Some new operational representations

By P. Humbert, Université de Montpellier.

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In this paper I give some operational representations, according to the Carson-Heaviside Calculus, which, as far as I know, are new; and I deduce some properties of the functions thus introduced.

§ 1. *The representation of* $\tan^{-1} x$.

It is known that the symbolic image of the sine-integral, si (x), is $\tan^{-1} p$, so that

$$\tan^{-1} p \doteqdot \operatorname{si}(x).$$

Now the question is, what is the symbolic image of $\tan^{-1} x$? We obtain it as follows.

We have, if x is small,

$$\tan^{-1} x = \Sigma (-1)^n \frac{x^{2n+1}}{2n+1}.$$

But, if x is small, p is large; and we can write symbolically

$$\tan^{-1} x \doteqdot \Sigma \frac{(-1)^n (2n+1)!}{(2n+1)\, p^{2n+1}} = \frac{1}{p} \Sigma \frac{(-1)^n (2n)!}{p^{2n}},$$

according to the well-known rules of the operational calculus: this formula will hold for p large.

Now if we consider the sine- and cosine-integrals under Nielsen's form, we have, if p is large,

$$\operatorname{si}(p) = -\frac{\cos p}{p} \Sigma \frac{(-1)^n (2n)!}{p^{2n}} - \frac{\sin p}{p} S,$$

$$\operatorname{ci}(p) = \frac{\sin p}{p} \Sigma \frac{(-1)^n (2n)!}{p^{2n}} - \frac{\cos p}{p} S,$$

when S is another series. We have then, eliminating S,

$$\frac{\operatorname{ci}(p)}{\cos p} - \frac{\operatorname{si}(p)}{\sin p} = (\operatorname{tg} p + \cot p) \Sigma \frac{(-1)^n (2n)!}{p^{2n}}$$

so that, when x is small, we obtain the following representation:

$$\tan^{-1} x \doteqdot \operatorname{ci}(p) \sin p - \operatorname{si}(p) \cos p.$$

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§ 2. *The representation of* ber (x) *and* bei (x).

Let us consider now Kelvin's functions ber (x) and bei (x) defined by

$$J_0(xi\sqrt{i}) = \text{ber}(x) + i\,\text{bei}(x),$$

where J_0 is the Bessel function of order zero. It has been proved by van der Pol that

$$\text{ber}(2\sqrt{x}) \doteqdot \cos 1/p$$
$$\text{bei}(2\sqrt{x}) \doteqdot \sin 1/p.$$

I propose to obtain the images of ber (x) and bei (x) by considering the image of $J_0(x)$, viz.

$$J_0(x) \doteqdot \frac{p}{\sqrt{(p^2+1)}}.$$

Writing $xi\sqrt{i}$ instead of x, and separating the real and imaginary parts on the right-hand side, we find that

$$\text{ber}(x) \doteqdot \frac{p}{\sqrt{2}}\left\{\frac{1}{\sqrt{(p^4+1)}} + \frac{p^2}{p^4+1}\right\}^{1/2},$$
$$\text{bei}(x) \doteqdot \frac{p}{\sqrt{2}}\left\{\frac{1}{\sqrt{(p^4+1)}} - \frac{p^2}{p^4+1}\right\}^{1/2}.$$

Some (probably) new properties of Kelvin's functions can be deduced from these operational representations. For instance, using the product theorem, we have

$$2\int_0^x \text{ber}(x-y)\,\text{bei}(y)\,dy \doteqdot \frac{p}{p^4+1}.$$

But

$$\sin \lambda x \doteqdot \frac{p\lambda}{p^2+\lambda^2},$$

$$\sinh \lambda x \doteqdot \frac{p\lambda}{p^2-\lambda^2};$$

hence

$$\sinh \lambda x - \sin \lambda x \doteqdot \frac{2p\lambda^3}{p^4-\lambda^4}.$$

