A ONE-REGULAR GRAPH OF DEGREE THREE

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1. Introduction. Soon after the publication of Tutte's paper [**5**] on m-cages, H. S. M. Coxeter asked in a letter to the author whether one-regular graphs of degree 3 exist. The purpose of the following paper is to show by an example that the answer is in the affirmative.

To avoid repetitions for the definitions of the terms: finite graph, group of automorphisms of a graph, regular graphs of degree 3 (or cubical graphs), etc. the reader is referred to a former paper by the author [**4**, pp. 365-366]. Let us add only the definition of a *symmetrical* graph, as it seems that this term has not yet been defined explicitly, although it has been used by Foster [**3**], who gave a list of the symmetrical graphs known to him, and defined implicitly by Tutte [**5**], whose "s-regular cubical graphs" are nothing else than those symmetrical graphs which are connected and regular of degree 3.

Definition. A finite and connected graph is called symmetrical if for any two arcs AB and CD of the graph its group contains at least one transformation which takes the vertex A into C, and B into D.

Finally, we shall need Tutte's definition [**5**] that a graph of degree 3 is *s-regular* if it is connected, and if for any two s-arcs S_1, S_2 there is a *unique* transformation of the graph which carries S_1 into S_2; here an s-arc is any path $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_s$ formed by s consecutive arcs $A_0 \rightarrow A_1$, $A_1 \rightarrow A_2, \ldots, A_{s-1} \rightarrow A_s$ of the graph taken in a definite sense; of course $A_i \neq A_j$ for $i \neq j$.

Tutte was especially interested in the problem of finding m-cages, i.e., those s-regular graphs of degree 3 where s takes its maximal value

$$s = [\tfrac{1}{2}m + 1],$$

m being here the girth of the graph, i.e., the least number of arcs forming a closed polygon (or m-circuit); he showed also that $s \leqslant 5$ for any symmetrical graph of degree 3.

The present paper is concerned with the other extreme case, that of the lowest possible degree of symmetry a symmetrical graph of degree 3 can possess. Coxeter [**1**, p. 421] had found that there are infinitely many cubical graphs with $s = 2$, but it seems that hitherto no example with $s = 1$ was known.

In §3 such an example will be given. Unfortunately this graph has 432 vertices (hence 648 edges) so that it is practically impossible to draw it on a sheet of paper. The author hopes however that someone else will find a one-regular graph of degree 3 with fewer vertices.

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The construction of the new graph is based on a general method (described in §2) which might be of some interest in itself apart from the use made of it here, as it would allow us to find also other symmetrical graphs of degree 3.

2. A general method for constructing symmetrical graphs of degree 3. In this section H will be any group of finite order g which can be *generated by three elements of order* 2. Let a_1, a_2, a_3 be these generators, a_0 the identity, and $a_4, a_5, \ldots, a_{g-1}$ the other elements of H.

Then a graph G with g vertices can be defined as follows:

(i) With every element a_i $(i = 0, 1, 2, \ldots, g - 1)$ of H we associate a vertex of G which shall be called a_i also, since there will be no danger of confusion.

(ii) Any two vertices a_i and a_j of G shall be joined by an arc if, and only if, in H the product $a_j a_i^{-1}$ is equal to one of the three generating elements.

In other words, the relation

$$\text{(R)} \qquad a_j a_i^{-1} = a_k$$

must hold in H with $k = 1$ or 2 or 3, if a_i and a_j are the endpoints of an arc in G; and no other arcs besides those just defined are introduced.

Note that by taking the inverse of both sides, relation (R) may be given the equivalent form:

$$\text{(R}'\text{)} \qquad a_i a_j^{-1} = a_k \qquad (k = 1, 2, 3),$$

since $a_k^{-1} = a_k$. It is thus seen that the relation (R) is only apparently asymmetrical in the subscripts i and j.

THEOREM 2.1. *The graph G just defined is regular of degree* 3.

(In Tutte's terminology such a graph, where each vertex is an endpoint of three arcs, is called "cubical.")

Proof. From the defining relations (R) or (R′) we have $a_j = a_k a_i$, whence it follows that for any given vertex a_i $(i = 0, 1, 2, \ldots, g - 1)$ of G there are just three arcs ending there, viz, those whose other endpoints are $a_1 a_i$, $a_2 a_i$, and $a_3 a_i$, respectively.

