < r < 1$, $f(g(rz))$ is convex [19]. Note that for every r, $\gamma = \partial g(|z| < r): z = z(t)$ is a smooth Jordan curve whose image under f is convex. Consider any point $a \in \gamma$; we may assume that $z(0) = a$. Then

$$0 \le \kappa_e(f(a), f \circ \gamma)|D_1 f(a)| = \kappa_h(a, \gamma) + \operatorname{Im}\left\{\frac{D_2 f(a)}{D_1 f(a)} \frac{z'(0)}{|z'(0)|}\right\}.$$

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We know that there exists f with $\alpha(f) = k/2$ such that

$$\operatorname{Im}\left\{\frac{D_2 f(a)}{D_1 f(a)} \frac{z'(0)}{|z'(0)|}\right\} = -k.$$

For example, we can take f to be a proper Koebe transformation of k_α with $\alpha = k/2$. Hence, we have $\kappa_h(a, \gamma) \geq k$. Since $a \in \gamma$ is arbitrary, it follows that $g(|z| < r)$ is hyperbolically k-convex for every r. If $\{r_n\}$ is an increasing sequence of positive numbers tending to 1, then it follows that $\Omega = g(\mathbb{D}) = \bigcup\{g(|z| < r_n) \colon n = 1, 2, \ldots\}$ is hyperbolically k-convex.

REMARK. For $k = 2$ this theorem was established independently by Heins [4] and Pommerenke [16].

5. Uniform local univalence

In this section we relate the families $S(\rho)$ and $\mathscr{F}(\alpha)$. Unlike the class $C(\rho)$, it is very difficult to obtain any sharp result for the class $S(\rho)$. The main tool we employ is an inequality for the Schwarzian derivative of a bounded univalent function. This inequality is a simple consequence of certain general inequalities of Nehari for bounded univalent functions ([11], [17, p. 99]). We include a simple proof of this inequality. The proof reveals that this bound on the Schwarzian derivative of bounded univalent functions is equivalent to the classical bound on the Schwarzian derivative of functions in the class S. Our proof is similar to work of Flinn and Osgood [2] showing that the bound on the second coefficient of a bounded univalent function is equivalent to the bound on the second coefficient for a function in the class S.

NEHARI'S INEQUALITY. *Suppose $g(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots$ is univalent in $\mathbb{D}$ and $g(\mathbb{D}) \subset \mathbb{D}$. Then*

$$\left|\frac{b_3}{b_1} - \left(\frac{b_2}{b_1}\right)^2\right| \leq 1 - |b_1|^2.$$

This inequality is sharp.

PROOF. Recall that if $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is a normalized univalent function in $\mathbb{D}$, then $|a_3 - a_2^2| \leq 1$, see [17, p. 20], with equality if and only if f is a rotation of the Koebe function k_2. For simplicity we shall write k in place of k_2 and let $k_\theta(z) = e^{-i\theta} k(e^{i\theta} z)$ denote a rotation of the Koebe

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function. Define $f(z) = e^{-i\varphi} b_1^{-1} k_\theta(g(e^{i\varphi} z))$, where the real numbers φ and θ will be specified later. Then f is a normalized univalent function and

$$f(z)=z+\left[\frac{b_2}{b_1}e^{i\varphi}+2b_1e^{i(\varphi+\theta)}\right]z^2+\left[\frac{b_3}{b_1}e^{2i\varphi}+4b_2e^{i(2\varphi+\theta)}+3b_1^2e^{2i(\varphi+\theta)}\right]z^3+\cdots.$$

Therefore,

$$\left|\left[\frac{b_3}{b_1}-\left(\frac{b_2}{b_1}\right)^2\right]e^{2i\varphi}-b_1^2e^{2i(\varphi+\theta)}\right|=|a_3-a_2^2|\le 1.$$

If we select φ with

$$\left[\frac{b_3}{b_1}-\left(\frac{b_2}{b_1}\right)^2\right]e^{2i\varphi}=\left|\frac{b_3}{b_1}-\left(\frac{b_2}{b_1}\right)^2\right|$$

and take θ so that $-b_1^2e^{2i(\varphi+\theta)}=|b_1|^2$, then we obtain the desired result.

Corollary. *If $g(z) = z + b_2z^2 + b_3z^3 + \cdots \in S$ and $|g(z)| \le M$ for $z \in \mathbb{D}$, then*

$$|b_3 - b_2^2| \le 1 - \frac{1}{M^2}.$$

Proof. Just apply Nehari's inequality to $g(z)/M$.

Theorem 6. *Let $\alpha(\rho)$ be the order of the linearly invariant family $S(\rho)$. Then*

$$\pi/(2\arctan R) \le \alpha(\rho) \le \sqrt{1 + 3(1 - 1/M)/R^2},$$

where $R = \tanh(\rho)$, and $M > 1$ satisfies

$$\left(\frac{1+R}{1-R}\right)^{\sqrt{1+3(1-1/M)/R^2}} - 2\sqrt{(R^2+3)M-3} - 1 = 0$$

when $\rho < \infty$, and $M = \infty$ when $\rho = \infty$.

