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IDEALS IN SIMPLE RINGS

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Abstract

In this article, we define the concept of a Malcev ideal in an alternative ring in a manner analogous to Lie ideals in associative rings. By using a result of Kleinfield's we show that a nonassociative alternative ring of characteristic not 2 or 3 is a ring sum of Malcev ideals Z and $[R, R]$ where Z is the center of R and $[R, R]$ is a simple non-Lie Malcev ideal of R. If R is a Cayley algebra over a field F of characteristic 3 then $[R, R]$ is a simple 7 dimensional Lie algebra. A similar result is obtained if R is a simple associative ring.

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1. Introduction

Jacobson and Rickart (1950) defined a Lie ideal in an associative ring. I. N. Herstein developed the abstract structure theory of Lie ideals in associative rings in the early 1950's. These results can be found in Herstein (1969). In section 2 we show how a simple associative ring may be decomposed into a sum of Lie ideals.

The concept of a Malcev ring goes back to A. I. Malcev (1955). A. A. Sagle (1961) developed the basic structure theory of Malcev rings. In section 3 the concept of a Malcev ideal in an alternative ring is defined in a manner analogous to a Lie ideal in an associative ring. Then we prove that: if R is a simple alternative ring which is not associative and of characteristic not 2 or 3, then $R = Z + [R, R]$ is a ring sum of Malcev ideals Z and $[R, R]$ where Z is the center of R.

2. Decomposition of a simple ring into Lie ideals

Let A be an associative ring. We form a new ring A^- called a *Lie ring* by introducing a new multiplication

$$[x, y] = xy - yx \quad \text{for } x, y \text{ in } A.$$

Z will denote the center of the ring A and if B and C are subsets of A then $[B, C]$ will denote the subgroup of A generated by the set $\{bc - cb \mid b \in B, c \in C\}$. If A is also a vector space we define $[B, C]$ to be the subspace spanned by all commutators $[b, c]$ for b in B and for c in C.

For completeness we collect the most essential results to be used below. A subset U of an associative ring A is called a *Lie ideal* of A provided

(i) U is an additive subgroup of A

and

(ii) $[x, y] = xy - yx \in U$ for x in U and y in A.

Hence U is a Lie ideal of A if and only if U is an ideal of the Lie ring A^-. If S is a subset of A define

$$S^L = \{x \in A \mid [x, u] = 0 \quad \text{for all } u \in S\}.$$

From this definition we see that if B and C are subsets of an associative ring A and $B \subset C$, then $C^L \subset B^L$.

LEMMA 1. *If U is a Lie ideal in A, then U^L is a Lie ideal.*

PROOF. If x is in U^L, then $xu = ux$ for u in U. Expanding commutators and using $xu = ux$, we see that $[[x, y], u] = [x, [y, u]]$ for y in A. $[y, u] \in U$ implies $[x, [y, u]] = 0$. So $[x, y] \in U^L$.

THEOREM 2. (Herstein) *Let A be a simple associative ring of characteristic not 2 and let U be a Lie ideal of A. Then either $U \subset Z$ the center of A or $U = [A, A]$.*

See Herstein (1969), Chapter 1 Theorem 1.12 for a proof of this result.

THEOREM 3. *Let U be a non-zero Lie ideal of a simple associative ring A which contains Z and suppose that if $U \supset U^L$ implies $U = A$. Then $U = A$ or $U = Z$.*

PROOF. Since U is a Lie ideal, Theorem 2 implies that $U \subset Z$ or $U = [A, A]$. Hence we need only consider the case where $U = [A, A]$ and show that $U = A$. Since U^L is a Lie ideal of A, $U^L = [A, A]$ or $U^L \subset Z \subset U$. If $U^L \subset U$ then by our hypotheses we have that $U = A$.

For the case $U^L = [A, A]$ we see that $U^L = U$ using $U = [A, A]$. In particular, $U^L \subset U$ and so by hypothesis, $U = A$, that is $A = [A, A]$.