THEOREM 2.2. *The graph G is connected.*

Proof. It will be sufficient to show that for any vertex a_i of the graph G there is some s-arc joining it with a_0 (the vertex corresponding to the identity of H). Since H is generated by a_1, a_2, a_3, the element a_i of H can be written as some product of generators, say

$$a_i = a_{k_1} a_{k_2} a_{k_3} \ldots a_{k_{s-1}} a_{k_s},$$

where all the suffices $k_1, k_2, \ldots, k_s$ can take only the values 1, 2, 3; moreover, it can be assumed that no "partial product"

$$a_{k_t} a_{k_{t+1}} a_{k_{t+2}} \ldots a_{k_{t+u}} \qquad (t \geqslant 1, t + u \leqslant s)$$

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be equal to the identity a_0 (because otherwise it could be omitted from the total product). Then

$$a_0 \to a_{k_s},\ a_{k_s} \to a_{k_{s-1}}a_{k_s},\ a_{k_{s-1}}a_{k_s} \to a_{k_{s-2}}a_{k_{s-1}}a_{k_s}, \ldots,$$

$$a_{k_2}a_{k_3} \ldots a_{k_s} \to a_{k_1}a_{k_2}a_{k_3} \ldots a_{k_s}$$

are arcs of the graph G since they satisfy the condition (R), and joining them in the sense indicated by the arrows we obtain the desired s-arc leading from a_0 to a_i.

THEOREM 2.3. *If a_j and a_n are any two vertices of the graph G, the group of automorphisms of G contains at least one transformation which takes a_j into a_n.*

Proof. Let T_ρ ($\rho = 0, 1, 2, \ldots, g - 1$) be that permutation of the vertices of G which takes a_i ($i = 0, 1, 2, \ldots, g - 1$) into the vertex corresponding to the product $a_i a_\rho$:

$$a_i^{T_\rho} = a_i a_\rho;$$

then T_ρ belongs to the transformations of the group of automorphisms of G. Indeed, if a_j is the other endpoint of an arc $a_i \to a_j$, we have only to show that $a_i^{T_\rho} \to a_j^{T_\rho}$ is also an arc of G. But by (R), $a_j a_i^{-1} = a_k$ (where $k = 1, 2,$ or 3); hence

$$a_j^{T_\rho} \cdot (a_i^{T_\rho})^{-1} = (a_j a_\rho)(a_i a_\rho)^{-1} = a_j a_\rho a_\rho^{-1} a_i^{-1} = a_j a_i^{-1} = a_k,$$

and this is, by (R), just the condition for $a_i^{T_\rho}$ and $a_j^{T_\rho}$ to be endpoints of an arc in G.

Now, if a_j and a_n are the two vertices considered in the theorem, the transformation T_j takes any a_i into $a_i a_j$, and hence a_0 into $a_0 a_j = a_j$. In an analogous manner T_n takes a_0 into a_n. Hence the product $T_j^{-1}T_n$, i.e., the inverse transformation T_j^{-1}, followed by T_n, takes a_j into a_n, and thus satisfies Theorem 2.3.

It might be remarked as a corollary that the g transformations $T_0, T_1, \ldots, T_{g-1}$ constitute a group simply isomorphic to H.

The theorem thus proved tells us, in other words, that the group of automorphisms of G is transitive on the vertices; note that this is less than symmetry (as defined in §1) which requires transitivity at least on the 1-arcs. In order to obtain symmetrical graphs of degree 3 it will be necessary to impose a further condition on the group H. (So far it had only been required that H be generated by three elements of order 2.) Since this condition has to do with the automorphisms of the group H, it will be convenient to state first the following theorem:

THEOREM 2.4. *Let the group H (generated by three elements of order 2) admit an automorphism ϕ which permutes the three generators of H. If this automorphism carries any element a_i of H into a_i^ϕ, then the corresponding permutation of the vertices of G is a transformation of G belonging to its group of automorphisms.*

