Appendix H
Pairing in a single *j*-shell

H.1 BCS solution

We shall discuss some of the consequences of pairing correlations in the case of particles moving in a single *j*-shell. The number of degenerate pair levels $(\nu, \bar{\nu})$ which can be accommodated in the shell is

$$\Omega = \frac{2j+1}{2}. \tag{H.1}$$

The value of the occupation numbers V_ν and U_ν must be the same for all the orbitals. In particular, the occupation probability of the level when the system is occupied with N particles is $N/2\Omega$. Consequently,

$$V_\nu = V = \sqrt{\frac{N}{2\Omega}} \tag{H.2}$$

and

$$U_\nu = U = \sqrt{1 - \frac{N}{2\Omega}}, \tag{H.3}$$

in keeping with the fact that $U_\nu^2 + V_\nu^2 = 1$. Making use of the above relation one finds

$$\begin{aligned}\Delta &= G \sum_{\nu>0} U_\nu V_\nu = G\Omega U V \\ &= \frac{G}{2}\sqrt{N(2\Omega - N)}.\end{aligned} \tag{H.4}$$

The pairing gap thus achieves its maximum value for the system with $N = \Omega$ particles (half-filled shell), in keeping with the fact that, owing to the degeneracy of the levels, pairs of particles and hole states are equivalent as far as pairing correlations are concerned (see Fig. H.1).

Making use of the condition $H_{20} = 0$, i.e.

$$2(\varepsilon_\nu - \lambda)U_\nu V_\nu = \Delta(U_\nu^2 - V_\nu^2), \tag{H.5}$$

https://doi.org/10.1017/9781009401920.020 Published online by Cambridge University Press

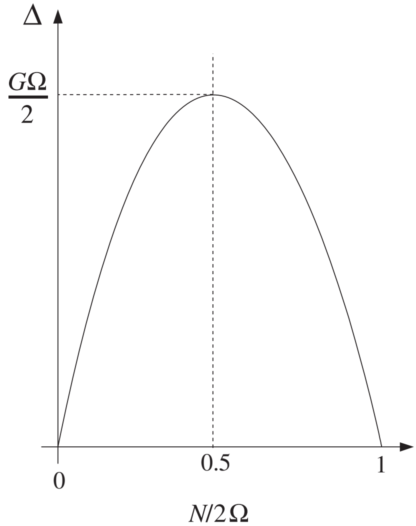

Figure H.1. Schematic representation of the pairing gap as a function of the number of particles (see equation (H.4)).

and assuming $\varepsilon_\nu = \varepsilon = 0$, one obtains

$$-\frac{2\lambda}{2\Omega}\sqrt{N(2\Omega - N)} = \frac{G}{2}\sqrt{N(2\Omega - N)}\frac{1}{\Omega}(\Omega - N), \qquad \text{(H.6)}$$

thus leading to

$$\lambda = -\frac{G}{2}(\Omega - N). \qquad \text{(H.7)}$$

Let us now calculate the ground-state energy

$$E_0 = U + \lambda N = 2\sum_{\nu>0} \varepsilon_\nu V_\nu^2 - \frac{\Delta^2}{G}.$$

Consequently,

$$E_0 = -\frac{\Delta^2}{G} = -\frac{G^2}{4}\frac{1}{G}N(2\Omega - N)$$
$$= -\frac{G\Omega}{2}N + \frac{G}{4}N^2. \qquad \text{(H.8)}$$

Assuming $\Omega \gg N$ one obtains from equation (H.7)

$$\lambda \approx -\frac{G\Omega}{2}$$

and equation (H.8) can be rewritten as

$$E_0 \approx \lambda N + \frac{G}{4}N^2. \qquad \text{(H.9)}$$

https://doi.org/10.1017/9781009401920.020 Published online by Cambridge University Press

