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QUADRILATERAL-TREE PLANAR RAMSEY NUMBERS

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Abstract

For two given graphs G_1 and G_2, the planar Ramsey number $PR(G_1, G_2)$ is the smallest integer N such that every planar graph G on N vertices either contains G_1, or its complement contains G_2. Let C_4 be a quadrilateral, T_n a tree of order $n \geq 3$ with maximum degree k, and $K_{1,k}$ a star of order $k + 1$. We show that $PR(C_4, T_n) = \max\{n + 1, PR(C_4, K_{1,k})\}$. Combining this with a result of Chen *et al.* ['All quadrilateral-wheel planar Ramsey numbers', *Graphs Combin.* **33** (2017), 335–346] yields exact values of all the quadrilateral-tree planar Ramsey numbers.

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1. Introduction

In this paper, all graphs are simple and finite. Let $G = (V(G), E(G))$ be a graph. The numbers of vertices and edges in G are called the *order* and *size* of G, respectively. The *neighbourhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v and the *degree* $d_G(v)$ of v is $|N_G(v)|$. Let $\Delta(G)$ and $\delta(G)$ be the *maximum degree* and *minimum degree* of G, respectively. A vertex of degree 1 is also said to be a *leaf*, and a leaf adjacent to v is also called a *leaf neighbour* of v. The *complement* of G is denoted by $\overline{G}$. Let $K_{1,n-1}$, P_n, C_n and K_n be a *star*, a *path*, a *cycle* and a *complete graph* of order n, respectively. (C_4 is also called a quadrilateral.)

For two given graphs G_1 and G_2, the planar Ramsey number $PR(G_1, G_2)$ is the smallest integer N such that every planar graph G on N vertices either contains G_1, or its complement contains G_2. The planar Ramsey number was introduced by Walker [7] in 1969 and is the usual Ramsey number $R(G_1, G_2)$ with the ground set restricted to planar graphs. It is easy to see that $PR(G_1, G_2) \leq R(G_1, G_2)$. Since many problems in graph theory become more tractable when restricted to the plane, we might hope that determining $PR(G_1, G_2)$ is tractable. Calculating $R(K_m, K_n)$ is a very challenging problem as it increases exponentially. However, based on the four colour theorem and

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Grümbaum's 3-colourings theorem [3], Steinberg and Tovey [6] determined all values of $PR(K_m, K_n)$. For the Ramsey number for C_4 versus a tree, Burr *et al.* [1] showed that $R(C_4, T_n) = \max\{4, n+1, R(C_4, K_{1,k})\}$, where T_n is a tree of order n with maximum degree k, which transfers the problem of determining the values of $R(C_4, T_n)$ to calculating the values of $R(C_4, K_{1,k})$. Unfortunately, the exact values for $R(C_4, K_{1,n})$ are far from being known (see [8, 9]). It is shown in [5] that $R(C_4, K_{1,n}) \le n + \lfloor \sqrt{n-1} \rfloor + 2$ for $n \ge 2$ and in [1] that $R(C_4, K_{1,n}) \ge n + \sqrt{n} - 6n^{11/40}$ for sufficiently large n.

In this paper, our aim is to determine $PR(C_4, T_n)$. The main result is as follows.

THEOREM 1.1. *Let T_n be a tree of order $n \ge 3$ with maximum degree k. Then* $PR(C_4, T_n) = \max\{n+1, PR(C_4, K_{1,k})\}$.

Theorem 1.1 tells us that the values of $PR(C_4, T_n)$ depend essentially on the values of $PR(C_4, K_{1,k})$, where $k = \Delta(T_n)$. Can we determine all the values of $PR(C_4, K_{1,n})$? Define $\delta(n, C_4) = \max\{\delta(G) \mid G$ is a planar graph of order n without $C_4\}$. Recently, Chen *et al.* [2] determined the values of $\delta(n, C_4)$ for all n.

THEOREM 1.2 [2]. *Let $n \ge 4$ be an integer. Then*

$$\delta(n, C_4) = \begin{cases} 1 & \text{if } n = 4, \\ 2 & \text{if } 5 \le n \le 9, \\ 3 & \text{if } 10 \le n \le 43 \text{ and } n \notin \{30, 36, 39, 42\}, \\ 4 & \text{otherwise}. \end{cases}$$

Now define

$$f(n) = \begin{cases} 4 & \text{if } n = 2, \\ n+3 & \text{if } 3 \le n \le 6, \\ n+4 & \text{if } 7 \le n \le 39 \text{ and } n \notin \{26, 32, 35, 38\}, \\ n+5 & \text{otherwise}. \end{cases}$$