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Proof. We have only to show that, if a_i and a_j are endpoints of an arc in G, so also are a_i^ϕ and a_j^ϕ. Since ϕ is an automorphism of H,

$$a_j^\phi \,.\, (a_i^\phi)^{-1} = a_j^\phi \,.\, (a_i^{-1})^\phi = (a_j a_i^{-1})^\phi,$$

and replacing $a_j a_i^{-1}$ by a_k (where $k = 1, 2$, or 3), we have

$$a_j^\phi \,.\, (a_i^\phi)^{-1} = a_k^\phi.$$

But we made the assumption that ϕ only permutes the generators a_1, a_2, a_3; hence a_k^ϕ is also one of the generating elements, say a_q $(q = 1, 2, 3)$. Thus we have obtained

$$a_j^\phi \,.\, (a_i^\phi)^{-1} = a_q \qquad (q = 1, 2, 3);$$

and this is, according to (R), the condition for a_i^ϕ and a_j^ϕ to be endpoints of an arc in G.

Having proved Theorem 2.4, we now give a sufficient condition for obtaining symmetrical graphs of degree 3.

THEOREM 2.5. *If the group H admits an automorphism such that the three generators of order* 2 *undergo a* cyclic *permutation, then the graph G is symmetrical.*

Proof. Let $a_i \rightarrow a_j$ and $a_q \rightarrow a_r$ be any two arcs in G; then Theorem 2.5 will be proved if we can show that the group of automorphisms of G contains at least one transformation θ fulfilling the two conditions:

$$\text{(C)} \qquad a_i^\theta = a_q \text{ and } a_j^\theta = a_r.$$

To obtain such a transformation θ we proceed as follows: as in the proof of Theorem 2.3, let T_p be that transformation of G which replaces any vertex a_t by $a_t a_p$, let ϕ be the transformation of G corresponding (by Theorem 2.4) to the automorphism ϕ of H whose existence is supposed in Theorem 2.5; then also the three products (read from left to right)

$$\theta_n = T_i^{-1}\phi^n T_q \qquad (n = 1, 2, 3)$$

are transformations belonging to the group of the graph. We will show that just one of them fulfils the two conditions (C).

It is obvious that the first of these conditions is satisfied for each value of n; indeed, T_i^{-1} replaces a_i by a_0, any power of ϕ leaves a_0 fixed, and T_q takes a_0 into a_q. As to the second condition, the following argument may be used:

By (R) we have not only $a_j = a_k a_i$ $(k = 1, 2, 3)$, but also that $a_r a_q^{-1}$ is a generator,

$$a_r a_q^{-1} = a_t \qquad (t = 1, 2, \text{ or } 3).$$

Now, since ϕ is supposed to produce a cyclic permutation on the generators a_1, a_2, a_3, some power of ϕ will replace a_k by a_t. Let ϕ^n be that power of ϕ, then

$$a_k^{\phi^n} = a_t \qquad (n = 1, 2, \text{or } 3),$$

and it follows readily that also the second condition (C) is satisfied by the transformation $\theta = \theta_n = T_i^{-1}\phi^n T_q$, since

$$a_j^{\theta_n} = (a_k a_i)^{T_i^{-1}\phi^n T_q} = a_k^{\phi^n T_q} = a_t^{T_q} = (a_r a_q^{-1})^{T_q} = a_r.$$

We close this section by giving three examples of symmetrical graphs of degree 3 which can be obtained by Theorem 2.5.

(1) The simplest example of a group H which can be generated by three (but not less than three) elements of order 2 (and which admits an automorphism of the kind established in Theorem 2.5) is the direct product $\{a_1\} \times \{a_2\} \times \{a_3\}$ of order 8. It is easy to see that the corresponding graph G is that of the vertices and edges of a cube.

(2) The symmetric group of degree 4 and order $4! = 24$ can be generated by $a_1 = (1\ 2)$, $a_2 = (1\ 3)$, $a_3 = (1\ 4)$. The corresponding graph turns out to be Foster's III-13 [**3**, Fig. 9]; it is called $\{12\} + \{12/5\}$ by Coxeter [**1**, pp. 439, 440].

(3) An apparently new symmetrical graph with 64 vertices and girth 8 corresponds to the group of order 64 which can be generated by the following permutations on twelve symbols:

$$\begin{aligned} a_1 &= (1\ 2)(3\ \ 4)(\ 9\ 11), \\ a_2 &= (1\ 3)(5\ \ 6)(\ 7\ \ 8), \\ a_3 &= (5\ 7)(9\ 10)(11\ 12). \end{aligned}$$

(As an abstract group, it can be characterized by the condition that all the commutators be invariant elements.)