If we now take $\lambda = \sqrt{i}$, we obtain

$$\sinh x\sqrt{i} - \sin x\sqrt{i} \doteqdot 2i\sqrt{i}\,\frac{p}{p^4+1},$$

and so

$$\int_0^x \text{ber}(x-y)\,\text{bei}(y)\,dy = \frac{\sinh x\sqrt{i} - \sin x\sqrt{i}}{4i\sqrt{i}}.$$

Again, from the operational formula

$$\frac{f(p)}{p} \doteqdot \int_0^x h(u)\,du,$$

we have

$$\int_0^x \operatorname{bei}(u)\,du \doteqdot \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{(p^4+1)}} - \frac{p^4}{p^4+1}\right\}^{1/2}.$$

If we denote the right-hand side by $f(p)$, we have

$$f(\sqrt{p}) = \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{(p^2+1)}} - \frac{p^2}{p^2+1}\right\}^{1/2}.$$

But we know that

$$f(\sqrt{p}) \doteqdot \sqrt{2}\,S(x)$$

where $S(x)$ is one of Fresnel's integrals, namely

$$S(x) = \int_0^x \frac{\sin u}{\sqrt{(2\pi u)}}\,du.$$

Now an operational rule states that, if

$$f(p) \doteqdot h(x),$$

then

$$f(\sqrt{p}) \doteqdot \frac{1}{\sqrt{\pi x}}\int_0^\infty \exp\left(-\tfrac{1}{4}s^2/x\right) h(s)\,ds$$

and we obtain the following formula, connecting Fresnel's integral with Kelvin's function

$$S(x) = \frac{1}{\sqrt{(2\pi x)}}\int_0^\infty \exp\left(-\tfrac{1}{4}s^2/x\right)\int_0^s \operatorname{bei}(u)\,du\,ds.$$

§ 3. *Bourget's extended Bessel functions.*

Bourget's extended Bessel function is

$$J_{n,k}(x) = \frac{1}{\pi}\int_0^\pi (2\cos\theta)^k \cos(n\theta - x\sin\theta)\,d\theta,$$

reducing, for $k = 0$, to the Bessel function J_n. The following properties of this function are well known:

(1) $$J_{n,0}(x) = J_n(x)$$

(2) $$J_{n,1}(x) = \frac{2n}{x}J_n(x)$$

(3) $$J_{n,k} = J_{n-1,k-1} + J_{n+1,k-1}$$

(4) $$J_{n,k}(0) = N_{-n,k,0},$$

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this being the Cauchy-number, which is equal to zero if $k < n$. From these we shall deduce the representation of Bourget's function and some new properties.

I shall prove that

$$J_{n,k}(x) \doteqdot 2^k p\,(p^2+1)^{(k-1)/2}\{\sqrt{(p^2+1)}-p\}^n.$$

For

(i) Making $k=0$, we obtain

$$p\,(p^2+1)^{-1/2}\{\sqrt{(p^2+1)}-p\}^n,$$

which is the image of $J_n(x)$.

(ii) Making $k=1$, we obtain

$$2p\,\{\sqrt{(p^2+1)}-p\}^n,$$

which is the image of $(2n/x)\,J_n(x)$.

(iii) If we write $n-1$ or $n+1$ instead of n, $k-1$ instead of k, we see readily that the above recurrence formula is verified.

Now, from this operational representation, we shall obtain some new formulae for Bourget's functions.

(a) We have

$$2^k p\,(p^2+1)^{(k-1)/2}\{\sqrt{(p^2+1)}-p\}^n$$
$$=\frac{2^k}{p}\,p(p^2+1)^{(\lambda-1)/2}\{\sqrt{(p^2+1)}-p\}^{\mu}\,p(p^2+1)^{(r-1)/2}\{\sqrt{(p^2+1)}-p\}^{s-1}$$

where $\lambda+r=k+1$

$\mu+s=n.$

Hence, by the product theorem,

$$J_{n,k}(x)=\frac{1}{2}\int_0^x J_{\mu,\lambda}(x-y)\,J_{n-\mu,\,k+1-\lambda}(y)\,dy$$

λ and μ being arbitrary numbers.

