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Abstract

We develop analogues of Green's N_p conditions for subvarieties of weighted projective space, and we prove that such N_p conditions are satisfied for high degree embeddings of curves in weighted projective space. A key technical result links positivity with low degree (virtual) syzygies in wide generality, including cases where normal generation fails.

1. Introduction

One of the foundational results connecting syzygies and geometry is Green's Theorem on linear syzygies of smooth curves.

Theorem 1.1 [Gre84a]. *Let C be a smooth curve of genus g embedded in $\mathbb{P}^n$ via a complete linear series $|L|$ and F be the minimal free resolution of the homogeneous coordinate ring of C. If $\deg(L) \geq 2g+1+p$ for some $p \geq 0$, then the embedding $C \hookrightarrow \mathbb{P}^n$ satisfies the N_p condition: that is, it is normally generated and F_i is generated in degree $i+1$ for $1 \leq i \leq p$.*

Let us recall the definitions of the terms in the theorem. The embedding $C \hookrightarrow \mathbb{P}^n$ is defined to be *normally generated* (or *projectively normal*) if the section ring $\bigoplus_{i\geq 0} H^0(C, L^i)$ is generated in degree 1. Theorem 1.1 gives a common generalization, in the language of syzygies, of two classical results showing that the algebraic presentation of a curve exhibits more rigid structure as its degree grows. Specifically, the $p=0$ case of Theorem 1.1 is Castelnuovo's Theorem, which states that $C \subseteq \mathbb{P}^r$ is normally generated if $\deg(L) \geq 2g+1$ [Cas93]; while the $p=1$ case is a result of Fujita and Saint-Donat stating that C is cut out by quadrics whenever $\deg(L) \geq 2g+2$ [Fuj77, SD72].

Theorem 1.1 helped launch the modern study of the geometry of syzygies. It led to numerous generalizations involving embeddings of surfaces [AKL19, BH13, GP96, GP99, KL19, NY22], smooth higher dimensional varieties [EGHP05, EL93, GP98, HSS06, HT13], abelian varieties [Chi19, Par00, PP03, PP04], and Calabi–Yau varieties [Niu19]; see also [DE22, ENP20].

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Theorem 1.1 also led to Green's Conjecture, which proposes a relationship between the Clifford index of a non-hyperelliptic curve over $\mathbb{C}$ and the linearity of the free resolution of its coordinate ring with respect to its canonical embedding [Gre84b]. This conjecture remains open in general and is a highly active area of research; see e.g. [AFP+19, AV03, Voi02, Voi05]. For an introduction to the wide circle of ideas on syzygies of curves, see Aprodu and Nagel's book [AN10], Ein and Lazarsfeld's survey [EL20], Eisenbud's book [Eis05], and more [EL18, Far17].

The past two decades have seen a flurry of activity devoted to generalizing work on syzygies to the nonstandard graded setting. For instance, Benson [Ben04] generalized Eisenbud–Goto's Theorem on Castelnuovo–Mumford regularity and linear free resolutions [EG84, Theorem 1.2(1)] to nonstandard $\mathbb{Z}$-graded polynomial rings, leading to breakthroughs in invariant theory due to Symonds [Sym10, Sym11]. Benson's resulting 'weighted' analogue of the Castelnuovo–Mumford regularity was generalized by Maclagan and Smith to multigraded polynomial rings in [MS04], with a view toward applications in toric geometry. This in turn led to much follow-up work on multigraded regularity [BC17, BHS21, BHS22, CH22, CN20], as well as a wide-ranging program on multigraded syzygies [BE21, BES20, BKLY21, BS22, BCN22, EES15, HNVT22, HS04, HSS06, SVT06, Yan21].

The present work is the first generalization of Green's Theorem that allows the target of the embedding to be a variety other than projective space, connecting with the literature on nonstandard gradings discussed above, and raising many new questions. For instance, it is far from obvious how to even state an analogue of Theorem 1.1 for curves embedded in weighted projective space. To define weighted N_p conditions for $p \geq 1$, one must ask: What does it mean for a complex of free modules over a nonstandard-$\mathbb{Z}$-graded polynomial ring to be *linear*? To illustrate the subtlety in this question, take a standard graded polynomial ring S and an S-module M generated in degree 0. The minimal free resolution F of M is linear if and only if it satisfies the following three equivalent conditions.

(1) The differentials in F are matrices of linear forms.

(2) The Betti table of F has exactly one row.

(3) The degrees of the syzygies grow no faster than those of the Koszul complex.

In the nonstandard $\mathbb{Z}$-graded case, each of these yields a *distinct* analogue of a linear resolution. Furthermore, there is no obvious 'best' choice, as each measures something meaningful: (1) leads to strong linearity (Definition 4.1) and the Bernstein–Gel'fand–Gel'fand (BGG) correspondence as in [BE22], (2) to weighted regularity and local cohomology as in [Ben04], and (3) to Koszul linearity (Definition 4.12) and connections with Koszul cohomology; see § 4 for details on these notions and how they are related. In fact, a central obstacle in our work is the technical challenge of interpolating between these nonequivalent notions of linearity, a challenge that simply is not present in the classical setting.

Phrasing an analogue of Theorem 1.1 also requires weighted versions of a complete linear series and of normal generation. A weighted notion of normal generation is fairly straightforward; see Definition 3.13. But defining a weighted version of a complete linear series turns out to be rather subtle and, as with linearity, there are multiple potential analogues, depending on which aspect of the classical notion one considers. The subtlety arises partly because there is a tension between nondegeneracy and normal generation (see Examples 3.7 and 3.16), and partly because any analogue must depend on data beyond just the line bundle. We propose a *log complete series* in Definition 3.8 as an analogue of a complete linear series that requires a minimal amount of extra data: a base locus and a degree. These lead to a rich family of examples of embeddings, where the underlying weighted spaces are fairly simple, involving just two distinct degrees.

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For our weighted N_p conditions, we use condition (3) from above.

Definition 1.2. Let S be the $\mathbb{Z}$-graded polynomial ring corresponding to a weighted projective space $\mathbb{P}(W)$. Write w^i for the maximal degree of an ith syzygy of the residue field. Let $Z \subseteq \mathbb{P}(W)$ be a variety and $F = [F_0 \leftarrow F_1 \leftarrow \cdots]$ the minimal S-free resolution of its coordinate ring. We say $Z \subseteq \mathbb{P}(W)$ satisfies the *weighted N_p condition* if it is normally generated (Definition 3.13), and F_i is generated in degree $\leq w^{i+1}$ for all $i = 1, 2, \ldots, p$ (i.e. the complex $[F_0 \leftarrow \cdots \leftarrow F_p]$ is Koszul 1-linear, in the sense of Definition 4.12).

In the standard grading, we have $w^{i+1} = i + 1$, so our definition extends Green's; e.g. if the variables of $S = k[x_1, x_2, x_3]$ have degrees 1, 2, and 5 then $w^1 = 5$, $w^2 = 7$, and $w^3 = 8$.

We now turn to our main results. We establish the following standing hypotheses.

Setup 1.3. Let C be a smooth curve of genus g, L a line bundle on C, D an effective divisor on C, and $d \geq 2$. Assume that $\deg(L \otimes \mathcal{O}(-D)) \geq 2g + 1$. Let W be the log complete series of type (D, d) for L (see Definition 3.8), $S = k[x_0, \ldots, x_n]$ the (nonstandard $\mathbb{Z}$-graded) coordinate ring of $\mathbb{P}(W)$, and $I_C \subseteq S$ the defining ideal of the induced embedding $C \subseteq \mathbb{P}(W)$.

Our first result is a weighted generalization of Castelnuovo's theorem, i.e. the $p = 0$ case of Green's theorem; see also [GL86, Mat61, Mum70].

Theorem 1.4. *Under Setup 1.3, the log complete series W is normally generated.*

The key to Theorem 1.4 is using a suitable generalization of the notion of a complete linear series (Definition 3.8), as many embeddings of curves into weighted projective spaces simply fail to enjoy any reasonable analogue of normal generation; see Example 3.16.

The following generalization of Green's theorem (Theorem 1.1) is our main result.

Theorem 1.5. *With Setup 1.3: if $\deg(L \otimes \mathcal{O}(-D)) \geq 2g + 1 + q$ for $q \geq 0$ then $C \subseteq \mathbb{P}(W)$ satisfies the weighted $N_{q+d\cdot\deg(D)}$ condition.*

The theorem shows that, even for embeddings into weighted projective spaces, geometric positivity continues to find expression via low degree syzygies, and in a manner that grows uniformly with $\deg(L)$. In other words, Green's fundamental insight from Theorem 1.1 continues to hold for embeddings into weighted projective spaces.

In fact, as the weighted setting has several distinct notions of 'linearity', the result even helps bring Green's result into sharper focus, clarifying that positivity is linked with linearity as defined in relation to the Koszul complex, as opposed to alternate notions of linearity, which are equivalent in the standard grading setting, but not in the weighted setting. In somewhat rough terms, Theorem 1.5 states that, as $q \to \infty$, the Betti table of the (weighted) homogeneous coordinate ring of C 'looks increasingly like the Koszul complex'. The ring S is standard graded if and only if $D = 0$, in which case Theorem 1.5 recovers Green's Theorem. The reader may find it helpful to look ahead to § 2, where we discuss several detailed examples.

As an immediate consequence of Theorem 1.5, we obtain a generalization of the aforementioned theorem of Fujita and Saint-Donat stating that a curve embedded by a complete linear series of degree $\geq 2g + 2$ is cut out by quadrics. It is too much to hope that C will be cut out by k-linear combinations of products $x_i x_j$ (see Example 2.1), but this intuition points towards the correct degree bound on the relations.

Corollary 1.6. *With Setup 1.3: if $\deg(L \otimes \mathcal{O}(-D)) \geq 2g + 2$ then I_C is defined by equations of degree at most $2d = \max_{i \neq j}\{\deg(x_i x_j)\}$.*

In other words, Corollary 1.6 implies that the degrees of the defining equations of C are bounded by the maximal degree of a syzygy of $\mathfrak{m}$. A number of results in the literature have a similar form to Corollary 1.6, showing that certain relations are generated in degree at most twice the degree of one of the k-algebra generators, e.g. [LRZ16, LRZ18, Sym11, VZB22].

Our proof of Theorem 1.5 relies on a far more general result relating geometric positivity to low degree syzygies of the section ring $R := \bigoplus_{e\geq 0} H^0(C, L^e)$.

Theorem 1.7. *Let C be a smooth curve of genus g, L a line bundle on C, and $f\colon C \to \mathbb{P}(W)$ a closed immersion induced by a weighted series W associated to L (see § 3.1 for the definition of a weighted series).*[1] *Assume that $\deg(L) \geq 2g+1$ and that $\dim S_1 > g$. Let F be the minimal free resolution of the section ring R over the coordinate ring S of $\mathbb{P}(W)$. The generators of each F_i lie in degree $\leq w^{i+1}$ for all $i \leq \dim W - g - 2$.*

Theorem 1.7 highlights that the connection between positivity of an embedding and low degree syzygies is quite robust, as it applies to many situations where normal generation fails. For instance, if we specialize to ordinary projective space, Theorem 1.7 may be applied to obtain low degree syzygies even in cases where a curve is embedded by an incomplete linear series. In fact, many of Green's results allow for an incomplete linear series, and in the case of an embedding into a standard projective space by an incomplete series, Theorem 1.7 follows from Green's Vanishing Theorem [Gre84a, Theorem 3.a.1]. In the weighted projective case, Theorem 1.7 can be applied quite broadly, as it does not involve a log complete hypothesis.

There is an important distinction between Theorems 1.7 and 1.5: the degree bounds hold for the section ring R and not for the coordinate ring S/I_C, respectively. For this reason, Theorem 1.7 yields something like virtual N_p conditions, where we use 'virtual' in the sense of the theory of virtual resolutions introduced in [BES20], as F is a virtual resolution of the structure sheaf $\mathcal{O}_C$. Theorem 1.7 shows that the connection between geometric positivity and low degree syzygies, which was first illuminated by Green, holds in tremendous generality as long as one considers *virtual* syzygies.

The main theme underlying the technical heart of this paper is the way that, when one passes from a standard to a nonstandard grading, notions of linearity split apart and yet remain subtly intertwined. More specifically, each of the three weighted notions of linearity of free complexes mentioned above, and discussed in detail in § 4, come into play in the following ways.

- Koszul linearity is closely linked to geometric positivity; that is, it is the right notion for weighted N_p conditions.
- Strong linearity, specifically the multigraded linear syzygy theorem of [BE22], is essential to our proof of our key technical result Theorem 1.7.[2]
- Weighted regularity plays a crucial role in our proof of Theorem 1.4.

In summary, our main results build on Green's insight that geometric positivity is expressed algebraically in terms of low degree syzygies, although this requires novel viewpoints on nearly all of the objects involved. Our results provide a proof of concept for the broader idea that the 'geometry of syzygies' literature has analogues in the weighted projective setting, and more

[1] We do not assume that W is a log complete series.

[2] In this way, the proofs of our main results echo the proof of Green's Theorem 1.1 via Green's linear syzygy theorem, as in [Eis05, Section 8]. We expect that one could also use an analogue of the M_L-bundle approach from [GL88] to obtain similar results, although such an approach is complicated by the fact that the weighted series W is generated in distinct degrees. See § 7.3.

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generally, for embeddings into toric varieties or beyond, bolstering the nascent homological theories for multigraded polynomial rings. Our work also raises a host of new questions: What might play the role of 'scrolls' in a weighted projective setting? Is there a weighted analogue of Green's Conjecture (perhaps for stacky curves)? See § 7 for an array of such questions related to the results in this paper.

Let us now give an overview of the paper. We begin in § 2 with a host of examples illustrating our main results. In § 3 we begin a detailed investigation of closed immersions into weighted projective spaces; we introduce in this section our notion of a 'log complete series' and prove a number of foundational results. Section 4 contains a detailed discussion of the various weighted flavors of linear free complexes discussed above. In § 5 we prove Theorem 1.7, the central technical result of the paper. In § 6 we prove the rest of our main results. Finally, in § 7 we outline some follow-up questions raised by this work.