3. Example of a one-regular graph of degree 3. The symmetrical graphs mentioned as examples at the end of the foregoing section are 2-regular. It is easy to see that this is due to the fact that in these examples the group H does not only admit an automorphism allowing a cyclic permutation of the generators, but is formally symmetrical in the three generators, admitting, e.g., an automorphism which leaves a_3 fixed and replaces a_1 by a_2.

This fact seemed to justify the author's hope of obtaining a one-regular graph of degree 3 by the method of Theorem 2.5, if a group H could be found which admits automorphisms producing cyclic permutations of the three generators of order 2, but no automorphism leaving a_3 fixed while interchanging a_1 and a_2. We will show that such a group is that generated by the following three permutations on nine symbols:

$$\begin{aligned} a_1 &= (1\ 2)(3\ 5)(4\ 8), \\ a_2 &= (1\ 3)(2\ 6)(5\ 9), \\ a_3 &= (1\ 4)(2\ 3)(6\ 7). \end{aligned}$$

In this section the letter H will be used to denote the group with these generators.

THEOREM 3.1. *The group H just defined admits an automorphism ϕ satisfying the conditions:*

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$$a_1^\phi = a_2, \quad a_2^\phi = a_3, \quad a_3^\phi = a_1;$$

but there is no automorphism ψ of H satisfying the conditions:

$$a_1^\psi = a_2, \quad a_2^\psi = a_1, \quad a_3^\psi = a_3.$$

Proof. If b is the following permutation:

$$b = (1\ 3\ 2)(4\ 5\ 6)(7\ 8\ 9),$$

it is immediately seen that

$$b^{-1}a_1b = a_2, \quad b^{-1}a_2b = a_3, \quad b^{-1}a_3b = a_1;$$

whence it follows that the automorphism

$$a_i^\phi = b^{-1}a_ib \qquad (i = 0, 1, 2, \ldots, g-1)$$

satisfies the conditions of Theorem 3.1. It may be remarked that this is an inner automorphism of the group H, since b is an element of H; indeed a somewhat lengthy computation shows that

$$b = \{(a_1a_3)^2(a_1a_2a_1a_3)^2\}^3 \cdot (a_1a_2a_1a_3)^3a_1a_3.$$

To prove that no automorphism ψ of H can exist which leaves a_3 fixed while interchanging a_1 and a_2, compute the product

$$e = a_3a_2(a_3a_1)^2(a_3a_2)^2a_1a_2,$$

from which it is found that $e = a_0$. Assuming the existence of an automorphism ψ and applying it to the product e just introduced, we would have

$$e^\psi = a_3a_1(a_3a_2)^2(a_3a_1)^2a_2a_1;$$

but since $e = a_0$, the left-hand side of the last equation must likewise be the identity a_0; the right-hand side however is found equal to

$$(1\ 6\ 8)(2\ 5\ 7)(3\ 4\ 9),$$

hence distinct from the identity. This contradiction shows the impossibility of the existence of ψ.

THEOREM 3.2. *The graph G of degree 3 which corresponds to the group H is one-regular.*

Proof. That G is at least one-regular follows from Theorems 2.5 and 3.1. It remains to be shown that G is not 2-regular. This can be done as follows:

We have already seen, in the proof of Theorem 2.2, that the endpoints of the s-arcs beginning with a_0 are the "non-trivial" products of s generating elements (where "non-trivial" means that there are no partial products equal to the identity); e.g., the six 2-arcs beginning with a_0 are

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$$a_0 \to a_1 \to a_2a_1, \quad a_0 \to a_1 \to a_3a_1, \quad a_0 \to a_2 \to a_1a_2,$$

$$a_0 \to a_2 \to a_3a_2, \quad a_0 \to a_3 \to a_1a_3, \quad a_0 \to a_3 \to a_2a_3.$$