Interpreting the second term in equation (H.8) or (H.9) as that corresponding to a rotor in two dimensions with moment of inertia

$$\frac{\hbar^2}{2\mathcal{I}} = \frac{G}{4} \tag{H.10}$$

or

$$\frac{\mathcal{I}}{\hbar^2} = \frac{2}{G} \tag{H.11}$$

we finally write

$$E_0 \approx \lambda N + \frac{\hbar^2}{2\mathcal{I}} N^2, \tag{H.12}$$

$$\left.\frac{\partial E_0}{\partial N}\right|_{N=0} = \lambda, \tag{H.13}$$

$$\frac{\partial^2 E_0}{\partial N^2} = \frac{\hbar^2}{\mathcal{I}}. \tag{H.14}$$

Making use of the estimate given in equation (2.27) (see also end of Section 2.5) for G and $A \approx 100$ (Sn-isotopes), we obtain $\hbar^2/2\mathcal{I} \approx 0.07$ MeV. This result is very close to the value needed to fit the experimental data (see Fig. 4.2, where the pairing rotational band is fitted with a parabola whose quadratic term is 0.1 MeV N^2).

In the single *f*-shell model, the quasiparticle energy is given by

$$\begin{aligned} E_\nu &= \sqrt{(\varepsilon_\nu - \lambda)^2 + \Delta^2} \\ &= \left[\frac{G^2}{4}(\Omega - N)^2 + \frac{G^2}{4} N(2\Omega - N)\right]^{1/2} \\ &= \frac{G}{2}\left[\Omega^2 - 2\Omega N + N^2 + 2\Omega N - N^2\right]^{1/2}, \end{aligned}$$

$$E_\nu = E = \frac{G\Omega}{2}. \tag{H.15}$$

H.2 Cranking moment of inertia

The cranking formula of the moment of inertia associated with pairing rotations (rotations in gauge space) is

$$\begin{aligned} \mathcal{I} &= 2\hbar^2 \sum_{\nu>0} \frac{|< \nu\bar{\nu}|N_\nu|\mathrm{BCS}>|^2}{2E_\nu} \\ &= \sum_{\nu} \frac{|< \nu\bar{\nu}|\hbar N_\nu|\mathrm{BCS}>|^2}{2E_\nu}. \end{aligned} \tag{H.16}$$

Making use of the relation

$$\begin{aligned} N_\nu &= a_\nu^\dagger a_\nu + a_{\bar{\nu}}^\dagger a_{\bar{\nu}} \\ &= (U_\nu^2 - V_\nu^2)(\alpha_\nu^\dagger \alpha_\nu + \alpha_{\bar{\nu}}^\dagger \alpha_{\bar{\nu}}) + 2U_\nu V_\nu (\alpha_\nu^\dagger \alpha_{\bar{\nu}}^\dagger + \alpha_{\bar{\nu}} \alpha_\nu) + 2V_\nu^2 \end{aligned}$$

one obtains

$$\langle \nu\bar{\nu}\,|N_\nu|\,\mathrm{BCS}\rangle = 2U_\nu V_\nu$$

leading to

$$\frac{\mathcal{I}}{\hbar^2} = 4\sum_{\nu>0}\frac{U_\nu^2 V_\nu^2}{E_\nu} = \sum_{\nu>0}\frac{\Delta^2}{E_\nu^3}. \tag{H.17}$$

Inserting (H.2), (H.3) and (H.15) into equation (H.17) one obtains

$$\begin{aligned}\frac{\mathcal{I}}{\hbar^2} &= 4\Omega\frac{N}{2\Omega}\left(1-\frac{N}{2\Omega}\right)\\ &= \frac{4N}{G\Omega}\left(1-\frac{N}{2\Omega}\right).\end{aligned} \tag{H.18}$$

Setting $N = \Omega$,

$$\frac{\mathcal{I}}{\hbar^2} = \frac{2}{G}, \tag{H.19}$$

which coincides with the result shown in equation (H.11).

Note that

$$\frac{(\hbar^2/2\mathcal{I})}{(G\Omega/2)} = \frac{1}{2\Omega},$$

implying that collective pairing rotations have much lower energy than two-quasiparticle excitation.