1.1 Notation

Throughout the paper, k denotes a field and the word 'variety' means 'integral scheme that is separated and a finite type over k'. Given a vector $\mathbf{d}=(d_0,\dots,d_n)$ of positive integers, we let $\mathbb{P}(\mathbf{d})$ denote the associated weighted projective space. We always assume that $d_0 \le d_1 \le \cdots \le d_n$. We often use exponents to indicate the number of weights of a particular degree; for instance, we write $\mathbb{P}(1^3, 2^2)$ for $\mathbb{P}(1,1,1,2,2)$. Given a weighted projective space $\mathbb{P}(\mathbf{d})$, we always denote its coordinate ring by S. That is, S is the $\mathbb{Z}$-graded ring $k[x_0,\dots,x_n]$ with $\deg(x_i)=d_i$. Alternatively, given a weighted vector space W, we write $\mathbb{P}(W)$ for the corresponding weighted projective space with coordinate ring $S=\operatorname{Sym}(W)$. We write $\mathfrak{m}$ for the homogeneous maximal ideal of S.

2. Examples

Before diving into the heart of the paper, we illustrate our main results with some examples. In particular, this section is intended to answer the question: What does a Betti table that satisfies the weighted N_p condition look like? All computations in Macaulay2 [GS] of Betti tables throughout this section were performed in characteristic 0.

Theorem 1.5 reduces to Green's Theorem (Theorem 1.1) when $\deg(D)=0$. The simplest new cases are therefore when $\deg(D)=1$ and $d=2$, and so we begin with such examples.

Example 2.1. Let $C=\mathbb{P}^1$, $D=[0:1]$ and $d=2$. If $L=\mathcal{O}_{\mathbb{P}^1}(2)$ then the corresponding log complete series W (see Definition 3.8) is $\langle s^2, st, st^3, t^4\rangle$. This induces a closed immersion

$$\mathbb{P}^1 \to \mathbb{P}(1^2,2^2) \quad \text{given by } [s:t]\mapsto [s^2:st:st^3:t^4].$$

The defining ideal I_C for the curve is generated by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1^2 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

where $\deg(x_0)=\deg(x_1)=1$ and $\deg(x_2)=\deg(x_3)=2$. Corollary 1.6 implies that I is generated in degree at most 4, and we can see that this holds and is sharp. A direct computation confirms that $C\subseteq \mathbb{P}(W)$ is normally generated (see Definition 3.13) and that S/I_C is a Cohen–Macaulay ring, as predicted by Theorem 1.4. Theorem 1.5 implies that this embedding satisfies the N_2 condition, and we can check this by observing the free resolution

$$F=\left[S \leftarrow S(-3)^2\oplus S(-4)\leftarrow S(-5)^2\leftarrow 0\right]$$

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of S/I_C. Indeed, F_1 is generated in degrees $\leq 4 = w^2$ and F_2 is generated in degrees $\leq 5 = w^3$ and these bounds are sharp.

Example 2.2. Let us continue with the setup of the previous example, but now take $L = \mathcal{O}_{\mathbb{P}^1}(8)$. The corresponding log complete series W is spanned by $s^8, s^7t, \ldots, st^7, st^{15}$ and t^{16}. This weighted series induces a map $\mathbb{P}^1 \to \mathbb{P}(W) = \mathbb{P}(1^8, 2^2)$. In this case, the defining ideal I_C is given by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7^2 & x_8 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{pmatrix}.$$

This ideal is generated in degree at most 4, as predicted by Corollary 1.6. Once again, one can directly check that the embedding is normally generated and that S/I_C is a Cohen–Macaulay ring. Theorem 1.5 implies that this embedding satisfies the N_8 condition, and one verifies this by inspecting the Betti table[3] of S/I_C.

	0	1	2	3	4	5	6	7	8
0 :	1	.	.	.	.	.	.	.	.
1 :	.	21	70	105	84	35	6	.	.
2 :	.	14	84	210	280	210	84	14	.
3 :	.	1	14	63	140	175	126	49	8

Indeed, F_i is generated in degree $\leq 3 + i = w^{i+1}$ for $1 \leq i \leq 8$; once again, the bounds imposed by the weighted N_p conditions are sharp.

Example 2.3. Let C be the genus 2 curve defined by the equation $z_2^2 - z_1^6 - 5z_0^3z_1^3 - z_0^6$ in $\mathbb{P}(1, 1, 3)$. Suppose that $d = 2$ and D is the single point $[1 : 0 : 1]$. Let L be a line bundle of degree 10 on C. We have

$$\deg(L \otimes \mathcal{O}(-D)) = 9 = 2g + 1 + q, \quad \text{with } q = 4.$$

It follows from the Riemann–Roch Theorem that $\mathbb{P}(W) = \mathbb{P}(1^8, 2^2)$, and so the associated log complete series induces an embedding $C \subseteq \mathbb{P}(1^8, 2^2)$. Theorem 1.4 shows that $C \subseteq \mathbb{P}(1^8, 2^2)$ is normally generated and that its coordinate ring is a Cohen–Macaulay ring, which does not seem obvious (at least to these authors). By Theorem 1.5, this embedding satisfies the $N_{q+d\cdot\deg D} = N_6$ condition. A computation in `Macaulay2` yields the following Betti table for S/I_C.

	0	1	2	3	4	5	6	7	8
0 :	1	.	.	.	.	.	.	.	.
1 :	.	19	58	75	44	5	.	.	.
2 :	.	14	80	186	220	136	26	2	.
3 :	.	1	14	61	128	145	98	23	2
4 :	.	.	.	.	.	.	.	6	2

Since $w^8 = 10$, we see that the 7th syzygies require a generator of degree $> w^8$, and thus C satisfies the weighted N_6 condition, but not the N_7 condition.[4]

The examples so far have been in the case where $d = 2$ and $\deg D = 1$. We now consider the shape of the Betti table in a slightly different setting.

[3] We follow standard Macaulay2 formatting of Betti tables, where the entry in the ith column and the jth row corresponds to $\dim \operatorname{Tor}_i(S/I, k)_{i+j}$, and a dot indicates an entry of 0.

[4] Those familiar with the Green–Lazarsfeld Gonality Conjecture [GL88, EL15] might wonder if one can 'see' the gonality of C in this Betti table. The answer is 'yes', but for a trivial reason; see § 7.

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Example 2.4. Let us consider the case of a genus g curve C, where $d=2$ but now $\deg D=2$. Assume that $\deg(L\otimes\mathcal{O}(-D))=2g+1+q$ for some $q\geq 0$. By the Riemann–Roch Theorem, we have $\dim W_1=g+q+2$ and $\dim W_2=4$, and so $C\subseteq\mathbb{P}(1^{g+q+2},2^4)$. Theorem 1.5 implies that the minimal free resolution of S/I_C satisfies the N_{q+4} condition. Since there are now four variables of degree >1, the shape of the Betti table is more complicated. Specifically, we have $w^2=4, w^3=6, w^4=8$, and $w^{i+1}=w^i+1$ for $i\geq 4$. This implies that the Betti table of the curve has the following shape, where a symbol $*$ indicates a potentially nonzero entry.

	0	1	2	3	4	5	…	$q+4$	$q+5$	…	$q+g+4$
0:	$*$	.	.	.	.	.	…	.	.	…	.
1:	.	$*$	$*$	$*$	$*$	$*$	…	$*$	$*$	…	$*$
2:	.	$*$	$*$	$*$	$*$	$*$	…	$*$	$*$	…	$*$
3:	.	$*$	$*$	$*$	$*$	$*$	…	$*$	$*$	…	$*$
4:	.	.	$*$	$*$	$*$	$*$	…	$*$	$*$	…	$*$
5:	.	.	.	$*$	$*$	$*$	…	$*$	$*$	…	$*$
6:	.	.	.	.	.	.	…	.	$*$	…	$*$

By combining Remark 4.8 and Corollary 6.6, we see that the kth row of the Betti table vanishes for $k>6$. The key moment for the weighted N_{q+4} condition is in column $q+5$, where Theorem 1.5 no longer guarantees that F_{q+5} is generated in degree $\leq w^{q+6}=q+10$.

Remark 2.5. For $C=\mathbb{P}^1$, the following minor variants of Example 2.1 both lead to non-Cohen–Macaulay examples: $W=\langle s^2,st,st^3,t^4,t^6\rangle$ and $W'=\langle s^2,st,st^5,t^6\rangle$. This underscores the challenge in finding an appropriate weighted analogue of a complete linear series.

3. Closed immersions into weighted projective spaces

We now begin to lay the technical foundation for this paper, starting with a study of closed immersions into a weighted projective space.

3.1 Weighted series

Let Z be a variety and L a line bundle on Z. A *weighted series* is a finite-dimensional, $\mathbb{Z}$-graded k-subspace $W\subseteq\bigoplus_{i\in\mathbb{Z}}H^0(Z,L^i)$. Choosing a basis $s_0,\dots,s_n$ of W, where $s_i\in W_{d_i}\subseteq H^0(Z,L^{d_i})$, induces a rational map $\varphi_W\colon Z\dashrightarrow\mathbb{P}(\mathbf{d})$ in exactly the same way as in the case of an ordinary projective space. When the intersection of the zero loci of the s_i is empty, φ_W is a well-defined morphism. Let $S=k[x_0,\dots,x_n]$ be the $\mathbb{Z}$-graded coordinate ring of $\mathbb{P}(\mathbf{d})$. We will only be interested in the case where φ_W is a closed immersion; we describe sufficient conditions for this in Proposition 3.2 below. In this case, let $I_Z\subseteq S$ denote the homogeneous prime ideal corresponding to the embedding of Z in $\mathbb{P}(\mathbf{d})$. The *homogeneous coordinate ring of* φ_W is the $\mathbb{Z}$-graded ring S/I_Z.

Remark 3.1. Before embarking on the results in this section, we highlight some key differences between the behavior of sheaves on weighted and ordinary projective spaces.

(1) We have $\mathrm{Pic}(\mathbb{P}(d))=\{\mathcal{O}_{\mathbb{P}(\mathbf{d})}(\ell m)\}_{\ell\in\mathbb{Z}}$, where $m=\mathrm{lcm}(d_0,\dots,d_n)$ [BR86, Theorem 7.1(c)]; in particular, not every sheaf $\mathcal{O}_{\mathbb{P}(\mathbf{d})}(i)$ is a line bundle.

(2) It can happen that $\mathcal{O}_{\mathbb{P}(\mathbf{d})}(i)\otimes\mathcal{O}_{\mathbb{P}(\mathbf{d})}(j)\not\cong\mathcal{O}_{\mathbb{P}(\mathbf{d})}(i+j)$; see e.g. [BR86, pp. 134]. However, it follows from [BR86, Corollary 4A.5(b)] that $\mathcal{O}_{\mathbb{P}(\mathbf{d})}(im)\otimes\mathcal{O}_{\mathbb{P}(\mathbf{d})}(j)\cong\mathcal{O}_{\mathbb{P}(\mathbf{d})}(im+j)$ for all $i,j\in\mathbb{Z}$.

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(3) Given a graded S-module M, it is not always the case that $\widetilde{M}(j) := \widetilde{M} \otimes \mathcal{O}_{\mathbb{P}(\mathbf{d})}(j)$ coincides with $\widetilde{M(j)}$. Indeed, taking $M = S(i)$, this follows from (2).

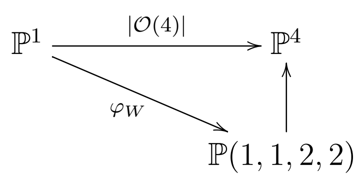
(4) Not every morphism $Z \to \mathbb{P}(\mathbf{d})$ arises as φ_W for some weighted series W. For instance, take $\mathbf{d} = (1,1,2)$. By (1), every line bundle on $\mathbb{P}(\mathbf{d})$ is of the form $\mathcal{O}_{\mathbb{P}(\mathbf{d})}(2\ell)$ for some $\ell \in \mathbb{Z}$. In particular, $\mathcal{O}_{\mathbb{P}(\mathbf{d})}(1)$ is not a line bundle, and so there is no line bundle that can induce the map $\mathbb{P}(\mathbf{d}) \xrightarrow{\mathrm{id}} \mathbb{P}(\mathbf{d})$.

PROPOSITION 3.2. *Let W be a weighted series with basis $s_i \in H^0(Z, L^{d_i})$ for $0 \le i \le n$. Assume that there exists $\ell > 0$ such that the map $S_\ell \to H^0(Z, L^\ell)$ induced by $\varphi_W : Z \to \mathbb{P}(\mathbf{d})$ is surjective and that L^ℓ is very ample. The morphism φ_W is a closed immersion.*

Proof. Let $f_0, \ldots, f_r$ be a basis of $H^0(Z, L^\ell)$. For $0 \le i \le r$, choose $F_i \in S_\ell$ such that $F_i(s_0, \ldots, s_n) = f_i$. The linear series determined by the F_i induces a rational map $\psi : \mathbb{P}(\mathbf{d}) \dashrightarrow \mathbb{P}^r$. Write U for the domain of definition of ψ. The image of φ_W lands in U since the restriction of the F_i to Z is $f_0, \ldots, f_r$, which is base-point free. By construction, the composition $\psi \circ \varphi_W$ is the morphism induced by $|L^\ell|$. Since L^ℓ is very ample, the map $\psi \circ \varphi_W$ is a closed immersion, and so φ_W is a closed immersion into U [Sta12, 0AGC]; since Z is proper, it follows that φ_W is a closed immersion as well. □

Remark 3.3. The pathologies described in Remark 3.1 all disappear when one works over the associated weighted projective stack; see [GS15, Section 7] or [Per08, Theorem 2.6]. However, the stack introduces its own complexities. For instance, the proof of Proposition 3.2 fails: letting $\mathbb{P}_{\text{stack}}(W)$ denote the associated stack and defining $\tau_W : Z \to \mathbb{P}_{\text{stack}}(W)$ in the same way as φ_W, the composition $Z \xrightarrow{\tau_W} \mathbb{P}_{\text{stack}}(W) \dashrightarrow \mathbb{P}^r$ being a closed immersion does not imply that τ_W is a closed immersion. For a simple counter example, one can let Z be a point and τ_W be any map to a stacky point.

