CERCLES DE REMPLISSAGE AND ASYMPTOTIC BEHAVIOUR ALONG CIRCUITOUS PATHS

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1. Introduction. In this paper we consider the value distribution of a meromorphic function whose behaviour is prescribed along a spiral. The existence of extremely wild holomorphic functions is established. Indeed a very weak form of one of our results would be that there are holomorphic functions (in the unit disc or the plane) for which every curve "tending to the boundary" is a Julia curve.

The theorems in this paper generalize results of Gavrilov [**7**], Lange [**9**], and Seidel [**11**].

I wish to express my thanks to Professor W. Seidel for his guidance and encouragement.

2. Preliminaries. For the most part we will be dealing with the metric space (D, ρ) where D is the unit disc, $|z| < 1$, and ρ is the non-Euclidean hyperbolic metric on D. The chordal metric on the Riemann sphere will be denoted by χ. For a subset $A \subset D$ and a non-negative number r, we write

$$\Delta(A, r) = \{z \in D: \rho(A, z) \leqq r\}.$$

If the set A is a singleton $A = \{z_0\}$, we write $\Delta(z_0, r)$ instead of $\Delta(\{z_0\}, r)$ and refer to $\Delta(z_0, r)$ as a disc.

Definition. Let $w = f(z)$ be a meromorphic function in D. A sequence of points $\{z_n\}$, $z_n \in D$, is called a sequence of ρ-points for the function $f(z)$ if there are sequences $\{L_n\}$ and $\{r_n\}$, where

(A) $L_1 > L_2 > \ldots > L_n > \ldots, L_n \to 0$, for $n \to \infty$,

(B) $r_1 > r_2 > \ldots > r_n > \ldots, r_n \to 0$, for $n \to \infty$,

such that the sequence $\{\Delta_n\}$ of discs, $\Delta_n = \Delta(z_n, r_n)$, in D has the following property:

(C) in each disc Δ_n, $n = 1, 2, \ldots$, the function $f(z)$ assumes all values of the Riemann sphere with the possible exception of two sets of values $E(n)$ and $G(n)$ whose chordal diameters do not exceed L_n.

If $\{z_n\}$ is a sequence of ρ-points for $f(z)$, then the associated sequence of discs $\{\Delta_n\}$ is called a sequence of *ρ-cercles de remplissage* for $f(z)$.

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THEOREM 1. *A sequence of points* $\{z_n\}$, $z_n \in D$, *is a sequence of* ρ*-points for a function* $f(z)$ *meromorphic in* D *if and only if the family*

$$f\left(\frac{z+z_n}{1+\bar{z}_n z}\right), \qquad n = 1, 2, \ldots, \tag{1}$$

is not normal at $z = 0$.

Proof. If $\{z_n\}$ is a sequence of ρ-points, then the family (1) is not normal at $z = 0$ since it is not even equicontinuous there.

The sufficiency follows from [**5**, Corollary 2].

It follows from Theorem 1 (see also [**4**, Theorem 3]) that a meromorphic function is normal in D if and only if it possesses no sequence of ρ-points.

For a subset $A \subset D$ and a value w_0 of the Riemann sphere, we say that a function $f(z)$ defined in D tends to w_0 in (or on) A if $f(z_n) \to w_0$ for each sequence $\{z_n\}$, $z_n \in A$, for which $|z_n| \to 1$. A boundary path α is a continuous curve $\alpha(t)$, $0 \leqq t < 1$, in D, for which $|\alpha(t)| \to 1$ as $t \to 1$. The end of α is the set of all points of the circumference $|z| = 1$ which are in the closure of α.

3. Meromorphic functions with asymptotic values.

THEOREM 2. *Let* $w = f(z)$ *be a non-constant meromorphic function which tends to a value along a boundary path* α *whose end contains more than one point. Then for each point* z_0 *in the end of* α, *there is a sequence of* ρ*-points whose limit is* z_0.

Proof. Suppose that z_0 is in the end of α and z_0 is not the limit of a sequence of ρ-points. It will suffice to show that in this case $f(z)$ is identically constant.

There is a number $r > 0$ such that $f(z)$ has no sequence of ρ-points in $G = \{z \in D: |z - z_0| < r\}$; for otherwise, by a diagonalization process, one could construct a sequence of ρ-points with limit z_0. Let $z = h(\xi)$ map the unit disc $|\xi| < 1$ conformally onto G. It follows from Pick's inequality [**8**, p. 239] that $f(h(\xi))$ has no sequence of ρ-points in $|\xi| < 1$ since $f(z)$ has no sequence of ρ-points in G. Hence $f(h(\xi))$ is a normal function in $|\xi| < 1$ which tends to a limit along a Koebe sequence of arcs. Bagemihl and Seidel [**1**] have shown that such a function is identically constant. This completes the proof.

Let α be a boundary path in D. We define $\mu'\alpha$ by the equation

$$\mu'\alpha = \inf\{\sup\{\rho(\alpha, z): r < |z| < 1\}: 0 < r < 1\}.$$

It turns out that if a meromorphic function tends to a value along a boundary path α for which $\mu'\alpha = 0$, then for any boundary sequence $\{z_n\}$, there are only two possibilities. Either the function is extremely well behaved near $\{z_n\}$ or extremely wildly behaved near $\{z_n\}$. Indeed we have the following result.

THEOREM 3. *Let* $w = f(z)$ *be a meromorphic function in* D *which tends to a value* w_0 *along a boundary path* α *for which* $\mu'\alpha = 0$. *Then for any boundary* *sequence* $\{z_n\}$ *in* D, *either* $\{z_n\}$ *is a sequence of* ρ-*points or for some* $r > 0$, $f(z)$ *tends to* w_0 *in* $\bigcup_{n=1}^{\infty} \Delta(z_n, r)$.