Then by Theorem 1.2, we compute

$$\delta(f(n), C_4) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } 3 \le n \le 6, \\ 3 & \text{if } 7 \le n \le 39, \\ 4 & \text{otherwise}. \end{cases}$$

$$\delta(f(n)-1, C_4) = \begin{cases} 2 & \text{if } 2 \le n \le 6, \\ 3 & \text{if } 7 \le n \le 39 \text{ and } n \notin \{26, 27, 32, 33, 35, 36, 38, 39\}, \\ 4 & \text{otherwise}. \end{cases}$$

Since $\delta(G) + \Delta(\overline{G}) = |V(G)| - 1$ for any graph G, it follows that $\overline{\Delta}(f(n), C_4) = n + 1$ if $n \in \{26, 32, 35, 38\}$ and $\overline{\Delta}(f(n), C_4) = n$ otherwise, and so $PR(C_4, K_{1,n}) \le f(n)$. Moreover, $\overline{\Delta}(f(n) - 1, C_4) = n - 2$ if $n \in \{2, 27, 33, 36, 39\}$ and $\overline{\Delta}(f(n) - 1, C_4) = n - 1$ otherwise, and so $PR(C_4, K_{1,n}) \ge f(n)$. So we have the following corollary.

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Corollary 1.3. *Let $n \geq 2$ be an integer. Then $PR(C_4, K_{1,n}) = f(n)$.*

By Theorem 1.1 and Corollary 1.3, we can completely determine all the quadrilateral-tree planar Ramsey numbers.

2. Preliminaries

In this section, we first introduce some operations on graphs, then give several lemmas that will be used in the proof of Theorem 1.1.

For any $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S in G, and $G - S$ the graph obtained from G by deleting all the vertices of S. When $S = \{v\}$, we simplify $G - \{v\}$ to $G - v$. Let $G - uv$ denote the graph obtained from G by deleting the edge $uv \in E(G)$. For $X, Y \subseteq V(G)$, we define $(X, Y)_G = \{uv \in E(G) \mid u \in X, v \in Y\}$. Let $G[X, Y]$ be a bipartite graph with vertex set $X \cup Y$ and edge set $(X, Y)_G$.

In [1], Burr *et al.* considered Ramsey numbers for C_4 versus some special trees and obtained $R(C_4, P_n) = n + 1$ for $n \geq 3$ and $R(C_4, F) \leq 2(q + 1)$ for any forest F of size q without isolated vertices. Since $PR(G_1, G_2) \leq R(G_1, G_2)$, we have the following two results.

Lemma 2.1. *Let $n \geq 3$ be an integer. Then $PR(C_4, P_n) \leq n + 1$.*

Lemma 2.2. *Suppose that F is a forest of size q without isolated vertices. Then $PR(C_4, F) \leq 2(q + 1)$.*

The following 'folklore lemma' gives a sufficient condition for a graph to contain all trees of given order.

Lemma 2.3. *Let G be a graph with $\delta(G) \geq n - 1$. Then G contains all trees of order n.*

In 1935, Hall [4] gave a necessary and sufficient condition for the existence of a matching in a bipartite graph $G[X, Y]$ which covers every vertex in X.

Lemma 2.4 [4]. *A bipartite graph $G = G[X, Y]$ has a matching which covers every vertex in X if and only if $|N_G(S)| \geq |S|$ for all $S \subseteq X$, where $N_G(S)$ is the set of all neighbours of the vertices in S.*

3. Proof of Theorem 1.1

Let T_n be a tree of order $n \geq 3$ with maximum degree k, and let $x \in V(T_n)$ be a vertex of degree k. Set $p = \max\{n + 1, PR(C_4, K_{1,k})\}$.

First we show that p is a lower bound for $PR(C_4, T_n)$. Observe that $PR(C_4, T_n) \geq PR(C_4, K_{1,k})$. On the other hand, $PR(C_4, T_n) \geq n + 1$ because $K_{1,n-1}$ is a planar graph of order n without C_4 and there is no tree of order n in its complement. Therefore $PR(C_4, T_n) \geq \max\{n + 1, PR(C_4, K_{1,k})\}$.

Next we show by induction on n that $PR(C_4, T_n) \leq \max\{n + 1, PR(C_4, K_{1,k})\}$. If $n = 3$ or 4, then T_n is a path or a star and the result holds by Lemma 2.1. Assume that $n \geq 5$ and that the result holds for all smaller values of n. Let G be a planar graph of

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order p without C_4. We will show that $\overline{G}$ contains T_n. Since the result holds for paths and stars, we may assume that $3 \le k \le n-2$.