Example 3.4. Let $Z = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(2)$. The weighted series W spanned by $s^2, st \in H^0(Z, L)$ and $st^3, t^4 \in H^0(Z, L^2)$ induces a map $\varphi_W : \mathbb{P}^1 \to \mathbb{P}(1,1,2,2)$ given by $[s : t] \mapsto [s^2 : st : st^3 : t^4]$ (this is the map in Example 2.1). Applying Proposition 3.2 with $\ell = 2$ implies that φ_W is a closed immersion. Indeed, we have a commutative diagram



where the vertical arrow is given by $[x_0 : x_1 : x_2 : x_3] \mapsto [x_0^2 : x_0 x_1 : x_1^2 : x_2 : x_3]$.

PROPOSITION 3.5. *Let W be a weighted series with basis $s_i \in H^0(Z, L^{d_i})$ for $0 \le i \le n$ and $f : S \to \bigoplus_{i \ge 0} H^0(Z, L^i)$ the ring homomorphism given by $x_i \mapsto s_i$. If φ_W is a closed immersion then $I_Z = \ker(f)$.*

Proof. The ideal I_Z is the unique homogeneous prime ideal $P \subseteq S$ such that $Z = V(P)$, where $V(-)$ is the assignment that sends any homogeneous ideal in S to its associated subvariety in $\mathbb{P}(\mathbf{d})$ (see e.g. [CLS11, Section 5.2]). Since $\bigoplus_{i \ge 0} H^0(Z, L^i)$ is a domain, $\ker(f)$ is prime, and clearly $V(\ker(f)) = Z$. □

DEFINITION 3.6. A closed immersion $Z \subseteq \mathbb{P}(\mathbf{d})$ is *nondegenerate* (respectively *degenerate*) if the defining ideal I_Z is (respectively is not) contained in $\mathfrak{m}^2$.

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Let $Z \subseteq \mathbb{P}(\mathbf{d})$ be a closed immersion with defining ideal $I_Z \subseteq S$. Since any basis of $\mathfrak{m}/\mathfrak{m}^2$ can be lifted to give algebra generators for S, the immersion $Z \subseteq \mathbb{P}(\mathbf{d})$ is degenerate if and only if I_Z contains an element in some minimal generating set for $\mathfrak{m}$. For instance, by examining their defining ideals, one can see that the curves in Examples 2.1 and 2.2 are both nondegenerate.

Example 3.7. Suppose that $Z = \mathbb{P}^1$, $L = \mathcal{O}(1)$, and $W = H^0(\mathbb{P}^1, \mathcal{O}(1)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(2))$. This is a fairly naive way to generalize a complete linear series, as we have simply taken all sections of degrees 1 and 2. The map $\varphi_W : \mathbb{P}^1 \to \mathbb{P}(1^2, 2^3)$ given by $[s:t] \mapsto [s:t:s^2:st:t^2]$ is a closed immersion, by Proposition 3.2. In this case, $Z \subseteq \mathbb{P}(1^2, 2^3)$ is degenerate since I_Z contains $z_2 - z_0^2$, $z_3 - z_0 z_1$, and $z_4 - z_1^2$.

3.2 Log complete series

We now ask: What is a weighted projective analogue of a *complete* linear series? Before we state our proposed definition, we fix the following notation: given a divisor L and an effective divisor D on a variety Z, we write $H^0(Z, L)_D$ for the subspace of sections that vanish along D.

Definition 3.8. Let Z be a smooth projective variety, L a line bundle on Z, D an effective divisor on Z, and $d \geq 2$. A weighted series W is a *log complete series of type* (D, d) if $W_1 = H^0(Z, L)_D$, $H^0(Z, L^d) = W_d \oplus \operatorname{im}(\operatorname{Sym}_d(W_1) \to H^0(Z, L^d))$, and $W_i = 0$ for $i \neq 1, d$.

Example 3.9. Let us revisit Example 2.1, where $C = \mathbb{P}^1$, $D = [0:1]$, and $d = 2$. In this case, $W_1 = \langle s^2 : st \rangle$ are the sections of L vanishing at D; and $W_2 = \langle st^3 : t^4 \rangle$.

Remark 3.10. While a log complete series provides a strong analogue of a complete linear series in the weighted setting, we do not claim that this is a comprehensive analogue. In fact, one could easily imagine minor variants of our setup that would be generated in 3 or more distinct degrees. As with analogues of linear resolutions, we expect that there are distinct analogues of a complete linear series that lead in different directions. We restrict attention to log complete series because they strike a good balance. On one hand, they are sufficiently rich to allow for a wide range of new applications and for our overarching goal of investigating the extent to which Green's results hold in nonstandard graded settings. On the other hand, they yield embeddings into fairly simple weighted projective spaces of the form $\mathbb{P}(1^a, d^b)$, thus avoiding some of the pathologies of arbitrary weighted spaces.

When $D = 0$, Definition 3.8 recovers the usual notion of a complete linear series. A log complete series W of type (D, d) is unique up to isomorphism of graded vector spaces; we will therefore refer to *the* log complete series of type (D, d). Observe that, when L^d is base-point free, the intersection of the zero loci of the sections in W is empty, so that W induces a well-defined morphism $\varphi_W : Z \to \mathbb{P}(\mathbf{d})$.

Lemma 3.11. *Let Z be a curve, L a line bundle on Z, and D an effective divisor on Z. There is an isomorphism $H^0(Z, L^a)_{bD} \cong H^0(Z, L^a \otimes \mathcal{O}(-bD))$ for all $a, b \in \mathbb{Z}$.*

Proof. Since $\mathcal{O}_{bD}$ is the structure sheaf of a zero-dimensional scheme, $L \otimes \mathcal{O}_{bD} \cong L$. Twisting the short exact sequence $0 \to \mathcal{O}(-bD) \to \mathcal{O} \to \mathcal{O}_{bD} \to 0$ by L, we therefore arrive at the short exact sequence $0 \to L^a \otimes \mathcal{O}(-bD) \to L^a \to \mathcal{O}_{bD} \to 0$. The long exact sequence in cohomology yields $H^0(Z, L^a)_{bD} := \ker(H^0(Z, L^a) \to H^0(Z, \mathcal{O}_{bD})) \cong H^0(Z, L^a \otimes \mathcal{O}(-bD))$. □

Proposition 3.12. *Let Z, L, D, and d be as in Definition 3.8 and W the log complete series of type (D, d). Assume that L^d is very ample. The following assertions hold.*

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(1) *The canonical map* $S_d \to H^0(Z, L^d)$ *is surjective.*

(2) *The induced map* $\varphi_W : Z \to \mathbb{P}(\mathbf{d})$ *is a nondegenerate closed immersion.*

(3) *The log complete series* W *is maximal in the following sense: any weighted series concentrated in degrees 1 and* d *that properly contains* W *is degenerate.*

Proof. Part (1) follows from the definition of a log complete series. We may apply Proposition 3.2, with $\ell = d$, to conclude that φ_W is a closed immersion. Nondegeneracy holds since the image of the map $W_1^{\otimes d} \to H^0(Z, L^d)$ intersects W_d trivially. This proves (2). If we were to add a section of $\mathrm{Sym}_d(W_1)$ to W_d then it would be in the image of the map $W_1^{\otimes d} \to H^0(Z, L^d)$, forcing degeneracy. Similarly, if we were to add a section s of $H^0(Z, L)$ to W_1, the image of $W_1^{\otimes d} \to H^0(Z, L^d)$ would intersect W_d nontrivially; this gives (3). □

For explicit examples of log complete series, see Examples 2.1 and 2.2 above.

3.3 A weighted analogue of normal generation

Classically, a closed immersion of a variety Z in $\mathbb{P}^n$ is projectively normal if the coordinate ring S/I_Z of the immersion is integrally closed [Har77, Example I.3.18]. If Z is normal, and the closed immersion is induced by the line bundle L, then the integral closure of S/I_Z is the section ring $\bigoplus_{i\in\mathbb{Z}} H^0(Z, L^i)$, and so the immersion is projectively normal if and only if the canonical map $S \to \bigoplus_{i\in\mathbb{Z}} H^0(Z, L^i)$ is surjective. In this case, the line bundle L is said to be *normally generated* [Mum70]. Since we have a short exact sequence

$$0 \to S/I_Z \hookrightarrow \bigoplus_{i\in\mathbb{Z}} H^0(Z, L^i) \to H^1_{\mathfrak{m}}(S/I_Z) \to 0$$

of graded S modules, one concludes that L is normally generated if and only if $H^1_{\mathfrak{m}}(S/I_Z) = 0$. With this in mind, we make the following definition.

Definition 3.13. Let Z be a variety and $Z \subseteq \mathbb{P}(\mathbf{d})$ a closed immersion defined by a weighted series W. We say W is *normally generated* if $H^1_{\mathfrak{m}}(S/I_Z) = 0$.

Remark 3.14. Let $Z \subseteq \mathbb{P}(\mathbf{d})$ be a closed immersion induced by a weighted series W. The following hold.

(1) The weighted series W is normally generated if and only if the depth of the S-module S/I_Z is at least 2. In particular, if Z is a smooth curve then W is normally generated if and only if S/I_Z is a Cohen–Macaulay ring.

(2) Let $T = S/I_Z$. Since I_Z is prime, $H^0_{\mathfrak{m}}(T) = 0$, and so, by [Eis95, Theorem A4.1], we have a short exact sequence of graded S modules

$$0 \to T \hookrightarrow \bigoplus_{i\in\mathbb{Z}} H^0(Z, \widetilde{T(i)}) \to H^1_{\mathfrak{m}}(T) \to 0.$$

Thus, W is normally generated if and only if the canonical map $S_i \to H^0(Z, \widetilde{T(i)})$ is surjective for all i, echoing the classical definition. □

Remark 3.15. Unlike the classical case, normal generation of W is not equivalent to S/I_Z being integrally closed, even when Z is normal. For instance, it follows from Theorem 1.4 that the weighted series from Example 2.1 is normally generated. However, using the notation of that example, the ring $S/I_C = k[s^2, st, st^3, t^4] \subseteq k[s,t]$ is not integrally closed. Indeed, $t^2 = st^3/st$ is in the field of fractions of S/I_C but not in S/I_C, and it is a root of the polynomial $z^2 - t^4 \in (S/I_C)[z]$.

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Example 3.16. As mentioned in the introduction, many weighted series fail to be normally generated. For instance, take a weighted series W such that W_1 is a base-point free, incomplete linear series that yields an embedding $Z \to \mathbb{P}(W_1)$. We have $H^1_{\mathfrak{m}}(S/I_Z)_1 \neq 0$, and thus, W fails to be normally generated. We thus see that, for a very positive linear series to have any hope of normal generation, adding a base locus to W_1 is necessary; this observation was a key motivation for our definition of a log complete series.

4. Linearity of free resolutions in the weighted setting

There are multiple ways to extend the definition of a linear free resolution to the weighted setting, each with its advantages and disadvantages. We consider three such notions.

(1) Perhaps the most obvious definition of linearity in the weighted setting is *strong linearity*, which requires all differentials in the resolution to be expressible as k-linear combinations of the variables; see Definition 4.1 below. This notion was defined and studied in our previous paper [BE22], and it is closely related to the multigraded generalization of the BGG correspondence [HHW12].

(2) We often find that strong linearity is too restrictive for our purposes. There is a weaker, and more well-known, notion of linearity based on a weighted analogue of the Castelnuovo–Mumford regularity and arising from invariant theory [Ben04, Sym11], which we call *weighted regularity*. It is determined by the number of rows in the Betti table of the resolution; see Definition 4.7 for details.

(3) Weighted regularity, however, is too weak of a condition for us; we therefore introduce in this paper an intermediate notion between (1) and (2) called *Koszul linearity* (Definition 4.12). Roughly speaking, a free resolution is Koszul linear if its Betti numbers grow no faster than those of the Koszul complex. Our definition of weighted N_p conditions (Definition 1.2) is based on Koszul linearity.

Each of (1)–(3) will be used in the proofs of our main results. In the standard graded case, each notion gives an alternative but equivalent way to view linear resolutions; Example 4.16 below illustrates how these notions diverge in the general weighted case. To briefly explain, while weighted regularity only depends on the number of rows in the Betti table of the resolution, Koszul linearity involves more granular information about the Betti numbers. Moreover, strong linearity cannot be detected from the Betti numbers of the resolution at all, as one can see from Example 4.2 below. See also [BE23], which explores the relationship between these notions in greater detail.

4.1 Strong linearity

Our most restrictive notion of linearity for nonstandard graded free resolutions is the following.

Definition 4.1 [BE22, Definition 1.1]. A complex F of graded free S modules is *strongly linear* if there exists a choice of basis of F with respect to which its differentials may be represented by matrices whose entries are k-linear combinations of the variables.

In the nonstandard graded setting, strong linearity of a free complex F cannot be detected by the degrees of its generators, as the following simple example illustrates.

Example 4.2. Suppose that $S = k[x_0, x_1]$, where the variables have degrees 1 and 2. Consider the complexes $S \xleftarrow{x_0^2} S(-2)$ and $S \xleftarrow{x_1} S(-2)$; only the second complex is strongly linear.

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The main goal of our previous paper [BE22] was to establish a theory of linear *strands* of free resolutions in the nonstandard graded context, culminating in a generalization of Green's linear syzygy theorem [Gre99]: that circle of ideas will play a key role in this paper. Before we recall the details, we briefly discuss some background on (a weighted analogue of) the BGG correspondence. We refer the reader to [BE21, Section 2.2] for a detailed introduction to the multigraded BGG correspondence, following the work of [HHW12].