Proof. From Example 1, we know that $\alpha(\rho) \ge \pi/(2\arctan R)$. We shall establish the upper bound via an iterative procedure. Since $S(\rho)$ is a rotationally invariant compact normal family which is also invariant under $\mathrm{Aut}(\mathbb{D})$, there exists a function $f(z) = z + a_2z^2 + \cdots \in S(\rho)$ which satisfies $\alpha(\rho) = a_2 > 0$. In fact, the function f maximizes the real part of the second coefficient over the family $S(\rho)$. Since $S(\rho)$ is linearly invariant, the Marty relation [14] yields $3a_3 - 2a_2^2 - 1 = 0$, or

$$a_3 - a_2^2 = (1 - a_2^2)/3. \tag{1}$$

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We start with an elementary upper bound on $\alpha(\rho)$ and then use Nehari's Inequality and (1) to obtain an improved estimate on $\alpha(\rho)$. The function $g(z) = f(Rz)/R$ belongs to the class S, so

$$\alpha(\rho) = a_2 \le 2/R. \tag{2}$$

This gives the elementary upper bound $\alpha(\rho) \le 2\coth(\rho)$. From [14, p. 115], we get

$$|f(z)| \le \frac{R}{4}\left\{\left(\frac{1+|z|}{1-|z|}\right)^{2/R} - 1\right\}.$$

Hence

$$|g(z)| < \frac{1}{4}\left\{\left(\frac{1+R}{1-R}\right)^{2/R} - 1\right\} \equiv \sqrt{M_0} \qquad (z \in \mathbb{D}).$$

By using the Corollary to Nehari's Inequality, we get

$$|a_2^2 - a_3| \le (1 - 1/M_0)/R^2.$$

Then (1) gives

$$|a_2^2 - 1| \le 3(1 - 1/M_0)/R^2,$$

so that

$$\alpha(\rho) = a_2 \le \sqrt{1 + 3(1 - 1/M_0)/R^2} < 2/R. \tag{3}$$

Now, if we repeat the above process with inequality (3) in place of (2), then we get

$$|g(z)| < \sqrt{M_1} \equiv$$
$$\frac{1}{2R\sqrt{1+3(1-1/M_0)/R^2}}\left\{\left(\frac{1+R}{1-R}\right)^{\sqrt{1+3(1-1/M_0)/R^2}} - 1\right\} \qquad (z \in \mathbb{D})$$

and

$$\alpha(\rho) = a_2 \le \sqrt{1 + 3(1 - 1/M_1)/R^2}.$$

By using induction, we obtain a decreasing sequence $\{M_n\}$ defined by

$$\sqrt{M_{n+1}} \equiv \frac{1}{2R\sqrt{1+3(1-1/M_n)/R^2}}\left\{\left(\frac{1+R}{1-R}\right)^{\sqrt{1+3(1-1/M_n)/R^2}} - 1\right\}$$

such that

$$\alpha(\rho) = a_2 \le \sqrt{1 + 3(1 - 1/M_{n+1})/R^2}.$$

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The decreasing sequence $\{M_n\}$ is bounded below by 1, so $M = \lim M_n$ exists and satisfies

$$\sqrt{M} = \frac{1}{2R\sqrt{1+3(1-1/M)/R^2}}\left\{\left(\frac{1+R}{1-R}\right)^{\sqrt{1+3(1-1/M)/R^2}} - 1\right\},$$

or equivalently,

$$\left(\frac{1+R}{1-R}\right)^{\sqrt{1+3(1-1/M)/R^2}} - 2\sqrt{(R^2+3)M-3} - 1 = 0.$$

In the limit we obtain

$$\alpha(\rho) = a_2 \leq \sqrt{1+3(1-1/M)/R^2}.$$

This completes the proof of Theorem 6.

REMARKS. (i) From the equation that M satisfies and the fact that $\alpha(\rho) \geq 2$ since k_2 belongs to $S(\rho)$, we have

$$1/(1-R^2) \leq M \leq \frac{1}{4(3+R^2)}\left\{\left[\frac{1+R}{1-R}\right]^{\sqrt{1+3/R^2}} - 1\right\}^2.$$

(ii) Note that $\alpha(\rho) \leq \sqrt{1+3(1-1/M)/R^2} \leq \sqrt{1+3/R^2} \approx \sqrt{3}/R$ as $\rho \to 0$. On the other hand, $\alpha(\rho) \geq \pi/(2\arctan R) \geq \pi/2R$. Thus, $\alpha(\rho) = O(1/R)$ as $\rho \to 0$. Let $\mu = \limsup_{\rho\to 0} R\alpha(\rho)$; then $\pi/2 \approx 1.571 \leq \mu \leq 1.732 \approx \sqrt{3}$. Therefore, we have replaced the original upper bound of 2 (which follows from $\alpha(\rho) \leq 2\coth(\rho)$) for μ by $\sqrt{3}$.