As a particular case, we can take $\lambda=0$, and deduce that

$$J_{n,k}(x)=\frac{1}{2}\int_0^x J_{\mu}(x-y)\,J_{n-\mu,\,k+1}(y)\,dy.$$

Similarly, with $\lambda=1$,

$$J_{n,k}(x)=\mu\int_0^x \frac{J_{\mu}(x-y)}{x-y}\,J_{n-\mu,\,k}(y)\,dy.$$

We also have the two following formulae, giving the *duplication* of the indices,

$$J_{2n,\,2k}(x) = \frac{1}{2}\int_0^x J_{n,\,k}(x-y)\,J_{n,\,k+1}(y)\,dy$$

$$J_{2n,\,k}(x) = \frac{1}{2}\int_0^x J_{n,\,k+1}(x-y)\,J_n(y)\,dy.$$

(*b*) The image of Bourget's function can also be written as

$$\frac{1}{p}\;2^k p\,(p^2+1)^{(k-1)/2}\{\sqrt{(p^2+1)}-p\}^{n-\nu}\,p\,\{\sqrt{(p^2+1)}-p\}^{\nu};$$

hence

$$J_{n,\,k}(x) = \nu\int_0^x J_{n-\nu,\,k}(x-y)\,\frac{J_\nu(y)}{y}\,dy$$

and, in a particular case,

$$J_{2n,\,k}(x) = n\int_0^x J_{n,\,k}(x-y)\,\frac{J_n(y)}{y}\,dy.$$

(*c*) Let us again write the image

$$\frac{1}{p}\;2^k p\,(p^2+1)^{(\lambda-1)/2}\{\sqrt{(p^2+1)}-p\}^n\,\frac{p}{(p^2+1)^{(2\mu+1)/2}}$$

with $\lambda - 2\mu - 1 = k$. We know that

$$\frac{p}{(p^2+1)^{(2\mu+1)/2}} \doteqdot \frac{\sqrt{\pi}}{2^\mu\,\Gamma(\mu+\frac{1}{2})}\,x^\mu J_\mu(x).$$

So we obtain

$$J_{n,\,k}(x) = \frac{\sqrt{\pi}}{2^{3\mu+1}\,\Gamma(\mu+\frac{1}{2})}\int_0^x J_{n,\,k+2\mu+1}(x-y)\,y^\mu J_\mu(y)\,dy,$$

where μ is an arbitrary number, 2μ being an integer.

If we take $\mu = 0$, we have

$$J_{n,\,k}(x) = \frac{1}{2}\int_0^x J_{n,\,k+1}(x-y)\,J_0(y)\,dy,$$

and with $\mu = \frac{1}{2}$,

$$J_{n,\,k}(x) = \frac{1}{4}\int_0^x J_{n,\,k+2}(x-y)\sin y\,dy.$$

(*d*) Let us now write, without giving the demonstration, which is easy when one is accustomed to Heaviside's methods and images, the following two formulae of differentiation with respect to the indices. They hold only for $k < n$, for, in this case, the value of $J_{n,\,k}(0)$ is zero, and the calculus is rather simple:

$$\frac{\partial J_{n,\,k}(x)}{\partial n} = \int_0^x \frac{\partial J_{n,\,k}(y)}{\partial y}\,Ji_0(x-y)\,dy$$

where Ji_0 is the Bessel-integral function

$$Ji_0(x) = -\int_x^\infty \frac{J_0(u)}{u}\,du;$$

$$\frac{\partial J_{n,k}(x)}{\partial k} = J_{n,k}(x)\log 2 - \int_0^x \operatorname{ci}(x-y)\,\frac{\partial J_{n,k}(y)}{\partial y}\,dy$$

where ci is the cosine-integral.

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