4.1.1 *The weighted BGG correspondence.* Let $E = \bigwedge_k(e_0, \dots, e_n)$ be an exterior algebra, equipped with the $\mathbb{Z}^2$ grading given by $\deg(e_i) = (-\deg(x_i); -1)$. Denote by $\mathrm{Com}(S)$ the category of complexes of graded S modules and $\mathrm{DM}(E)$ the category of *differential E modules*, i.e. $\mathbb{Z}^2$-graded E-modules D equipped with a degree $(0;-1)$ endomorphism that squares to 0. The weighted BGG correspondence is an adjunction

$$\mathbf{L} : \mathrm{DM}(E) \leftrightarrows \mathrm{Com}(S) : \mathbf{R}$$

that induces an equivalence on derived categories. We will only be concerned in this paper with the functor $\mathbf{L}$ applied to E modules: if N is a $\mathbb{Z}^2$-graded E module, the complex $\mathbf{L}(N)$ has terms and differential given by

$$\mathbf{L}(N)_j = \bigoplus_{a \in \mathbb{Z}} S(-a) \otimes_k N_{(a;j)} \quad \text{and} \quad s \otimes n \mapsto \sum_{i=0}^{n} x_i s \otimes e_i n.$$

The complex $\mathbf{L}(N)$ is strongly linear, and in fact, every strongly linear complex of $\mathbb{Z}$-graded S modules is of the form $\mathbf{L}(N)$ for some E-module N [BE22].

4.1.2 *Strongly linear strands.*

Definition 4.3 [BE22]. Let M be a graded S module such that there exists $a \in \mathbb{Z}$ with $M_a \neq 0$ and $M_{<a} = 0$. We set $E^* = \mathrm{Hom}_k(E, k)$, considered as an E module via contraction. The *strongly linear strand* of the minimal free resolution of M is $\mathbf{L}(K)$, where $\mathbf{L}$ is the BGG functor defined above, and

$$K = \ker\left(M_a \otimes_k E^*(-a;0) \xrightarrow{\sum_{i=0}^n x_i \otimes e_i} \bigoplus_{i=0}^{n} M_{a+d_i} \otimes_k E^*(-a-d_i;-1) \right).$$

In the standard graded case, Definition 4.3 recovers the classical notion of the linear strand of a free resolution [Eis05, Corollary 7.11]. When M is generated in a single degree, the strongly linear strand of the minimal free resolution F of M may be alternatively defined as follows. it is the unique maximal strongly linear subcomplex F' of F such that F' is a summand (as an S module, but not necessarily as a complex) of F [BE22].

A main result of [BE22] is a multigraded generalization of Green's Linear Syzygy Theorem [Gre99]. We recall the statement of this theorem in the nonstandard $\mathbb{Z}$-graded case.

Theorem 4.4 [BE22, Theorem 6.2]. *Let M be a finitely generated $\mathbb{Z}$-graded S module and F its minimal free resolution. Suppose that $M_0 \neq 0$, and $M_i = 0$ for $i < 0$. The length of the strongly linear strand of F is at most* $\max\{\dim M_0 - 1, \dim R_0(M)\}$, *where $R_0(M)$ is the variety of rank one linear syzygies of M, i.e.*

$$R_0(M) = \{ w \otimes m \in W \otimes_{\mathbf{k}} M_0 : wm = 0 \text{ in } M \}.$$

The following geometric consequence of Theorem 4.4 plays a crucial role in all of our main results. It extends to weighted projective spaces a result originally proven by Green [Gre84a] over projective space; see also [Eis05, Corollary 7.4].

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Theorem 4.5. *Let Z be a variety, L a line bundle on Z, and W a weighted series associated to L such that the associated map $\varphi_W : Z \to \mathbb{P}(\mathbf{d})$ is a nondegenerate closed embedding. Let V be a vector bundle on Z and M the S-module $\bigoplus_{i\in\mathbb{Z}} H^0(Z, V \otimes L^i)$. Assume that $M_0 \neq 0$, and $M_i = 0$ for $i < 0$. The strongly linear strand of the minimal S-free resolution of M has length at most* $\dim M_0 - 1$.

While Theorem 4.5 follows directly from ideas in our previous paper [BE22] (cf. [BE22, Corollary 1.5]), we include a detailed proof here.

Proof. This follows from essentially the same argument as in [Eis05, Corollary 7.4] (see also the proof of [BE22, Corollary 1.5]). Let $m \in M_0$ and $w \in W$; recall that $W \subseteq S$ is the $\mathbf{k}$-vector subspace of S generated by the variables. Note that $m \otimes w \in R_0(M)$ if and only if $m \otimes w_i \in R_0(M)$ for all homogeneous components w_i of w. Assume that $m \otimes w \in R_0(M)$ and that w is homogeneous; by Theorem 4.4, it suffices to show that this syzygy is trivial, i.e. either $m = 0$ or $w = 0$. Suppose that $m \neq 0$, and let Q be a maximal ideal of S such that the image m_Q of m in the localization M_Q is nonzero. Let I_Z be the defining ideal of Z in $\mathbb{P}(W)$; since Z is integral, I_Z is prime. Let w_Q denote the image of w in $(S/I_Z)_Q$. Note that M_Q is a free R_Q-module, where R is the ring $\bigoplus_{i\in\mathbb{Z}} H^0(Z, L^i)$. Since R is a domain, and the natural map $S/I_Z \to R$ is injective by Proposition 3.5, the relation $w_Q m_Q = 0$ forces $w_Q = 0$, which implies that $w \in P$. By the nondegeneracy of the embedding, P does not contain a homogeneous linear form; we conclude that $w = 0$. □

We need one additional result concerning strongly linear strands.

Lemma 4.6. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of S modules. Assume that M'_a and M_a are nonzero, and $M'_i = M_i = 0$ for $i < a$. Moreover, assume that $M''_a = 0$. There is a natural isomorphism between the strongly linear strands of M' and M.*

Proof. We assume, without loss, that $a = 0$. Let L be the $\mathbb{Z}^2$-graded E-module $\bigoplus_{i=0}^n M_{d_i} \otimes_k E^*(-d_i; -1)$, and define L' and L'' similarly. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_0 \otimes_k E^* & \xrightarrow{\cong} & M_0 \otimes_k E^* & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

of $\mathbb{Z}^2$-graded E modules, where the rows are exact, and the vertical maps are given by multiplication on the left by $\sum_{i=0}^n x_i \otimes e_i$. Let K (respectively K') denote the kernel of the middle (respectively left-most) vertical map. By the Snake lemma, the natural map $K' \to K$ is an isomorphism, and hence, the natural map $\mathbf{L}(K') \to \mathbf{L}(K)$ is as well. □

4.2 Weighted regularity

Benson introduced in [Ben04] an analogue of the Castelnuovo–Mumford regularity for nonstandard $\mathbb{Z}$-graded polynomial rings, which we call 'weighted regularity' to emphasize its connection with weighted projective space.[5]

Definition 4.7. Let M be a finitely generated graded S module. For each $i \geq 0$, set

$$a_i(M) = \sup\{j \in \mathbb{Z} : H^i_{\mathfrak{m}}(M)_j \neq 0\}.$$

[5]This is also a special case of the notion of multigraded regularity defined by Maclagan and Smith [MS04].

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The *weighted regularity of* M is $\sup\{i \geq 0 : a_i(M) + i\}$.

Remark 4.8. By a result of Symonds [Sym11, Proposition 1.2], if M has weighted regularity r, and F is the minimal free resolution of M, then F_j is generated in degree at most $r + j + \sum_{i=0}^{n}(\deg(x_i) - 1)$. Equivalently, the kth row of the Betti table of any such module vanishes for $k > r + \sum_{i=0}^{n}(\deg(x_i) - 1)$.

Example 4.9. Let us revisit the two resolutions from Example 4.2. Recall that $S = k[x_0, x_1]$, where the variables have degrees 1 and 2. Both $S/(x_0^2)$ and $S/(x_1)$ have weighted regularity 0, and their Betti tables are both as follows.

	0	1
0	1	.
1	.	1

In particular, while $S/(x_0^2)$ has weighted regularity 0, its minimal free resolution is not strongly linear. By contrast, any module that is generated in degree 0 and has a strongly linear free resolution is weighted 0-regular (see Remark 4.13(1) and Proposition 4.14 below).

Example 4.10. In Corollary 6.6, we prove that, under Setup 1.3, the weighted regularity of S/I_C is 2 if $g > 0$ and 1 if $g = 0$. For instance, consider the genus 2 curve from Example 2.3 embedded in $\mathbb{P}(1^8, 2^2)$. Its coordinate ring has weighted regularity 2, and so, by Remark 4.8, the Betti table has $2 + \sum_{i=0}^{9}(\deg(x_i) - 1) = 2 = 2 + 2 = 4$ rows.

4.3 Koszul linearity

We fix the following.

Notation 4.11. Let w^i (respectively w_i) be the sum of the i largest (respectively smallest) degrees of the variables: that is, $w^i := \sum_{j=n-i+1}^{n} d_j$ and $w_i := \sum_{j=0}^{i-1} d_j$. □

If $K = K_0 \leftarrow K_1 \leftarrow \cdots$ is the Koszul complex on $x_0, \ldots, x_n$ then w_i is the smallest degree of a generator of K_i, and w^i is the largest such degree.

Definition 4.12. A minimal free complex $[F_0 \overset{\varphi_1}{\leftarrow} F_1 \overset{\varphi_2}{\leftarrow} F_2 \cdots]$ of graded S modules is *Koszul* a*-linear* if each F_i is generated in degrees $< w^{i+1} + a$; by minimal we mean $\varphi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. We sometimes abbreviate Koszul 0-linear to simply 'Koszul linear'.

Remark 4.13. We observe the following.

(1) If M is as in Definition 4.12, and the free resolution of F is Koszul a-linear, then it follows from Remark 4.8 that M is weighted a-regular. The converse is false; see Example 4.16.

(2) The weighted N_p condition from Definition 1.2 is equivalent to normal generation of the weighted series and Koszul 1-linearity of the complex $[F_0 \leftarrow \cdots \leftarrow F_p]$. □

Of course, the Koszul complex on $x_0, \ldots, x_n$ is Koszul 0-linear. More generally, we have the following proposition.

Proposition 4.14. *Let M be a graded S module that is generated in a single degree a. If the minimal free resolution F of M is strongly linear then it is Koszul a-linear.*

Proof. Since F is strongly linear and M is generated in a single degree, F is equal to its strongly linear strand $\mathbf{L}(K)$, where K is as in Definition 4.3. It therefore follows from the definition of K that F is a summand (as an S module, but not as a complex) of a direct sum of copies

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of $\mathbf{L}(E^*(-a;0))$. Finally, observe that $\mathbf{L}(E^*(-a;0))$ is the Koszul complex with the 0th term generated in degree a; the result immediately follows. □

Example 4.15. The converse of Proposition 4.14 is false. Returning Example 4.2, the complex $S \xleftarrow{x_0^2} S(-2)$ is Koszul 0-linear but not strongly linear.

Example 4.16. Let $C = \mathbb{P}^1$, $L = \mathcal{O}_C(5)$, and D the divisor $[0:1]+[1:0]$. The associated log complete seres induces an embedding $\mathbb{P}^1 \subseteq \mathbb{P}(1^4, 2^4)$ given by

$$[s:t] \mapsto [s^4t : s^3t^2 : s^2t^3 : st^4 : s^9t : s^{10} : st^9 : t^{10}].$$

The Betti table is as follows.

	0	1	2	3	4	5	6
total:	1	21	70	105	84	35	6
0 :	1	.	.	.	.	.	.
1 :	.	3	2	.	.	.	.
2 :	.	12	24	12	.	.	.
3 :	.	6	36	54	24	.	.
4 :	.	.	8	36	48	20	.
5 :	.	.	.	3	12	15	6

From this Betti table, one can check that this resolution is Koszul 1-linear. For instance, F_1 has generators of degree $< 5 = w^2 + 1$, F_2 has generators of degree $< 7 = w^3 + 1$, and so on.

The defining ideal I_C is given by the 2×2 minors of the matrix $\begin{pmatrix} x_0 & x_1 & x_2 & x_4 & x_5 & x_3^2 & x_6 \\ x_1 & x_2 & x_3 & x_0^2 & x_4 & x_6 & x_7 \end{pmatrix}$. It follows that the minimal free resolution of S/I_C is the Eagon–Northcott complex of this matrix. Since this matrix includes the entries x_3^2 and x_0^2, this minimal free resolution is not strongly linear. Thus, even in the case of a rational curve, strong linearity is too restrictive to capture the linearity of the free resolution of the coordinate ring.

Finally, let us analyze the example from the perspective of weighted regularity. By Remark 4.13(1), S/I_C is 1-regular; by Remark 4.8, this states precisely that the kth row of the Betti table vanishes for $k > 5$. Thus, for instance, the weighted regularity computation would imply that F_1 is generated in degree at most 6. We therefore see that weighted regularity is too weak to fully describe the situation.

5. Proof of Theorem 1.7

We begin by establishing several technical results. The first is a simple calculation.

Lemma 5.1. *Let S be as in Theorem 1.7 and M be a finitely generated S module. Assume that $M_0 \neq 0$ but $M_i = 0$ for $i < 0$. The following assertions hold.*

(1) *If the Betti number $\beta_{i,j}(M)$ is nonzero then $j \geq w_i$ (see Notation 4.11).*

(2) *Suppose that there is a variable $x_\ell \in S$ that is a nonzero divisor on M. Define*

$$w'_i = \begin{cases} w_i, & i < \ell, \\ w_{i+1} - \deg(x_\ell), & i \geq \ell. \end{cases}$$

If $\beta_{i,j}(M) \neq 0$ then $j \geq w'_i$.

Proof. If K is the Koszul complex on the variables $x_0, \ldots, x_n$ then the minimal degree of an element of $\operatorname{Tor}_i(M,k) = H_i(M \otimes_S K)$ is w_i. This proves (1). For (2), let F denote the minimal

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S-free resolution of M. Since x_ℓ is a nonzero divisor on M, $F/x_\ell F$ is the minimal $S/(x_\ell)$-free resolution of $M/x_\ell M$. Now apply (1) to the $S/(x_\ell)$-module $M/x_\ell M$. □

The following lemma is an analogue of a well-known result in the standard graded case and is proven in the same way as its classical counterpart.