Proof. Let $\{z_n\}$ be an arbitrary boundary sequence, and let $f_n(z)$ be the function given by formula (1), $n = 1, 2, \ldots$. If for some $r > 0$, the family $\{f_n(z)\}$ is normal in $\Delta(0, r)$, then familiar arguments on normal families show that $f(z)$ tends to w_0 in $\bigcup_{n=1}^{\infty} \Delta(z_n, r)$.

If on the other hand the family $\{f_n(z)\}$ is not normal at $z = 0$, then we invoke Theorem 1. This completes the proof.

We define a spiral α in D to be a boundary path $\alpha(t)$, $0 \leqq t < 1$, for which $\arg \alpha(t)$ tends to $+\infty$ (or $-\infty$) as t tends to 1. Seidel [**11**] has introduced a non-negative number (possibly $+\infty$), denoted by $\bar{\mu}\alpha$, which indicates the "tightness" of a spiral α. It is easily seen that $\mu'\alpha = 0$ whenever $\bar{\mu}\alpha = 0$ and so we have the following corollary.

Corollary 1. *Let* $w = f(z)$ *be a meromorphic function in* D *which tends to a value* w_0 *along a spiral* α *for which* $\bar{\mu}\alpha = 0$. *Then the conclusion of Theorem* 3 *holds.*

Consider now the class $V(\alpha)$ of unbounded holomorphic functions in the disc D which are bounded on a spiral α. It is well known, [**2**, Lemma 1; **12**], that if $w = f(z)$ is in the class $V(\alpha)$, then there is a spiral α' along which $f(z)$ tends to the value ∞. Since $\bar{\mu}\alpha'$ cannot exceed $\bar{\mu}\alpha$, we have the following result.

Corollary 2. *If* $w = f(z)$ *is a holomorphic function in the class* $V(\alpha)$ *with* $\bar{\mu}\alpha = 0$, *then each boundary sequence is a sequence of* ρ-*points.*

All the classes of functions which are considered in this paper are non-empty [**3**, Theorem 1]. In particular, Corollary 2 establishes the existence of holomorphic functions whose behaviour is extremely wild.

4. Examples. So as not to interrupt the main line of thought, we have postponed giving examples until the present section.

For many questions, the tightness of a spiral α as measured by $\bar{\mu}\alpha$ is of great use. We wish to justify our introduction in § 3 of a further tightness measure $\mu'\alpha$. First of all $\bar{\mu}\alpha$ is defined only for spirals, yet there are boundary paths α which are not spirals and for which $\mu'\alpha = 0$. Moreover, our first example shows that there is a spiral α for which $\bar{\mu}\alpha = +\infty$ but $\mu'\alpha = 0$. Hence Theorem 3 is truly more general than its Corollary 1.

Example 1. Let $\{x_n\}$ be a sequence of points in D for which

$$0 < x_1 < x_2 < \ldots < x_n < \ldots < 1 \quad \text{and} \quad \rho(x_n, x_{n+1}) \to +\infty.$$

Let $I_n = (-x_n, x_n)$, $n = 1, 2, \ldots$. Let $K_1 = (-x_1, x_1)$, and let C_1 be the

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boundary of $\Delta(K_1, 1)$. Having defined $K_1, K_2, \ldots, K_n$ and $C_1, C_2, \ldots, C_n$, we now define K_{n+1} and C_{n+1}. Let $K_{n+1} = \Delta(K_n, 1/n) \cup I_{n+1}$, and let C_{n+1} be the boundary of $\Delta(K_{n+1}, 1/(n+1))$. $\{C_n\}$ is a sequence of simple closed curves which tend uniformly to the circumference $|z| = 1$. Hence one can construct a spiral α by appropriately joining subarcs of these closed curves. Moreover, α may be so constructed that $\mu'\alpha = 0$ and $\bar{\mu}\alpha = +\infty$.

The second example shows that we cannot relax the tightness of the spiral in Corollary 2 and still hope to retain the same conclusion.

Example 2. Let α be a spiral in D with monotonic argument and modulus and such that $0 < \mu\alpha = \bar{\mu}\alpha < +\infty$ (see [**11**] for a definition of $\mu\alpha$). Choose $r > 0$ such that $4r < \bar{\mu}\alpha$. Schneider has shown [**10**] that there is an unbounded holomorphic function $w = f(z)$ which is bounded on $\Delta(\alpha, r)$. There is of course no sequence of ρ-points on α.

We remark that Theorem 3 and its corollaries hold in the plane as well as in the disc. It suffices to replace the non-Euclidean metric by the Euclidean metric wherever the former occurs and to replace formula (1) by

$$f(z + z_n), \qquad n = 1, 2, \ldots.$$

The Bagemihl-Seidel existence theorem which we invoked also holds in the plane. Indeed, it follows that there exists an entire function $w = f(z)$ with the following property. For every sequence $\{z_n\}$ of points tending to ∞, there are sequences $\{L_n\}$ and $\{r_n\}$ tending to zero such that in each disc $(|z - z_n| < r_n)$, $f(z)$ assumes every value of the Riemann sphere with the possible exception of two sets of values $E(n)$ and $G(n)$ whose chordal diameters do not exceed L_n. A very weak consequence is that every path tending to ∞ is a Julia path for $f(z)$.

In the present investigation we have concerned ourselves with the "shape" of asymptotic paths. In a forthcoming paper [**6**] we will consider how the speed of asymptotic approach affects the distribution of values of a function.

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