(iii) For $R = 1/\sqrt{3}$, the lower bound is $\pi/2\arctan R = 3$. On the other hand, the elementary upper bound is $2/R = 2\sqrt{3} \approx 3.464$, while our improvement in Theorem 6 yields the upper bound

$$\sqrt{1+3(1-1/M)/R^2} \leq \sqrt{1+3/R^2} = \sqrt{10} \approx 3.1623.$$

This shows that our upper bound is a real improvement on the elementary upper bound and can be quite close to the lower bound.

(iv) For $\rho < \infty$, it is unlikely that our upper bound on $\alpha(\rho)$ is sharp. What is the actual order of the linearly invariant family $S(\rho)$? Or what is the best possible value of μ? It is plausible to conjecture that $\alpha(\rho) = \pi/(2\arctan R)$.

We know that f is univalent in $|z| < R = \tanh\rho$ if $f \in S(\rho)$; so trivially f is convex in $|z| < (2-\sqrt{3})R$. From Theorems 3 and 6 we get a better result.

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COROLLARY. $S(\rho) \subset C(\operatorname{arctanh}(\sqrt{1+3/R^2} - \sqrt{3}/R))$. *In particular, each* $f \in S(\rho)$ *is convex univalent in* $|z| < \sqrt{1+3/R^2} - \sqrt{3}/R$.

PROOF. From Theorem 6, $\alpha(\rho) \leq \sqrt{1+3/R^2}$. By making use of Theorem 3, we have

$$S(\rho) \subset \mathscr{F}(\sqrt{1+3/R^2}) = C(\operatorname{arctanh}(\sqrt{1+3/R^2} - \sqrt{3}/R)).$$

From the proof of Theorem 6, it follows that $g(z) = f(Rz)/R$ satisfies $|g(z)| < M^{1/2}$. By making use of Example 1 and applying Nehari's Inequality to $g(z)$, we have

THEOREM 7. *Let* $f(z) = z + a_2 z^2 + \cdots \in S(\rho)$. *Then*

$$(\pi^2/(2\arctan R)^2 - 1)/3 \leq \max\{|a_2^2 - a_3| : f \in S(\rho)\} \leq (1 - 1/M)/R^2,$$

where $R = \tanh(\rho)$ *and* M *is given in Theorem* 6.

Recently, Harmelin [3] obtained an upper bound on the Schwarzian derivative of $f \in \mathscr{F}(\alpha)$. Theorem 6 in conjunction with this bound of Harmelin will yield an upper bound on the Schwarzian derivative of $f \in S(\rho)$. If we use Nehari's Inequality, we can obtain a better estimate.

COROLLARY. *Let* $f \in S(\rho)$. *Then*

$$(1 - |z|^2)^2 |S_f(z)| \leq 6(1 - 1/M)/R^2,$$

where $R = \tanh(\rho)$ *and* M *is given in Theorem* 6.

PROOF. For every $f \in S(\rho)$ and $a \in \mathbb{D}$, $F(z) = f(z, a) = z + A_2 z^2 + \cdots \in S(\rho)$, and $(1 - |a|^2)^2 S_f(a) = 6(A_3 - A_2^2)$. The desired result follows from Theorem 7.

In view of the fact that $C(\rho) = \mathscr{F}(\alpha(C(\rho)))$, it is natural to inquire whether $S(\rho) = \mathscr{F}(\alpha(S(\rho)))$. The answer is negative.

THEOREM 8. *Let* $\alpha(\rho)$ *be the order of the linearly invariant family* $S(\rho)$. *Then* $S(\rho)$ *is a proper subset of* $\mathscr{F}(\alpha(\rho))$.

PROOF. Suppose $S(\rho) = \mathscr{F}(\alpha(\rho))$. Then $k_{\alpha(\rho)} \in S(\rho)$ and so Example 1 implies $\alpha(\rho) \leq \pi/(2\arctan R)$, where $R = \tanh(\rho)$. Since Theorem 6 gives the opposite inquality, we would have $\alpha(\rho) = \pi/(2\arctan R)$. All that remains is to construct a function in the class $\mathscr{F}(\pi/(2\arctan R))$ which does

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not belong to $S(\rho)$. From Example 2 we know that $h_\beta \in \mathcal{F}(\pi/(2\arctan R))$ for $\beta = \pi/2\arctan R + \sqrt{\pi^2/(4\arctan^2 R) - 1}$. Note that $\beta > \pi/R$. In fact, this inequality is equivalent to $H(R) \equiv \pi^2 R/(\pi^2 + R^2) - \arctan R > 0$, which is true since $H'(R) > 0$ for $0 < R < 1$ and $H(0) = 0$. Because $\beta > \pi$, we know that $\rho(h_\beta) = \operatorname{artanh}(\pi/\beta)$. But $\pi/\beta < R$, so $\rho(h_\beta) < \rho$, or $h_\beta \notin S(\rho)$. Therefore, $S(\rho)$ is a proper subfamily of $\mathcal{F}(\alpha(\rho))$.

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