Lemma 5.2. *Let C, L, R, S, W, and $f\colon C \to \mathbb{P}(W)$ be as in Theorem 1.7. The following hold.*

(1) *The graded S-module R has depth 2. In particular, R is a Cohen–Macaulay S module and a maximal Cohen–Macaulay S/I_C module.*

(2) *Let $\omega_R = \bigoplus_{i\in\mathbb{Z}} H^0(C, \omega_C \otimes L^i)$, and denote by $|\mathbf{d}|$ the sum of the degrees of the variables in S. We have $Ext_S^{n-1}(R, S(-|\mathbf{d}|)) \cong \omega_R$.*

Proof. We observe that the canonical map $R \to \bigoplus_{i\in\mathbb{Z}} H^0(\mathbb{P}(W), \widetilde{R(i)})$ is an isomorphism, i.e. R is $\mathfrak{m}$ saturated. By [Eis95, Theorem A4.1], we have an exact sequence

$$0 \to H^0_{\mathfrak{m}}(R) \to R \xrightarrow{\cong} \bigoplus_{i\in\mathbb{Z}} H^0(\mathbb{P}(W), \widetilde{R(i)}) \to H^1_{\mathfrak{m}}(R) \to 0,$$

and isomorphisms

$$H^{j+1}_{\mathfrak{m}}(R) \cong \bigoplus_{i\in\mathbb{Z}} H^j(\mathbb{P}(W), \widetilde{R(i)}) = \bigoplus_{i\in\mathbb{Z}} H^j(C, L^i), \tag{5.3}$$

for $j > 0$. In particular, we have $H^i_{\mathfrak{m}}(R) = 0$ for $i = 0, 1$; that is, R has depth 2. Part (1) now follows from the observation that $\dim S/I_C = 2$. As for (2), given a $\mathbb{Z}$-graded k-vector space V, let V^* denote its graded dual. We have $Ext_S^{n-1}(R, S(-|\mathbf{d}|)) \cong H^2_{\mathfrak{m}}(R)^* \cong \bigoplus_{i\in\mathbb{Z}} H^1(C, L^i)^* \cong \omega_R$, where the first isomorphism follows from local duality, the second from (5.3), and the third from Serre duality. □

Next, we need the following strengthening of Theorem 4.5.

Lemma 5.4. *Suppose we are in the setting of Theorem 4.5, and assume that* $\dim W_1 > \dim M_0$. *Let F be the minimal S-free resolution of M. Any summand of F_i generated in degree j for some $j < w_{i+1}$ (see Notation 4.11) lies in the strongly linear strand of F. In particular, if $\beta_{i,j}(M) \neq 0$ for some $j < w_{i+1}$ then $i \le \dim M_0 - 1$.*

In the standard graded case, the first statement in Lemma 5.3 is tautological: it says that, if a summand of F_i is generated in degree i then it is in the linear strand. However, in the weighted setting, the strongly linear strand cannot be interpreted in terms of Betti numbers (see, for instance, Example 4.2), and so Lemma 5.3 is not at all obvious in general; indeed, our proof is a bit delicate.

Proof of Lemma 5.4. The second statement follows immediately from the first, by Theorem 4.5. Let K be the Koszul complex on the variables of S. We consider classes in $\operatorname{Tor}^S_*(k, M)$ as homology classes in $K \otimes_S M \cong \bigwedge W \otimes_k M$, and we fix once and for all an embedding $\operatorname{Tor}^S_*(k, M) \hookrightarrow Z(\bigwedge W \otimes_k M)$ of $\mathbb{Z}^2$-graded k-vector spaces, where the target denotes the cycles in $\bigwedge W \otimes_k M$. In this proof, we identify classes in $\operatorname{Tor}^S_*(k, M)$ with cycles in $\bigwedge W \otimes_k M$ via this embedding. We may decompose any element $\sigma \in \bigwedge W \otimes_k M$ as $\sum_{i\ge 0} \sigma_i$, where $\sigma_i \in \bigwedge W \otimes_k M_i$. Let $W_{>1} = \bigoplus_{i>1} W_i$, so that $\bigwedge W = \bigwedge W_1 \otimes_k \bigwedge W_{>1}$. We may write any $\sigma \in \bigwedge W \otimes_k M$ as $\sum \alpha \otimes \beta \otimes m_{\alpha,\beta}$, where the sum ranges over all pairs (α, β) such that α is an exterior product of basis elements of W_1, and β is an exterior product of basis elements of $W_{>1}$; here, each $m_{\alpha,\beta}$ is an element of M. We call each nonzero $\alpha \otimes \beta \otimes m_{\alpha,\beta}$ in this sum a *term* of σ. It is possible

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that α (respectively β) is an empty product of basis elements, in which case α (respectively β) is $1\in\bigwedge^0 W_1$ (respectively $1\in\bigwedge^0 W_{>1}$). Given a nonzero element $\sigma\in\bigwedge W\otimes_k M$, we define

$$\nu(\sigma)=\max\left\{m : \text{a term of } \sigma \text{ lies in } \bigwedge^{\dim W_1-m} W_1\otimes_k\bigwedge W_{>1}\otimes_k M\right\}.$$

The function ν measures the maximal number of degree 1 elements that do not appear in one of the exterior forms α. For instance, if $\nu(\sigma)=0$ then, for every term $\alpha\otimes\beta\otimes m_{\alpha,\beta}$ of σ, α is the product of all of the degree 1 variables. Let us now prove the following.

Claim. *If σ is a nonzero class in $\operatorname{Tor}^S_*(k,M)$ then $\nu(\sigma)\neq 0$.*

Proof of Claim. Indeed, let x_i be a degree 1 variable, $\overline{W}$ the quotient of W by the span of x_i, and $\overline{M}$ the corresponding module $M/(x_i)$ over $\overline{S}=S/(x_i)$. Since x_i is a regular element on M, the surjection $\bigwedge W\otimes_k M\twoheadrightarrow\bigwedge\overline{W}\otimes_k\overline{M}$ induces an isomorphism $\theta:\operatorname{Tor}^S_*(k,M)\xrightarrow{\cong}\operatorname{Tor}^{\overline{S}}_*(k,\overline{M})$ on homology. Since $\theta(\sigma)\neq 0$, $\nu(\sigma)$ must be nonzero; this proves the claim. □

Now, let σ be a nonzero class in $\operatorname{Tor}^S_i(k,M)_j$, where $j<w_{i+1}$. It suffices to show that $\sigma=\sigma_0$; this implies that σ lies in the strongly linear strand. Assume, toward a contradiction, that $\sigma_\ell\neq 0$ for some $\ell>0$. Since $\sigma_\ell\in\bigwedge^i W\otimes M_\ell$, we have $w_i+\ell\le j<w_{i+1}$. Recalling that $w_{i+1}-w_i=d_{i+1}:=\deg(x_{i+1})$, this implies that $d_{i+1}>\ell\ge 1$. We conclude that

$$i\ge\dim W_1. \tag{5.5}$$

There are two cases to consider.

Case 1: $\nu(\sigma_\ell)>0$ *for some* $\ell>0$. In this case, σ_ℓ has some term $\alpha\otimes\beta\otimes m_{\alpha,\beta}$ such that α is not divisible by a degree 1 variable; without loss of generality, let us say α is not divisible by x_0. It follows that $\deg(\alpha\otimes\beta)\ge\deg(x_1x_2\cdots x_i)=w_{i+1}-1$. Thus,

$$\deg(\sigma_\ell)=\deg(\alpha\otimes\beta)+\ell\ge w_{i+1}-1+\ell\ge w_{i+1}.$$

This is impossible, since $\deg(\sigma_\ell)=\deg(\sigma)<w_{i+1}$.

Case 2: $\nu(\sigma_\ell)=0$ *for all* $\ell>0$. For every term $\alpha\otimes\beta\otimes m_{\alpha,\beta}$ of σ_ℓ for $\ell>0$, we have $\beta\in\bigwedge^{i-\dim W_1}W_{>1}$. On the other hand, it follows from the claim above that there must be some term $\alpha'\otimes\beta'\otimes m_{\alpha',\beta'}$ of σ_0 such that $\beta'\in\bigwedge^{i-\dim W_1+t}W_{>1}$ for some $t>0$; recall that, by (5.5), $i-\dim W_1\ge 0$. Let $E=\bigwedge W^*$, and note that $\bigwedge W\otimes_k M$ is an E module via the contraction action of E on $\bigwedge W$. We may choose $f\in\bigwedge^{i-\dim W_1+1}W^*_{>1}\subseteq E$ such that $f\sigma_0\neq 0$; note, however, that $f\sigma_\ell=0$ for all $\ell>0$. Thus, $f\sigma=f\sigma_0=(f\sigma)_0\in\bigwedge W\otimes M_0$. Moreover, since $\sigma\in\bigwedge W\otimes_k M$ is a cycle, $f\sigma$ is also a cycle, as the Koszul differential on $\bigwedge W\otimes_k M$ is E-linear. Thus, since $f\sigma=(f\sigma)_0$, it follows from the definition of the strongly linear strand (Definition 4.3) that $f\sigma$ determines a summand of the strongly linear strand of F. But $f\sigma$ has homological degree $i-(i-\dim W_1+1)=\dim W_1-1$, and so $\dim W_1-1\le\dim M_0-1$, by Theorem 4.5. This contradicts our assumption that $\dim W_1>\dim M_0$. □

In the standard graded case, the proof of Green's Theorem (Theorem 1.1) via the linear syzygy theorem (cf. [Eis05, Theorem 8.8.1]) makes use of numerous statements about linear strands that rely on degree arguments. These break down in the nonstandard graded situation, and Lemmas 5.1–5.3 act to fill that gap. Thus, with these lemmas in hand, we can now turn to the proof of Theorem 1.7.

Proof of Theorem 1.7. Recall that $d_0,\dots,d_n$ are the degrees of the variables $x_0,\dots,x_n$ in S, and we assume that $d_0\le d_1\le\cdots\le d_n$. As in Lemma 5.2(2), we let $|\mathbf{d}|=\sum_{i=0}^n d_i$ and

$\omega_R = \bigoplus_{i\in\mathbb{Z}} H^0(C, \omega_C \otimes L^i)$. We remark, for later use, that $\dim(\omega_R)_0 = H^0(C,\omega_C) = g$. By Lemma 5.2(2), we have $Ext_S^{n-1}(R, S(-|\mathbf{d}|)) \cong \omega_R$. Letting F be the minimal S-free resolution of R and $F^\vee = \operatorname{Hom}_S(F,S)$, it follows that $F^\vee(-|\mathbf{d}|)[-n+1]$ is the minimal free resolution of ω_R. In particular, we have $\beta_{i,j}(R) = \beta_{n-1-i,|\mathbf{d}|-j}(\omega_R)$. Now, suppose that $\beta_{i,j}(R) = \beta_{n-1-i,|\mathbf{d}|-j}(\omega_R) \neq 0$, and assume that $j > w^{i+1}$. We now compute

$$|\mathbf{d}| - w_{n-i} = \sum_{j=0}^{n} d_j - \sum_{j=0}^{n-1-i} d_j = \sum_{j=n-i}^{n} d_j = w^{i+1} < j.$$

Rearranging this inequality, we have $|\mathbf{d}| - j < w_{n-i}$. There are now two cases to consider.

Case 1: $g = 0$. In this case, $(\omega_R)_1 \neq 0$, and $(\omega_R)_i = 0$ for $i < 1$. Every variable $x_i \in S$ is a nonzero divisor on ω_R. In particular, x_0 has this property; recall that $\deg(x_0) = 1$. Applying Lemma 5.1(2), with $\ell = 0$, we arrive at the inequality $|\mathbf{d}| - j \geq w_{n-i}$, a contradiction. We therefore conclude that if $\beta_{i,j}(R) \neq 0$ then $j \leq w^{i+1}$.

Case 2: $g > 0$. We now have $(\omega_R)_0 \neq 0$, and $(\omega_R)_i = 0$ for $i < 0$. Applying Lemma 5.3 to ω_R implies that $n - 1 - i < \dim(\omega_R)_0 = g$, i.e. $i > n - 1 - g = \dim W - g - 2$. $\square$

Let us illustrate the proofs of both Theorem 1.7 and Lemma 5.3 via an example.

Example 5.6. Suppose we are in the setting of Theorem 1.7, and assume that $g = 2$ and $\mathbb{P}(W) = \mathbb{P}(1^6, 2^4)$. Let ω_R be as in Lemma 5.2(2). To prove Theorem 1.7 in this example, we must show that the columns of the Betti table of ω_R are bounded above by the dots in the diagram below.[6]

	0	1	2	3	4	5	6	7	8
0 :	•	•	.	.	.	.	.	.	.
1 :	.	.	•	•	•	•	†	.	.
2 :	.	.	.	.	.	.	•	.	.
3 :	.	.	.	.	.	.	.	•	.
4 :	.	.	.	.	.	.	.	.	.
5 :	.	.	.	.	.	.	.	.	.
6 :	.	.	.	.	.	.	.	.	•

For degree reasons alone, entries in the 0th row must lie in the strongly linear strand of the minimal free resolution of ω_R, and the length of that strand is $\leq g - 1 = 1$ by [BE22, Corollary 1.4]. So the first entry that could potentially pose an issue is the one in the position marked by a †, as we cannot conclude, for purely degree reasons, that such an entry lies in the strongly linear strand. Let us use the argument in the proof of Lemma 5.3 to show this entry must be 0.

We adopt the notation of the proof of Lemma 5.3. Say we have a cycle $\sigma \in \bigwedge^6 W \otimes \omega_R$ corresponding to a nonzero syzygy in position †. For degree reasons, we have $\sigma_i = 0$ for $i \neq 0, 1$; and $\nu(\sigma_1) = 0$. In particular, we have $\sigma_1 = x_0 x_1 \cdots x_5 \otimes y$ for some $y \in (\omega_R)_1$. It follows that, for every $f \in W_2^*$, we have $f\sigma_1 = 0$. The claim in the proof of Lemma 5.3 implies that $\nu(\sigma) \neq 0$, and thus, σ_0 must be nonzero and satisfy $\nu(\sigma_0) > 0$. In particular, every term of σ_0 must involve at least one variable from $W_{>1}$. We can thus choose an element $f \in W_{>1}^*$ such that $f\sigma_0 \neq 0$. We therefore have $f\sigma = f\sigma_0 + f\sigma_1 = f\sigma_0 \neq 0$, which means $f\sigma$ corresponds to a summand of the strongly linear strand that lies in the position of the entry marked $\star$ below.

[6] We are using here the fact that the kth row in the Betti table of R must vanish for $k > 6$. One sees this by combining Remark 4.8 with the fact that the weighted regularity of R is 2, which we prove in Corollary 6.6.