Let $v \in V(G)$ with $d_G(v) = \delta(G)$. Then v is a vertex of maximum degree in $\overline{G}$. Set $S = N_{\overline{G}}(v)$ and $\overline{\Delta} = |S|$. Since $p \ge PR(C_4, K_{1,k})$, we have $\overline{\Delta} \ge k$.

Case 1: $\overline{\Delta} \ge n$.

Let $F = T_n - x$. Then F is a forest of order $n-1$ with $n-k-1$ edges. If $n \le 2k$, then $2(n-k) \le n$. By Lemma 2.2, $\overline{G}[S]$ contains all forests of size $n-k-1$ without isolated vertices, so $\overline{G}[S]$ contains F. Since $\overline{\Delta} \ge n$ and v is adjacent to all the vertices in S in $\overline{G}$, it follows that $\overline{G}[S \cup \{v\}]$ contains T_n. Now assume that $n \ge 2k+1$. If $k \ge 4$, then $PR(C_4, K_{1,k}) \le k+5 \le 2k+1 \le n$ by Corollary 1.3. If $k = 3$, then $PR(C_4, K_{1,3}) = 6 < n$. So in either case, $PR(C_4, K_{1,k}) \le n$. Thus F is contained in a tree T' of order $n-1$ with $\Delta(T') \le k$. Consequently, $PR(C_4, F) \le PR(C_4, T') \le \max\{n, PR(C_4, K_{1,k})\} = n$ by the induction hypothesis. Thus $\overline{G}[S]$ contains F. Since $\overline{\Delta} \ge n$ and v is adjacent to all the vertices in S in $\overline{G}$, it follows that $\overline{G}[S \cup \{v\}]$ contains T_n.

Case 2: $\overline{\Delta} \le n-1$.

We consider two subcases according to the relationship between $\overline{\Delta}$ and k.

Subcase 2.1: $\overline{\Delta} \le 2(k-1)$.

Since $\overline{\Delta} \le 2(k-1)$, we have $\overline{\Delta} \ge 2(\overline{\Delta} - k + 1)$. By Lemma 2.2, $\overline{G}[S]$ contains all forests of size $\overline{\Delta} - k$ without isolated vertices. Let T' be a tree of order $\overline{\Delta} + 1$ $(\le n)$ obtained from T by successively deleting leaves other than those which are adjacent to x and let $F' = T' - x$. Then F' is a forest of order $\overline{\Delta}$ with $\overline{\Delta} - k$ edges. Thus, $\overline{G}[S]$ contains F'. Since v is adjacent to all the vertices in S in $\overline{G}$, it follows that $\overline{G}[S \cup \{v\}]$ contains T'. Since $p \ge n+1$ and G contains no C_4, the tree T' can be extended to T in $\overline{G}$.

Subcase 2.2: $\overline{\Delta} \ge 2k-1$.

In this case, $2k-1 \le \overline{\Delta} \le n-1$, so $2k \le n$. Let

$$W = \{v \in V(T_n) \mid v \text{ has a leaf neighbour of } T_n\}.$$

Since T_n is not a star, we have $|W| \ge 2$.

First we assume $|W| = 2$. Then T_n is a tree obtained from two disjoint stars by joining their centres with a path. Let T' be a tree obtained from T_n by deleting x and all leaf neighbours of x, and let T'' be a tree of order $\overline{\Delta} - k$ obtained from T' by successively deleting leaves (but keeping the vertex which is adjacent to x). Since T' is a tree of order $n-k$ and $\overline{\Delta} - k \le n-k-1$, we have $\Delta(T'') \le k-1$. Let T^* be the tree obtained from T'' by adding x and all leaf neighbours of x in T_n. Then $|V(T^*)| = \overline{\Delta}$.

Claim 1. $\overline{\Delta} \ge PR(C_4, T'')$.

Proof of Claim 1. For $3 \le k \le 4$, we have $PR(C_4, K_{1,k-1}) \le k+2 \le 2k-1 \le \overline{\Delta}$ by Corollary 1.3, so $PR(C_4, T'') \le \max\{\overline{\Delta} - k + 1, PR(C_4, K_{1,k-1})\} \le \overline{\Delta}$ by induction. For $k \ge 5$, we have $PR(C_4, K_{1,k-1}) \le k+4 \le 2k-1 \le \overline{\Delta}$ by Corollary 1.3, and again $PR(C_4, T'') \le \max\{\overline{\Delta} - k + 1, PR(C_4, K_{1,k-1})\} \le \overline{\Delta}$ by induction. □

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By Claim 1, $\overline{G}[S]$ contains T''. Since v is adjacent to all the vertices in S in $\overline{G}$, it follows that $\overline{G}[S \cup \{v\}]$ contains T^*. Since $p \geq n+1$ and G contains no C_4, we see that T^* can be extended to T_n in $\overline{G}$.