THEOREM 4. *Let A be a simple associative ring in which any Lie ideal U*

that has the property that $U \supset U^L$ implies $U = A$. Then one of the following holds:

(i) $A = [A, A] + Z$
(ii) $A = [A, A]$
(iii) $A = Z$.

Proof. *Case 1*. Assume that $Z \subset [A, A]$. Letting $U = [A, A]$ which is a Lie ideal of A, Theorem 3 implies conditions (ii) or (iii) holds.
Case 2. Suppose now that $Z \not\subset [A, A]$ and let $T = [A, A] + Z$, a ring sum of the ideals Z and $[A, A]$. T is a Lie ideal of A. Since T^L is a Lie ideal, we have $T^L = [A, A]$ or $T^L \subset Z \subset T$ by Theorem 2. If $T^L \subset T$, then $T = A$ by hypothesis. If $T^L = [A, A]$, then since $T = [A, A] + Z$ we have $T^L \subset T$. So in this case we also have $T = A$.

Example 5. Let A be the algebra of quaternions over a field F of characteristic not 2. See Lam (1973), Chapter 3 for the properties of the quaternion algebra. As an algebra A is a 4-dimensional vector space over F with basis l, i, j, k. Lam (1973), Chapter 3 Proposition 1.1 gives that A is a simple algebra and has $F = F.1$ as its center Z. $[A, A]$ is a simple Lie ideal of A which does not contain Z. Hence we see that $A = Z + [A, A]$ as a ring direct sum of minimal Lie ideals Z and $[A, A]$.

3. Malcev ideals in alternative rings

Max Zorn in 1930 defined an alternative ring as a ring in which the associator $(x, y, z) = (xy)z - x(yz)$ changes sign on the interchange of two of its arguments. This is equivalent to the following identity (See Sagle (1971)):

$$x^2y = x(xy);\ yx^2 = (yx)x \quad \text{for } x, y \in A.$$

Sagle (1971) defines a *Malcev ring* A by the following identities:

(i) $x^2 = 0$ for each x in A

and

(ii) $(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$

for all x, y, z in A. In an alternative ring A we form a new ring A^- by introducing a new multiplication:

$$[x, y] = xy - yx \quad \text{for } x, y \text{ in } A.$$

Theorem 6. (Malcev) *If A is an alternative ring, then A^- is a Malcev ring.*

Theorem 7. (Sagle) *Any Lie ring A is a Malcev ring.*

THEOREM 8. *Let A be an alternative ring of characteristic 2 or 3. Then the Malcev ring A^- obtained from A is a Lie ring.*

THEOREM 9. (Kleinfield) *A simple alternative ring R which is not of characteristic 3 is either a Cayley–Dickson algebra over its center or associative.*

The proofs of Theorem 6 and 8 may be found in Davenport (1971), Chapter 1 Theorem 2.5 and 2.6 respectively. Theorem 7 is given in Sagle (1961), p. 429 and Theorem 9 is the main result of Kleinfield (1957), p. 399.

DEFINITION 10. A subset A of a ring R is a *Malcev subring* of R if A is an additive subgroup of R such that $[x, y] \in A$ for each x, y in R.

DEFINITION 11. A subset U of a Malcev subring A of a ring R is called a *Malcev ideal* of A provided U is an additive subgroup of A and $[u, x] \in U$ for u in U and x in A.

Hence we see that U is a Malcev ideal of a ring R if and only if U is a Malcev ideal of the Malcev subring R^-.

Schafer (1966), Chapter 1 gives a multiplication table for a Cayley algebra C over a field F of characteristic unequal to 2. We introduce a new multiplication in C by the commutator $[x, y] = xy - yx$ for each $x, y \in C$. Define $v_1 = u_1$ and $v_i = u_i/2$ for $i = 2, \cdots, 8$. Thus we obtain the following multiplication table.