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	0	1	2	3	4	5	6	7	8
0 :	●	●	.	.	.	⋆	.	.	.
1 :	.	.	●	●	●	●	†	.	.
2 :	.	.	.	.	.	.	●	.	.
3 :	.	.	.	.	.	.	.	●	.
4 :	.	.	.	.	.	.	.	.	.
5 :	.	.	.	.	.	.	.	.	.
6 :	.	.	.	.	.	.	.	.	●

This is impossible, because the strongly linear strand has length at most $g-1=1$.

6. Normal generation and the weighted N_p results

We use the notation/assumptions in Setup 1.3 throughout this entire section. Recall that $\varphi_W\colon C \to \mathbb{P}(\mathbf{d})$ is a closed embedding, by Proposition 3.12(2). As above, we denote by R the section ring $\bigoplus_{i\in\mathbb{Z}} H^0(C, L^i)$, and we write $H^0(C, L^a)_{bD}$ for the space of sections of L^a that vanish along the divisor bD. We begin with several technical results.

Lemma 6.1. *We have $(S/I_C)_{\ell d} \cong R_{\ell d}$ for all $\ell \geq 0$.*

Proof. By Proposition 3.5, we need only show that the ring map $\alpha\colon S \to \bigoplus_i H^0(C, L^i)$ given by $x_i \mapsto s_i$ induces surjections $\alpha_{\ell d}\colon S_{\ell d} \twoheadrightarrow H^0(C, L^{\ell d})$ for all $\ell \geq 0$. By Proposition 3.12(1), α_d is surjective. Let $V = H^0(C, L^d)$, $f_0, \dots, f_r$ a basis of V, and $F_0, \dots, F_r \in S_d$ elements such that $\alpha_d(F_i) = f_i$. Let $\ell \geq 0$. By our assumption on $\deg(L \otimes \mathcal{O}(-D))$, the embedding $C \hookrightarrow \mathbb{P}(V)$ determined by $|L^d|$ is normally generated, and so the induced map $h\colon \mathrm{Sym}^\ell(V) \to H^0(C, L^{\ell d})$ is surjective. Let $s \in H^0(C, L^{\ell d})$, and choose $p \in \mathrm{Sym}^\ell(V)$ such that $h(p) = s$, i.e. $p(f_0, \dots, f_r) = s$. We have $\alpha_{\ell d}(p(F_0, \dots, F_r)) = s$. ☐

Lemma 6.2. *Let $e \geq 0$, and write $e = qd + e'$ for $0 \leq e' < d$. The following assertions hold.*

1. *The natural map $H^0(C, L^{qd}) \otimes H^0(C, L^{e'} \otimes \mathcal{O}(-e'D)) \to H^0(C, L^e \otimes \mathcal{O}(-e'D))$ is surjective.*
2. *The image of the injection $(S/I_C)_e \hookrightarrow H^0(C, L^e)$ is given by the sections that vanish with multiplicity $\geq e'$ along D.*

Proof. Part (1) is immediate from [Gre84a, Corollary 4.e.4]. As for (2): let ι denote the injection $(S/I_C)_e \hookrightarrow H^0(C, L^e)$. Because C is embedded by a log complete series of type (D, d), the variables of $S = k[x_0, \dots, x_n]$ have degrees 1 and d. Say $x_0, \dots, x_r$ are the variables of degree 1. Every element of S_e, and hence $(S/I_C)_e$, lies in $(x_0, \dots, x_r)^{e'}$. It follows that every section in the image of g vanishes with multiplicity $\geq e'$ along D; that is, $\mathrm{im}(\iota) \subseteq H^0(C, L^e)_{e'D}$. By Lemma 3.11, there is a natural isomorphism $H^0(C, L^e)_{e'D} \cong H^0(C, L^e \otimes \mathcal{O}(-e'D))$. Since $\deg(L \otimes \mathcal{O}(-D)) \geq 2g+1$, the complete linear series on $L \otimes \mathcal{O}(-D)$ induces a normally generated embedding into projective space, i.e. the natural map $H^0(C, L \otimes \mathcal{O}(-D))^{\otimes a} \to H^0(C, L^a \otimes \mathcal{O}(-aD))$ is surjective for all $a \geq 0$.

We first consider the case where $e < d$, so that $e = e'$ and $q = 0$. We have isomorphisms $(S/I_C)_1 = H^0(C, L)_D \cong H^0(C, L \otimes \mathcal{O}(-D))$ and $H^0(C, L^e \otimes \mathcal{O}(-eD)) \cong H^0(C, L^e)_{eD}$. Combining these identifications with the surjection $H^0(C, L \otimes \mathcal{O}(-D))^{\otimes e} \twoheadrightarrow H^0(C, L^e \otimes \mathcal{O}(-eD))$ yields a surjection $\pi\colon (S/I_C)_1^{\otimes e} \twoheadrightarrow H^0(C, L^e)_{eD}$. We have a commutative diagram

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$$\begin{array}{ccccc} (S/I_C)_1^{\otimes e} & \longrightarrow & (S/I_C)_e & \xrightarrow{\iota} & H^0(C, L^e) \\ & \searrow^{\pi} & & & \uparrow \\ & & & & H^0(C, L^e)_{eD} \end{array}$$

where the vertical map is the inclusion, and the left-most horizontal map is given by multiplication. This proves (2) when $e < d$. Finally, suppose that $e \geq d$. By Lemma 6.5, we have $(S/I_C)_{\ell d} \cong R_{\ell d} = H^0(C, L^{\ell d})$ for all $\ell \geq 0$, and we have shown above that $(S/I_C)_{e'} \cong H^0(C, L^{e'} \otimes \mathcal{O}(-e'D))$. Part (1) yields a surjection

$$H^0(C, L^{qd}) \otimes H^0(C, L^{e'} \otimes \mathcal{O}(-e'D)) \twoheadrightarrow H^0(C, L^e \otimes \mathcal{O}(-e'D)) \cong H^0(C, L^e)_{e'D}.$$

Combining these observations, we see that there is a surjection $\pi : (S/I_C)_{qd} \otimes (S/I_C)_{e'} \twoheadrightarrow H^0(C, L^e)_{e'D}$ such that the diagram

$$\begin{array}{ccccc} (S/I_C)_{qd} \otimes (S/I_C)_{e'} & \longrightarrow & (S/I_C)_e & \xrightarrow{\iota} & H^0(C, L^e) \\ & \searrow^{\pi} & & & \uparrow \\ & & & & H^0(C, L^e)_{e'D} \end{array}$$

commutes, where the vertical map is the inclusion, and the left-most horizontal map is multiplication. The result follows. $\square$

PROPOSITION 6.3. *Let Q denote the cokernel of the injection $S/I_C \hookrightarrow R$. The following hold.*

(1) *We have $Q_{qd} = 0$ for all $q \geq 0$. In particular, if $0 \leq j < d$ then any element of S_{d-j} annihilates any element of Q_{qd+j}.*
(2) *For all $e \geq 0$, we have $\dim Q_e = \dim Q_{e+d}$.*
(3) *The support of the sheaf $\widetilde{Q}$ is the set of points in D. In particular, $\widetilde{Q}$ is a zero-dimensional sheaf on $\mathbb{P}(W)$, and Q is a one-dimensional S module.*
(4) *We have $H^j_{\mathfrak{m}} Q = 0$ for $j \neq 1$, and $(H^1_{\mathfrak{m}} Q)_e = 0$ for $e \geq 0$. In particular, Q is a Cohen–Macaulay S module, and its weighted regularity (Definition 4.7) is at most 0.*

Before beginning the proof, we discuss a simple example.

Example 6.4. Consider Example 2.1, where $S/I_C \cong k[s^2, st, st^3, t^4]$ and $R \cong k[s^2, st, t^2]$, so that $Q = t^2 \cdot k[t^4]$. In other words, letting $M = S/(x_0, x_1, x_2)$, we have $Q \cong M(-1)$. Observe that Q is concentrated in positive odd degrees, and each of its nonzero homogeneous components is a one-dimensional k-vector space. Its support is the point $V(x_0, x_1, x_2)$ in $\mathbb{P}(W)$, which is the point in D. Clearly, $H^0_{\mathfrak{m}} Q = 0$ because x_3 is a nonzero divisor on Q. A local duality argument implies that $H^1_{\mathfrak{m}} Q = t^{-2} \cdot k[t^{-4}]$, which is zero in nonnegative degrees.

Proof of Proposition 6.3. Part (1) is clear from Lemma 6.5, and part (3) is immediate from Lemma 6.2(2). For part (2), we write $e = qd + e'$ with $0 \leq e' < d$ and $q \geq 0$. When $e = 0$, this is clear from part (1). Assume $e > 0$. We have

$$\dim Q_e = \dim H^0(C, L^e) - \dim H^0(C, L^e \otimes \mathcal{O}(-e'D)) = e' \cdot \deg D,$$

where the first equality follows from Lemmas 3.11 and 6.2(2), and the second follows from the Riemann–Roch Theorem. In particular, we see that $\dim Q_e$ only depends on the remainder of e modulo d; this proves part (2). Finally, we consider part (4). The inclusion $D \subseteq C$ yields a short exact sequence

$$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \mathcal{O}_D \to 0.$$

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Twisting by L^d, and noting that $\mathcal{O}_D \otimes L^d = \mathcal{O}_D$ because D is zero dimensional, we obtain a short exact sequence

$$0 \to L^d \otimes \mathcal{O}_C(-D) \to L^d \to \mathcal{O}_D \to 0.$$

Noting that $H^1(C, L^d \otimes \mathcal{O}(-D)) = 0$ since $\deg(L^d \otimes \mathcal{O}(-D)) \geq \deg(L \otimes \mathcal{O}(-D)) \geq 2g+1$, this short exact sequence induces a surjection

$$(S/I_C)_d \cong H^0(C, L^d) \twoheadrightarrow H^0(D, \mathcal{O}_D). \tag{6.5}$$

Since D is a finite collection of points, it is an affine scheme, and so $H^0(D, \mathcal{O}_D)$ contains a unit. Choose a degree d element $u \in S_d$ such that the surjection

$$S_d \twoheadrightarrow (S/I_C)_d \xrightarrow{(6.5)} H^0(D, \mathcal{O}_D)$$

sends u to a unit. This implies that the map $Q \to Q(d)$ given by multiplication by u does not alter the multiplicity of vanishing along D and, thus, induces an isomorphism $Q_e \to Q_{e+d}$ for all $e \geq 0$. In particular, any nonzero element of Q_e with $0 < e < d$ cannot be annihilated by the entire maximal ideal $\mathfrak{m}$, and so $H^0_{\mathfrak{m}} Q = 0$. Since $\dim Q = 1$, we also have $H^i_{\mathfrak{m}} Q = 0$ for $i > 1$. It remains to consider $H^1_{\mathfrak{m}} Q$. Using the fact that $H^0_{\mathfrak{m}} Q = 0$, [Eis95, Theorem A4.1] yields a short exact sequence

$$0 \to Q_e \to H^0(\mathbb{P}(W), \widetilde{Q(e)}) \to (H^1_{\mathfrak{m}} Q)_e \to 0.$$

We know $\dim Q_e = \dim Q_{e+d}$ for all $e \geq 0$. In fact, since the map $Q(e) \xrightarrow{u} Q(e+d)$ is injective and has a finite-dimensional cokernel for all $e \in \mathbb{Z}$, we have $\widetilde{Q(e)} \cong \widetilde{Q(e+d)}$ for all $e \in \mathbb{Z}$. It follows that $(H^1_{\mathfrak{m}} Q)_e \cong (H^1_{\mathfrak{m}} Q)_{e+d}$ for all $e \geq 0$. However, $(H^1_{\mathfrak{m}} Q)_e = 0$ for $e \gg 0$, and so we must have $(H^1_{\mathfrak{m}} Q)_e = 0$ for all $e \geq 0$. □

Proof of Theorem 1.4. From the short exact sequence $0 \to S/I_C \to R \to Q \to 0$, we get a long exact sequence in local cohomology. Since $H^0_{\mathfrak{m}} Q = 0$ by Proposition 6.3, and $H^1_{\mathfrak{m}} R = 0$ by Lemma 5.2(1), we conclude that $H^1_{\mathfrak{m}}(S/I_C) = 0$. Thus, S/I_C is normally generated, and it follows from Remark 3.14(1) that S/I_C is a Cohen–Macaulay ring. □

Corollary 6.6. *The weighted regularity of S/I_C and R is 2 if $g > 0$ and 1 if $g = 0$.*

Proof. By Theorem 1.4, S/I_C is a Cohen–Macaulay ring, and so $H^0_{\mathfrak{m}}(S/I_C) = H^1_{\mathfrak{m}}(S/I_C) = 0$. Since R is a Cohen–Macaulay S module by Lemma 5.2(1), and Q is a one-dimensional S module by Proposition 6.3(3), the short exact sequence $0 \to S/I \to R \to Q \to 0$ yields the short exact sequence $0 \to H^1_{\mathfrak{m}} Q \to H^2_{\mathfrak{m}}(S/I_C) \to H^2_{\mathfrak{m}} R \to 0$. Proposition 6.3(4) implies that $(H^1_{\mathfrak{m}} Q)_e = 0$ for $e \geq 0$, and (5.3) implies that $(H^2_{\mathfrak{m}} R)_e \cong H^1(C, L^e)$. We have $H^1(C, L^e) = 0$ if and only if $e > 0$ (respectively $e \geq 0$) when $g > 0$ (respectively $g = 0$). The statement immediately follows. □

Proof of Theorem 1.5. Normal generation follows from Theorem 1.4. Let us record the following computation:

$$\begin{aligned} \dim W &= \dim W_1 + \dim W_d \\ &= \dim H^0(C, L \otimes \mathcal{O}(-D)) + \dim H^0(C, L^d) - \dim H^0(C, L^d \otimes \mathcal{O}(-dD)) \\ &= \deg(L \otimes \mathcal{O}(-D)) - g + 1 + d \deg(D) \\ &\geq g + 2 + q + d \deg(D). \end{aligned}$$

Here the first two equalities follows from the definition of a log complete series along with Lemma 3.11, the third from the Riemann–Roch Theorem, and the inequality by hypothesis.