In an analogous manner we can form the twelve 3-arcs beginning with a_0, etc. By computing all the products of 2, 3, 4, ... generators of our group H, it turns out that these endpoints of the s-arcs beginning with a_0 are all different *so long as* $s \leqslant 5$. But of the endpoints of the 96 six-arcs beginning with a_0 only 77 are distinct, since it turns out that there are fifteen pairs of such 6-arcs with a common endpoint, and two triples of such 6-arcs, viz, those ending with

$$a_1a_3a_1a_2a_1a_2 = a_2a_1a_2a_3a_2a_3 = a_3a_2a_3a_1a_3a_1 = (1\ 5\ 9)(2\ 4\ 8)(3\ 6\ 7)$$

or

$$a_1a_3a_1a_3a_2a_3 = a_2a_1a_2a_1a_3a_1 = a_3a_2a_3a_2a_1a_2 = (1\ 9\ 5)(2\ 8\ 4)(3\ 7\ 6).$$

These two vertices are thus characterized as the only endpoints of *triples* of 6-arcs beginning with a_0; therefore no transformation of G can exist that leaves a_0 fixed but carries one of these 6-arcs into a 6-arc ending at some other vertex. Now the two triples of 6-arcs just considered begin with one of the following three 2-arcs:

$$a_0 \to a_2 \to a_1a_2, \quad a_0 \to a_3 \to a_2a_3, \quad \text{or} \quad a_0 \to a_1 \to a_3a_1;$$

hence no transformation of G can take one of these three 2-arcs into any of the other three 2-arcs beginning with a_0. This means that G is not 2-regular.

COROLLARY. *The graph G is of girth* 12.

Indeed a 12-circuit can be formed with two of the 6-arcs of one of the triples just mentioned, but no m-circuit exists with $m < 12$.

THEOREM 3.3. *The one-regular graph just found has* 432 *vertices.*

Proof. Since the number of vertices of G is equal to the order of the group H, we have only to prove that the group generated by

$$a_1 = (1\ 2)(3\ 5)(4\ 8), \quad a_2 = (1\ 3)(2\ 6)(5\ 9), \quad a_3 = (1\ 4)(2\ 3)(6\ 7),$$

is of order 432.

It is easy to see that the order of this group H is either 432 *or some factor contained in* 432. Indeed, the generators a_1, a_2, a_3, and hence any permutation of H, leave fixed the following triple system of 9 elements:

129; 137; 145; 168; 238; 246; 257; 349; 356; 478; 589; 679;

and it has been pointed out by Emch [**2**] that the group of all the permutations which leave a triple system of nine elements invariant, has the order 432, since it is simply isomorphic to the holomorph of the noncyclic group of order 9.

Hence our group H is simply isomorphic either to the same holomorph of order 432 *or to some proper subgroup of it.* It remains only to be shown that this second alternative does not take place, i.e., that the order of H cannot be less than 432.

Our group H is transitive, since the eight permutations a_1, a_2, a_3,

$$a_1a_3a_1 = (1\,5)(2\,8)(6\,7), \quad a_1a_2a_1 = (1\,6)(2\,5)(3\,9),$$

$$a_3a_1a_3 = (1\,8)(2\,5)(3\,4), \quad a_2a_1a_2 = (1\,9)(3\,6)(4\,8),$$

$$a_2a_3a_2a_3 = (1\,7\,3)(2\,6\,4),$$

replace the symbol "1" respectively by 2, 3, . . . , 9. According to a well-known theorem, the order of H will be nine times that of the subgroup H_1 formed by all the permutations of H each of which leaves the symbol "1" fixed. But this subgroup H_1 contains the following three permutations:

$$a_3a_2'a_3 = (24)(37)(59), \quad a_3a_2a_1a_3a_1a_3 = (2\,5\,9\,4)(3\,6\,7\,8),$$

$$(a_1a_3)^4 \cdot a_3a_2a_1a_3a_1a_2 = (2\,7\,8\,9\,3\,6)(4\,5).$$

It is readily shown by computing all the products of the last two permutations (which are even) that it is possible to form 24 different even permutations which leave the symbol "1" fixed. Hence H_1 (containing also the odd permutation $a_3a_2a_3$) cannot have an order less than 48, and H cannot be of order less than 432.

COROLLARY. *The group of automorphisms of the graph described in this section is of order* 1296.

Indeed it is a regular permutation group on the 1296 one-arcs.

Addendum.

It was kindly pointed out by Coxeter (in a letter to the author) that the one-regular graph described in the last section could be derived from the regular hyperbolic tesselation $\{12, 3\}$ by making appropriate identifications; in other words, it could be embedded in a surface of characteristic -108 (or genus 55) so as to form a map of 108 dodecagons (in agreement with the Corollary to Theorem 3.2).

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