Now we assume $|W| \geq 3$. Let $a, b, c \in W$ and let a_1 be a leaf neighbour of a, b_1 a leaf neighbour of b and c_1 a leaf neighbour of c. Assume without loss of generality that a is a vertex of W with $d_T(a)$ as large as possible. Then $3 \leq k \leq n-3$. If $T' = T - \{a_1, b_1, c_1\}$, then T' is a tree of order $n-3$ with maximum degree at most k.

Claim 2. *$PR(C_4, T') \leq p-3$ unless $n = 7$, $p = 8$, $k = 3$ and T' is a $K_{1,3}$.*

Proof of Claim 2. For $k = n-3$, we have $n = 6$ because $2k \leq n$ and so $k = 3$ and $p = \max\{7, PR(C_4, K_{1,3})\} = 7$. In this case, T' is a $K_{1,2}$ and $PR(C_4, T') = 4 = p-3$. For $k = n-4$, we have $n = 7$ or 8 as $2k \leq n$. If $n = 7$, then $k = 3$ and

$$p = \max\{8, PR(C_4, K_{1,3})\} = 8.$$

In this case, T' is a $K_{1,3}$ or a P_4. If T' is a $K_{1,3}$, then $PR(C_4, T') = 6 = p-2$ by Corollary 1.3. If T' is a P_4, then $PR(C_4, T') = 5 = p-3$ by Lemma 2.1. If $n = 8$, then $k = 4$ and $p = \max\{9, PR(C_4, K_{1,4})\} = 9$. In this case, $a = x$ by the choice of a and T' is a tree of order 5 with maximum degree 3. Thus

$$PR(C_4, T') = \max\{6, PR(C_4, K_{1,3})\} = 6 = p-3$$

by induction. For $k \leq n-5$, we have $PR(C_4, K_{1,k}) = k+3 \leq n-2$ if $3 \leq k \leq 6$, and $PR(C_4, K_{1,k}) \leq k+5 \leq 2k-2 \leq n-2$ if $k \geq 7$. Then

$$PR(C_4, T') \leq \max\{n-2, PR(C_4, K_{1,k})\} = n-2 \leq p-3$$

by induction. □

If $\delta(\overline{G}) \geq n-1$, then $\overline{G}$ contains T_n by Lemma 2.3, so we assume $\delta(\overline{G}) \leq n-2$, and so $\Delta(G) \geq 2$. Let u be a vertex of maximum degree of G and let u_1, u_2 be two neighbours of u. Let $G' = G - \{u, u_1, u_2\}$. Then G' is a planar graph of order $p-3$ without C_4.

First we assume $\overline{G'}$ contains T'. Set $\{w\} = V(G') - V(T')$. Let $X = \{a, b, c\}$ and $Y = \{u, u_1, u_2, w\}$. Consider the bipartite graph $\overline{G}[X, Y]$. Note that G contains no C_4, and so $|N(S)| \geq |S|$ for any $S \subseteq X$. By Lemma 2.4, $\overline{G}[X, Y]$ has a matching covering every vertex in X. Then T' together with this matching is a T in $\overline{G}$.

Next assume $\overline{G'}$ contains no T'. Then $n = 7$, $p = 8$, $k = 3$ and T' is a $K_{1,3}$ by Claim 2. In this case, $V(T) = \{x, a, a_1, b, b_1, c, c_1\}$ and $E(T) = \{xa, xb, xc, aa_1, bb_1, cc_1\}$. Note that G' is a planar graph of order 5 without C_4, so $\Delta(\overline{G'}) = 2$. Let w be a vertex of maximum degree of $\overline{G'}$ and w_1, w_2 two neighbours of w in $\overline{G'}$. Set $\{w_3, w_4\} = V(G') - \{w, w_1, w_2\}$. Then $ww_3, ww_4 \in E(G)$. Since G contains no C_4, then $wu_1 \in E(\overline{G})$ or $wu_2 \in E(\overline{G})$, say $wu_1 \in E(\overline{G})$. Hence the subgraph induced by $\{wu_1, ww_1, ww_2\}$ is a $K_{1,3}$. Let $X = \{u_1, w_1, w_2\}$, $Y = \{u, u_2, w_3, w_4\}$. Consider the bipartite graph $\overline{G}[X, Y]$. Since G contains no C_4, we have $|N(S)| \geq |S|$ for any $S \subseteq X$. By Lemma 2.4, $\overline{G}[X, Y]$ has a matching covering every vertex in X. Then $K_{1,3}$ together with this matching is a T in $\overline{G}$.

This completes the proof of Theorem 1.1.

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