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Also,

$$\dim W_1 = \dim H^0(C, L \otimes \mathcal{O}(-D)) = \deg(L \otimes \mathcal{O}(-D)) - g + 1 \geq g + 2 + q > g,$$

and so the assumption $\dim S_1 > g$ in Theorem 1.7 holds here.

Let Q be as in Proposition 6.3. By Theorem 1.4, Lemma 5.2(1), and Proposition 6.3(4), we have a short exact sequence $0 \to S/I_C \to R \to Q \to 0$ of Cohen–Macaulay S modules of dimensions 2, 2, and 1, respectively. Recall that $S = k[x_0, \dots, x_n]$ and $|\mathbf{d}| = \sum_{i=0}^n \deg x_i$. Write $\omega_R := Ext_S^{n-1}(R, S(-|\mathbf{d}|))$, $\omega_{S/I_C} := Ext_S^{n-1}(S/I_C, S(-|\mathbf{d}|))$, and $\omega_Q := Ext_S^n(Q, S(-|\mathbf{d}|))$ for the Matlis duals of these modules. We have a short exact sequence

$$0 \to \omega_R \to \omega_{S/I_C} \to \omega_Q \to 0.$$

Just as in our proof of Theorem 1.7, we must consider the $g = 0$ and $g > 0$ cases separately.

Case 1: $g = 0$. While we argue as in the proof of Theorem 1.7, we recapitulate the details for completeness. Since S/I_C is a Cohen–Macaulay ring of dimension 2, we have $\beta_{i,j}(S/I_C) = \beta_{n-1-i,|\mathbf{d}|-j}(\omega_{S/I_C})$ for all i, j. Now, suppose that $\beta_{i,j}(S/I_C) = \beta_{n-1-i,|\mathbf{d}|-j}(\omega_{S/I_C}) \neq 0$, and assume that $j > w^{i+1}$. We have

$$|\mathbf{d}| - w_{n-i} = \sum_{j=0}^{n} d_j - \sum_{j=0}^{n-1-i} d_j = \sum_{j=n-i}^{n} d_j = w^{i+1} < j.$$

Rearranging, we get $|\mathbf{d}| - j < w_{n-i}$. Corollary 6.6 (along with local duality) implies that $(\omega_{S/I_C})_1 \neq 0$ and $(\omega_{S/I_C})_{<1} = 0$. Since I_C is prime, every variable $x_i \in S$ is a nonzero divisor on ω_{S/I_C}. In particular, x_0 has this property. Applying Lemma 5.1(2), with $\ell = 0$, we get $|\mathbf{d}| - j \geq w_{n-i}$, a contradiction. Thus, if $\beta_{i,j}(S/I_C) \neq 0$ then $j \leq w^{i+1}$. It follows that the embedding $C \subseteq \mathbb{P}(W)$ satisfies the weighted N_p condition for all $p \geq 0$.

Case 2: $g > 0$. We first prove that

$$\operatorname{Tor}_i^S(\omega_R, k)_j = \operatorname{Tor}_i^S(\omega_{S/I_C}, k)_j \tag{6.7}$$

for all $j < w_{i+1}$. Proposition 6.3(4) (along with local duality) implies that $(\omega_Q)_i = 0$ for $i \leq 0$, while Corollary 6.6 (along with local duality) implies that $(\omega_{S/I_C})_0 \neq 0$ and $(\omega_{S/I_C})_{<0} = 0$, and similarly for ω_R. Lemma 4.6 therefore implies that the strongly linear strands of the minimal free resolutions of ω_R and ω_{S/I_C} are isomorphic. The identification (6.7) now follows by applying Lemma 5.3 to both ω_R and ω_{S/I_C}. (Note that $\dim(\omega_R)_0 = \dim(\omega_{S/I_C})_0 = g$, and so, since $\dim W_1 > g$, the assumption '$\dim W_1 > M_0$' in Lemma 5.3 holds for both $M = \omega_R$ and $M = \omega_{S/I_C}$.) Finally, as in the proof of Theorem 1.7 (and Case 1), we have $\beta_{i,j}(R) = \beta_{n-1-i,|\mathbf{d}|-j}(\omega_R)$, and similarly for S/I_C. The equality (6.7) implies that $\operatorname{Tor}_i(R, k)_j = \operatorname{Tor}_i(S/I_C, k)_j$ whenever $|\mathbf{d}| - j < w_{n-i}$, i.e. $j > |\mathbf{d}| - w_{n-i} = w^{i+1}$. Applying Theorem 1.7, we therefore conclude that if $i \leq \dim W - g - 2$ and $\beta_{i,j}(S/I_C) \neq 0$, then $j \leq w^{i+1}$. Since $\dim W - g - 2 \geq q + d\deg(D)$, it follows that the embedding $C \subseteq \mathbb{P}(W)$ satisfies the weighted $N_{q+d\deg(D)}$ property. □

Proof of Corollary 1.6. Immediate from Theorem 1.5. □

7. Questions

7.1 Higher-dimensional varieties

Mumford famously showed that any high degree Veronese of a projective variety is 'cut out by quadrics' [Mum70]; see also the generalization in [SS11]. Corollary 1.6 is an analogue of Mumford's result for curves in weighted projective spaces; it is natural to ask if this result can be extended to other varieties in weighted projective spaces.

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Question 7.1. Can one prove results like Corollary 1.6 for higher-dimensional varieties embedded in weighted projective spaces? □

We can also ask about normal generation and the N_p conditions for higher-dimensional varieties. Here, the central results are those of [EL93], which prove N_p results for embeddings by line bundles of the form $K_X + L^d + B$, where K_X is the canonical bundle, L is very ample, and B is effective.

Question 7.2. Can one obtain N_p conditions for higher-dimensional varieties embedded by a log complete series, under hypotheses similar to those in [EL93]? □

Embeddings into weighted spaces also provide an intermediate case for investigating asymptotic syzygy-type questions, as in [EL12].

Question 7.3. With notation as in Question 7.1, can one prove asymptotic nonvanishing results, similar to what happens in the main results of [EL12]? At the other extreme, can one prove asymptotic vanishing results as in [Par21]? □

7.2 Scrolls and the gonality conjecture

There is a rather trivial sense in which curves embedded via log complete series of high degree satisfy an analogue of Green–Lazarsfeld's gonality conjecture. Recall that a high degree curve in $\mathbb{P}^r$ has regularity 2, and so the Betti table looks as follows.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{cccccccccc} 0 & 1 & 2 & \cdots & a & a+1 & \cdots & b & b+1 & \cdots \end{array} \atop \begin{pmatrix} * & - & - & \cdots & - & - & \cdots & - & - & \cdots \\ - & * & * & \cdots & * & * & \cdots & * & - & \cdots \\ - & - & - & \cdots & - & * & \cdots & * & * & \cdots \end{pmatrix}$$

The N_p conditions are about the moment we first get nonzero entries in row 2, i.e. column $a+1$ in the picture. In [GL88], Green–Lazarsfeld conjectured that the moment where we first get a zero entry in row 1, i.e. column $b+1$ in the picture, is determined by the gonality $\operatorname{gon}(C)$ of the curve. This is the Green–Lazarsfeld gonality conjecture, and it was proven in [EL15], utilizing techniques originally developed by Voisin [Voi02].

In the standard graded setting, b is the maximal index such that F_i has a minimal generator of degree $i+1$. In the weighted setting, a natural analogue of the invariant b would be to let $b(C) := \max\{i : F_i \text{ has a generator of degree } w_i + 1\}$. However, the main result of [EL15] immediately implies that, with notation as in Theorem 1.5, we have $b(C) = \dim W_1 - 2 - \operatorname{gon}(C)$ for $\deg L \gg 0$. Since this only depends on the degree 1 part of W, it tells us nothing new about the relationship between the geometry of curves and the algebraic properties of syzygies. So if we want to find a meaningful weighted analogue of the gonality conjecture, we will need to look in a different direction.

The Green–Lazarsfeld gonality conjecture is one of a series of conjectures about the extent to which the syzygies of a curve C are determined by embeddings of C into special varieties such as scrolls or other varieties of minimal degree (or minimal regularity). To develop a meaningful weighted analogue of the Green–Lazarsfeld gonality conjecture, a natural first question to tackle would be as follows.

Question 7.4. Can we develop a weighted theory of rational normal scrolls, or varieties of minimal degree, or varieties of minimal regularity? More specifically, can one develop such theories for the weighted spaces $\mathbb{P}(1^a, d^b)$ that arise in Theorem 1.5? □

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There is a famous classical connection between varieties of minimal degree and the N_p condition: a variety has minimal degree if and only if it satisfies the N_p condition for the maximal possible p, i.e. if and only if its resolution is purely linear. If one can answer parts of Question 7.4, it would be interesting to then investigate how that answer is related to the weighted N_p conditions explored in this paper.

In a different direction, a famous result of Gruson, Lazarsfeld and Peskine [GLP83] bounds the regularity of any nondegenerate irreducible curve $C \subseteq \mathbb{P}^r$ in terms of its degree. It would be interesting to explore an analogue of such a theorem.

Question 7.5. Let C be a smooth (or irreducible) curve in $\mathbb{P}(d_0, \ldots, d_n)$. Can one bound the regularity of I_C via a Gruson–Lazarsfeld–Peskine-type formula? □

7.3 M_L bundles

Green, Lazarsfeld and others have used positivity of M_L bundles to obtain N_p results for syzygies of a curve C embedded by a line bundle L [AKL19, EL93, GL88, GLP83, KL19, Par00, Par21]. Let $C \subseteq \mathbb{P}^n$ be a curve embedded by the complete linear series for L. The vector bundle M_L is defined by the short exact sequence $0 \leftarrow L \leftarrow H^0(C, L) \otimes_k \mathcal{O}_C \leftarrow M_L \leftarrow 0$. Vanishing results about exterior powers of M_L can be used to obtain N_p results about syzygies of the embedding $C \hookrightarrow \mathbb{P}^r$ by the complete linear series $|L|$.

In the nonstandard graded case, the setup is more subtle, as the linear series involves sections of different degrees. This would require altering the basic framework, and it would be interesting to see whether N_p results for varieties could be proven via weighted analogues of this approach.

7.4 Stacky curves

Stacky curves have arisen in recent work on Gromov–Witten theory, mirror symmetry, the study of modular curves and more; see [VZB22] and the references therein. In [VZB22], Voigt and Zureick-Brown prove analogues of classical results like Petri's Theorem for stacky curves; in fact, their results can be viewed as showing that the canonical embeddings satisfy our weighted N_1 condition. Stacky curves cannot generally be embedded into standard projective space; rather, they embed into weighted projective stacks. The only relevant N_p conditions for such curves are therefore in the weighted projective setting.

Question 7.6. Prove an analogue of Theorem 1.5 for stacky curves embedded into weighted projective stacks by high degree line bundles. □

We highlight one aspect where stacky curves differ from smooth curves. For a line bundle of high enough degree on a smooth curve, the rank of the global sections depends only on the degree of the line bundle. This is not the case for stacky curves, as the space of global sections also depends on the behavior of the corresponding divisor at the stacky points. So instead of simply fixing the degree of the line bundle, a more natural setup for a stacky curve might be to follow the recipe from Ein and Lazarsfeld [EL93] and focus on N_p conditions for divisors of the form $K + L^d + B$, where K is the canonical divisor, L is very ample, and B is effective.

In a slightly different direction, one could focus on canonical embeddings. Green's Conjecture relates classical N_p conditions to the intrinsic geometry of a canonical curve, specifically to its Clifford index. The canonical embedding of a stacky curve lands in a weighted projective stack, and thus, our weighted N_p conditions provide a natural setting for considering a stacky analogue of Green's Conjecture.

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Question 7.7. Can one use weighted N_p conditions to articulate an analogue of Green's conjecture for stacky curves? □

7.5 Nonstandard Koszul rings

Koszul rings were defined by Priddy [Pri70] and now play a fundamental role within commutative algebra [AE92, AP01, ACI15, Con00]. One rich source of Koszul rings comes from high degree Veronese embeddings. Let $X \subseteq \mathbb{P}^r$ be a smooth variety embedded by a complete linear series for L^d, where L is very ample and $d \gg 0$; it is known that the homogeneous coordinate ring of $X \subseteq \mathbb{P}^r$ is a Koszul ring [Bac86, ERT94].

It would be interesting to know if high degree embeddings into weighted spaces (via a log complete series) can provide more exotic examples of Koszul rings, or related concepts. The following example shows some of the subtle behavior that might arise.

Example 7.8. Let $\mathbb{P}^1 \to \mathbb{P}(1^3, 2^2)$ be the map $[s:t] \mapsto [s^3 : s^2t : st^2 : st^5 : t^6]$ be the map determined by the log complete series for $\mathcal{O}_{\mathbb{P}^1}(3)$ with $d=2$ and $\deg(D)=1$. Let $S = k[x_0, x_1, x_2, y_0, y_1]$ be the Cox ring of $\mathbb{P}(1^3, 2^2)$. The defining ideal of the image is $I = \langle x_1^2 - x_0x_2,\ x_2y_0 - x_1y_1,\ x_1y_0 - x_0y_1,\ x_2^3 - x_0y_1,\ x_1x_2^2 - x_0y_0,\ y_0^2 - x_2^2y_1\rangle$. The ring S/I_C is isomorphic to the subalgebra $k[s^3, s^2t, st^2, st^5, t^6]$, and it is a variant of a Veronese subring; for instance, it contains the degree 6 Veronese subring. The ring $T = S/I_C$ does not satisfy the standard definition of a graded Koszul ring, as the minimal free resolution of the residue field has the form $[T \leftarrow T(-1)^3 \oplus T(-2)^2 \leftarrow \cdots]$. However, if we consider the grevlex order with $y_0 > y_1 > x_0 > x_1 > x_2$ then the initial ideal is $\text{in}(I) = \langle x_1^2,\ x_1y_1,\ x_1y_0,\ x_0y_1,\ y_0x_0,\ y_0^2\rangle$. Thus, I has a quadratic Gröbner basis; if S were standard graded then this would imply that S/I_C is G quadratic and, therefore, Koszul [Con00]. Given the nonstandard grading, it implies that S/I_C is a sort of nonstandard graded deformation of a Koszul ring.