Multiplication table for the Malcev ring C^-
when C is a Cayley algebra

Note: The remainder of the table may be computed by using $[v_i v_j] = -[v_j v_i]$ for $i, j \in \{1, 2, \cdots, 8\}$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	0	0	0	0	0	0	0	0
v_2		0	$-v_4$	$-\alpha v_3$	$-v_6$	$-\alpha v_5$	v_8	αv_7
v_3			0	βv_2	$-v_7$	$-v_8$	$-\beta v_5$	$-\beta v_6$
v_4				0	$-v_8$	$-\alpha v_7$	βv_6	$\alpha\beta v_5$
v_5					0	γv_2	γv_3	γv_4
v_6						0	$-\gamma v_4$	$-\alpha\gamma v_3$
v_7							0	$\beta\gamma v_2$
v_8								0

THEOREM 12. *If R is a Cayley algebra over a field F of characteristic not equal to 2 or 3, then $[R, R]$ is a simple non-Lie Malcev ring, and for characteristic equal to 3 $[R, R]$ is a simple Lie algebra.*

PROOF. Clearly $v_1 \notin [R, R]$. Let $I \neq (0)$ be an ideal in $[R, R]$. Suppose $x \in I$ where $x = \sum_{i=2}^{8} x_i v_i$, and some $x_i \neq 0$.

If $x_2 \neq 0$, a simple calculation shows the following:

$$xv_2 = \sum_{i=1}^{8} x_i (v_i v_2)$$

$$= x_3 v_4 + x_4 \alpha v_3 + x_5 v_6 + x_6 \alpha v_5 - x_7 v_8 - x_8 \alpha v_7$$

$$(xv_1)v_1 = x_2 \alpha v_2 + x_3 \alpha v_3 + x_4 \alpha v_4 + x_5 \alpha v_5 + x_6 \alpha v_6 + x_7 \alpha v_7.$$

Thus, $x_2 \alpha v_2 = \alpha x - (xv_2)v_2 \in I$, since I is an ideal. Hence $v_2 \in I$. In a similar manner, the following equations can be shown:

$$\beta x_3 v_3 = \beta x - (xv_3)v_3, \quad \text{if } x_3 \neq 0;$$

$$\alpha\beta x_4 v_4 = (xv_4)v_4 + 4\alpha\beta x, \quad \text{if } x_4 \neq 0;$$

$$\gamma x_5 v_5 = \gamma x - (xv_5)v_5, \quad \text{if } x_5 \neq 0;$$

$$\alpha\gamma x_6 v_6 = (xv_6)v_6 + \alpha\gamma x, \quad \text{if } x_6 \neq 0;$$

$$\beta\gamma x_7 v_7 = (xv_7)v_7 + \beta\gamma x, \quad \text{if } x_7 \neq 0;$$

and

$$\beta\gamma x_8 v_8 = \alpha\beta\gamma x - (xv_8)v_8, \quad \text{if } x_8 \neq 0.$$

In either case, $x_i \neq 0$ implies $v_i \in I$. Thus $I \neq (0)$ implies $v_i \in I$ for some i. A simple calculation gives that $v_i \in I$ for $i \in \{2, \cdots, 8\}$ implies $v_2, v_3, \cdots, v_8 \in I$. Therefore, $I = [R, R]$, and it follows that $[R, R]$ is a simple Malcev ring. If the characteristic of F is 3, then $[R, R]$ is a Lie algebra by Theorem 8.

A brief calculation gives that

$$J(v_2, v_4, v_5) = (v_2 v_4)v_5 + (v_4 v_5)v_2 + (v_5 v_2)v_4 = 3\alpha v_7.$$

Thus $J(v_2, v_4, v_5) \neq 0$ and it follows that $[R, R]$ is a non-Lie ideal of R.

THEOREM 13. *Let R be a simple alternative ring of characteristic not equal to 2 or 3 which is not associative. Then $R = Z + [R, R]$ as a ring sum of Malcev ideals Z, the center of R, and $[R, R]$ is a simple non-Lie Malcev ideal of R.*

PROOF. Theorem 9 implies that R is a Cayley–Dickson algebra over its center. Clearly the center Z of R is a Malcev ideal of R. The center and the nucleus of a Cayley algebra are the same. $[R, R]$ is a simple non-Lie Malcev ring by Theorem 7. So $[R, R]$ is a minimal Malcev ideal of R. Thus $R = Z + [R, R]$.

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