Question 7.9. Let X be a smooth variety, and consider an embedding $X \hookrightarrow \mathbb{P}(W)$ given by a log complete series for L^e, where L is very ample and $e \gg 0$. Let $I_X \subseteq S$ be the defining ideal in the corresponding nonstandard graded polynomial ring. Does I_X admit a quadratic Gröbner basis? What sort of Koszul-type properties are satisfied by S/I_X? □

7.6 N_p conditions for curves in other toric varieties

Another natural direction is to ask whether smooth curves in other toric varieties also satisfy N_p conditions. To approach this, one must consider the following.

Question 7.10. Let S be the $\mathbb{Z}^r$-graded Cox ring of a simplicial toric variety X and B the corresponding irrelevant ideal.

(1) What is a good analogue of a complete, or log complete, linear series?
(2) What is a good analogue of normal generation?
(3) What is the appropriate analogue of the N_p conditions in this setting?
(4) Does the answer to (1) or (2) depend only on the grading of S, or does it also depend on the choice of the irrelevant ideal B?
(5) When defining the N_p conditions, should one focus on minimal free resolutions or on virtual resolutions? □

Even for a product of projective spaces, some of these questions are open.

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Question 7.11. Can one develop analogues of the main results of this paper for a smooth curve in a product of projective spaces? □

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References

ACI15 L. L. Avramov, A. Conca and S. B. Iyengar, *Subadditivity of syzygies of Koszul algebras*, Math. Ann. **361** (2015), 511–534.

AE92 L. L. Avramov and D. Eisenbud, *Regularity of modules over a Koszul algebra*, J. Algebra **153** (1992), 85–90.

AFP$^+$19 M. Aprodu, G. Farkas, S. Papadima, C. Raicu and J. Weyman, *Koszul modules and Green's conjecture*, Invent. Math. **218** (2019), 657–720.

AKL19 D. Agostini, A. Küronya and V. Lozovanu, *Higher syzygies of surfaces with numerically trivial canonical bundle*, Math. Z. **293** (2019), 1071–1084.

AN10 M. Aprodu and J. Nagel, *Koszul cohomology and algebraic geometry*, University Lecture Series, vol. 52 (American Mathematical Society, Providence, RI, 2010).

AP01 L. L. Avramov and I. Peeva, *Finite regularity and Koszul algebras*, Amer. J. Math. **123** (2001), 275–281.

AV03 M. Aprodu and C. Voisin, *Green-Lazarsfeld's conjecture for generic curves of large gonality*, C. R. Math. Acad. Sci. Paris **336** (2003), 335–339.

Bac86 J. Backelin, *On the rates of growth of the homologies of Veronese subrings*, in *Algebra, algebraic topology and their interactions (Stockholm, 1983)* (Springer, Berlin, 1986), 79–100.

BC17 N. Botbol and M. Chardin, *Castelnuovo Mumford regularity with respect to multigraded ideals*, J. Algebra **474** (2017), 361–392.

BCN22 L. Busé, M. Chardin and N. Nemati, *Multigraded Sylvester forms, duality and elimination matrices*, J. Algebra **609** (2022), 514–546.

BE21 M. K. Brown and D. Erman, *Tate resolutions on toric varieties*, J. Eur. Math. Soc. (JEMS) (2021), to appear.

https://doi.org/10.1112/S0010437X25007092 Published online by Cambridge University Press

BE22 M. K. Brown and D. Erman, *Linear strands of multigraded free resolutions*, Preprint (2022), arXiv:2202.00402.

BE23 M. K. Brown and D. Erman, *Positivity and nonstandard graded Betti numbers*, Preprint (2023), arXiv:2302.07403.

Ben04 D. Benson, *Dickson invariants, regularity and computation in group cohomology*, Illinois J. Math. **48** (2004), 171–197.

BES20 C. Berkesch, D. Erman and G. G. Smith, *Virtual resolutions for a product of projective spaces*, Algebraic Geom. **7** (2020), 460–481.

BH13 P. Banagere and K. Hanumanthu, *Syzygies of surfaces of general type*, Geom. Dedicata **167** (2013), 123–149.

BHS21 J. Bruce, L. C. Heller and M. Sayrafi, *Characterizing multigraded regularity on products of projective spaces*, Preprint (2021), arXiv:2110.10705.

BHS22 J. Bruce, L. C. Heller and M. Sayrafi, *Bounds on multigraded regularity*, Preprint (2022), arXiv:2208.11115.

BKLY21 C. Berkesch, P. Klein, M. C. Loper and J. Yang, *Homological and combinatorial aspects of virtually Cohen-Macaulay sheaves*, Trans. London Math. Soc. **8** (2021), 413–434.

BR86 M. Beltrametti and L. Robbiano, *Introduction to the theory of weighted projective spaces*, Exposition. Math. **4** (1986), 111–162.

BS22 M. K. Brown and M. Sayrafi, *A short resolution of the diagonal for smooth projective toric varieties of Picard rank*, Preprint (2022), arXiv:2208.00562.

Cas93 G. Castelnuovo, *Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica*, Red. Circ. Mat. Palermo **7** (1893), 89–110.

CH22 M. Chardin and R. Holanda, *Multigraded Tor and local cohomology*, Preprint (2022), arXiv:2211.14357.

Chi19 S. Chintapalli, *On syzygies of projective bundles over abelian varieties*, J. Pure Appl. Algebra. **223** (2019), 2413–2424.

CLS11 D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, vol. 124 (American Mathematical Society, Providence, RI, 2011).

CN20 M. Chardin and N. Nemati, *Multigraded regularity of complete intersections*, Preprint (2020), arXiv:2012.14899.

Con00 A. Conca, *Universally Koszul algebras*, Math. Ann. **317** (2000), 329–346.

DE22 H. Dao and D. Eisenbud, *Linearity of free resolutions of monomial ideals*, Res. Math. Sci. **9** (2022), 35.

EES15 D. Eisenbud, D. Erman and F.-O. Schreyer, *Tate resolutions for products of projective spaces*, Acta Math. Vietnam. **40** (2015), 5–36.

EG84 D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), 89–133.

EGHP05 D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Restricting linear syzygies: algebra and geometry*, Compos. Math. **141** (2005), 1460–1478.

Eis05 D. Eisenbud, *The geometry of syzygies: a second course in algebraic geometry and commutative algebra*, vol. 229 (Springer Science & Business Media, New York, 2005).

Eis95 D. Eisenbud, *Commutative algebra, with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150 (Springer-Verlag, New York, 1995).

EL18 L. Ein and R. Lazarsfeld, *Syzygies of projective varieties of large degree: recent progress and open problems*, in *Algebraic geometry: Salt Lake City 2015* (American Mathematical Society, Providence, RI, 2018), 223–242.

EL20 L. Ein and R. Lazarsfeld, *Tangent developable surfaces and the equations defining algebraic curves*, Bull. Amer. Math. Soc. (N.S.) **57** (2020), 23–38.

EL12 L. Ein and R. Lazarsfeld, *Asymptotic syzygies of algebraic varieties*, Invent. Math. **190** (2012), 603–646.

EL15 L. Ein and R. Lazarsfeld, *The gonality conjecture on syzygies of algebraic curves of large degree*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 301–313.

https://doi.org/10.1112/S0010437X25007092 Published online by Cambridge University Press

EL93 L. Ein and R. Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math. **111** (1993), 51–67.

ENP20 L. Ein, W. Niu and J. Park, *Singularities and syzygies of secant varieties of nonsingular projective curves*, Invent. Math. **222** (2020), 615–665.

ERT94 D. Eisenbud, A. Reeves and B. Totaro, *Initial ideals, Veronese subrings, and rates of algebras*, Adv. Math. **109** (1994), 168–187.

Far17 G. Farkas, *Progress on syzygies of algebraic curves*, in *Moduli of curves* (Springer, Cham, 2017), 107–138.

Fuj77 T. Fujita, *Defining equations for certain types of polarized varieties*, in *Complex analysis and algebraic geometry* (Iwanami Shoten, Tokyo, 1977), 165–173.

GL86 M. Green and R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83** (1986), 73–90.

GL88 M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. **67** (1988), 301–314.

GLP83 L. Gruson, R. Lazarsfeld and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), 491–506.

GP96 F. J. Gallego and B. P. Purnaprajna, *Higher syzygies of elliptic ruled surfaces*, J. Algebra **186** (1996), 626–659.

GP98 F. J. Gallego and B. P. Purnaprajna, *Very ampleness and higher syzygies for Calabi-Yau threefolds*, Math. Ann. **312** (1998), 133–149.

GP99 F. J. Gallego and B. P. Purnaprajna, *Projective normality and syzygies of algebraic surfaces*, J. Reine Angew. Math. **506** (1999), 145–180.

Gre84a M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differ. Geom. **19** (1984), 125–171.

Gre84b M. L. Green, *Koszul cohomology and the geometry of projective varieties. II*, J. Differ. Geom. **20** (1984), 279–289.

Gre99 M. Green, *The Eisenbud-Koh-Stillman conjecture on linear syzygies*, Invent. Math. **136** (1999), 411–418.

GS15 A. Geraschenko and M. Satriano, *Toric stacks I: The theory of stacky fans*, Trans. Amer. Math. Soc. **367** (2015), 1033–1071.

Har77 R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer-Verlag, New York–Heidelberg, 1977).

HHW12 F. T. Hawwa, J. W. Hoffman and H. Wang, *Koszul duality for multigraded algebras*, Eur. J. Pure Appl. Math. **5** (2012), 511–539.

HNVT22 M. Harada, M. Nowroozi and A. Van Tuyl, *Virtual resolutions of points in,* $\mathbb{P}^1 \times \mathbb{P}^1$, J. Pure Appl. Algebra **226** (2022), 107140.

HS04 M. Haiman and B. Sturmfels, *Multigraded Hilbert schemes*, J. Algebraic Geom. **13** (2004), 725–769.

HSS06 M. Hering, H. Schenck and G. G. Smith, *Syzygies, multigraded regularity and toric varieties*, Compositio Math. **142** (2006), 1499–1506.

HT13 J.-M. Hwang and W.-K. To, *Syzygies of compact complex hyperbolic manifolds*, J. Algebraic Geom. **22** (2013), 175–200.

KL19 A. Küronya and V. Lozovanu, *A Reider-type theorem for higher syzygies on abelian surfaces*, Algebraic Geom. **6** (2019), 548–570.

LRZ16 A. Landesman, P. Ruhm and R. Zhang, *Spin canonical rings of log stacky curves*, Ann. Inst. Fourier (Grenoble) **66** (2016), 2339–2383.

LRZ18 A. Landesman, P. Ruhm and R. Zhang, *Section rings of* $\mathbb{Q}$*-divisors on minimal rational surfaces*, Math. Res. Lett. **25** (2018), 1329–1357.

GS D. Grayson and M. Stillman, *Macaulay2, a system for computation in algebraic geometry and commutative algebra*, https://macaulay2.com/.

Mat61 A. Mattuck, *Symmetric products and Jacobians*, Amer. J. Math. **83** (1961), 189–206.

MS04 D. Maclagan and G. G. Smith, *Multigraded Castelnuovo-Mumford regularity*, J. Reine Angew. Math. **571** (2004), 179–212.

https://doi.org/10.1112/S0010437X25007092 Published online by Cambridge University Press

Mum70 D. Mumford, *Varieties defined by quadratic equations*, in *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)* (Ed. Cremonese, Rome, 1970), 29–100.

Niu19 W. Niu, *On syzygies of Calabi-Yau varieties and varieties of general type*, Adv. Math. **343** (2019), 756–788.

NY22 S.-C. Ng and S.-K. Yeung, *On the syzygies of ample line bundles on fake projective planes*, Taiwanese J. Math. **26** (2022), 49–67.

Par00 G. Pareschi, *Syzygies of abelian varieties*, J. Amer. Math. Soc. **13** (2000), 651–664.

Par21 J. Park, *Asymptotic vanishing of syzygies of algebraic varieties*, Preprint (2021), arXiv:2110.12419.

Per08 F. Perroni, *A note on toric Deligne-Mumford stacks*, Tohoku Math. J. (2). **60** (2008), 441–458.

PP03 G. Pareschi and M. Popa, *Regularity on abelian varieties I.*, J. Amer. Math. Soc. **16** (2003), 285–302.

PP04 G. Pareschi and M. Popa, *Regularity on abelian varieties. II. Basic results on linear series and defining equations*, J. Algebraic Geom. **13** (2004), 167–193.

Pri70 S. B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

SD72 B. Saint-Donat, *Sur les équations définissant une courbe algébrique*, C. R. Acad. Sci. Paris Sér. A-B. **274** (1972), A324–A327.

SS11 J. Sidman and G. G. Smith, *Linear determinantal equations for all projective schemes*, Algebra Number Theory **5** (2011), 1041–1061.

Sta12 The Stacks Project Authors, *Stacks project* (2012), http://stacks.math.columbia.edu/.

SVT06 J. Sidman and A. Van Tuyl, *Multigraded regularity: syzygies and fat points*, Beiträge Algebra Geom. **47** (2006), 67–87.

Sym10 P. Symonds, *On the Castelnuovo-Mumford regularity of the cohomology ring of a group*, J. Amer. Math. Soc. **23** (2010), 1159–1173.

Sym11 P. Symonds, *On the Castelnuovo-Mumford regularity of rings of polynomial invariants*, Ann. Math. (2) **174** (2011), 499–517.

Voi02 C. Voisin, *Green's generic syzygy conjecture for curves of even genus lying on a, K3 surface*, J. Eur. Math. Soc. (JEMS) **4** (2002), 363–404.

Voi05 C. Voisin, *Green's canonical syzygy conjecture for generic curves of odd genus*, Compositio Math. **141** (2005), 1163–1190.

VZB22 J. Voight and D. Zureick-Brown, *The canonical ring of a stacky curve*, Mem. Amer. Math. Soc. **277** (2022), 1362.

Yan21 J. Yang, *Virtual resolutions of monomial ideals on toric varieties*, Proc. Amer. Math. Soc. Ser. B. **8** (2021), 100–111.

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