H.3 Two-particle transfer

The transfer operator is

$$\begin{aligned}P^\dagger &= \sum_{\nu>0} a_\nu^\dagger a_{\bar{\nu}}^\dagger\\ &= \sum_{\nu>0}\Big(U_\nu^2\alpha_\nu^\dagger\alpha_{\bar{\nu}}^\dagger - U_\nu V_\nu\left(\alpha_\nu^\dagger\alpha_\nu + \alpha_{\bar{\nu}}^\dagger\alpha_{\bar{\nu}}\right)\\ &\quad - V_\nu^2\alpha_{\bar{\nu}}\alpha_\nu + U_\nu V_\nu\Big).\end{aligned} \tag{H.20}$$

Consequently

$$\left\langle \mathrm{BCS}\left|P^\dagger\right|\mathrm{BCS}\right\rangle = \sum_{\nu>0} U_\nu V_\nu = \frac{\Delta}{G}, \tag{H.21}$$

and the two-particle transfer cross-section can be written as

$$\sigma(\mathrm{gs}\to\mathrm{gs}) \approx \left(\frac{\Delta}{G}\right)^2 = \left(\frac{12}{\sqrt{A}}\frac{A}{28}\right)^2 \approx \frac{A}{4}. \tag{H.22}$$

https://doi.org/10.1017/9781009401920.020 Published online by Cambridge University Press

On the other hand

$$\langle \nu\bar{\nu} \left| P^{\dagger} \right| BCS \rangle = U_{\nu}^{2} \approx 1, \tag{H.23}$$

leading to

$$\sigma(\text{gs} \rightarrow 2\text{qp}) \approx U_{\nu}^{4} \approx 1. \tag{H.24}$$

From the above equations one obtains

$$R = \frac{\sigma(\text{gs} \rightarrow \text{gs})}{\sigma(\text{gs} \rightarrow 2\text{qp})} \approx \frac{A}{4}. \tag{H.25}$$

For Sn-isotopes ($A \approx 100$) one thus expects

$$R = 25 \qquad \text{(BCS model)}. \tag{H.26}$$

Making use of the experimental results displayed in Fig. 4.2 one can calculate the average value of the ten observed two-particle transfer cross-sections connecting the members of the Sn-ground-state pairing rotational band ($64 \leq N \leq 76$), normalized to the $^{116}_{50}$Sn(gs) $\leftrightarrow$ $^{118}_{50}$Sn(gs) (p, t) and (t, p) cross-sections. One obtains,

$$\begin{aligned} &\sigma(\text{gs} \rightarrow \text{gs})_{\text{exp}} \\ &= \frac{1.3 + 1.2 + 1.0 + 1.3 + 1.5 + 1.1 + 1.4 + 1.1 + 1.2 + 1.1}{10} = 1.22. \end{aligned} \tag{H.27}$$

Similarly, the calculation of the average of the six two-particle (relative) cross-sections connecting members of the ground-state pairing rotational band to members of the two-quasiparticle (2qp) pairing vibrational bands leads to

$$\sigma(\text{gs} \rightarrow 2\text{qp})_{\text{exp}} = \frac{0.04 + 0.03 + 0.04 + 0.06 + 0.05 + 0.08}{6} = 0.05. \tag{H.28}$$

Consequently,

$$R_{\text{exp}} \approx \frac{1.22}{0.05} \approx 24.4, \tag{H.29}$$

essentially as predicted by theory (see also (2.58)).

H.4 Polarization effects

In the following we summarize in simple terms the results obtained in sub-section 10.4.1. The relation in equation (H.4) with $N = \Omega$ leads to

$$\Delta = \frac{1}{2} G\Omega. \tag{H.30}$$

We are particularly concerned with the role of polarization effects on the renormalization of the value of the pairing gap in a superfluid nucleus like e.g. ^{120}Sn.

We shall call G_b and Ω_b the bare pairing strength and degeneracy (closely related to the density of levels) associated with an effective mass equal to the k-mass ($m_k \approx 0.7$ m) (see equations (8.20) and (8.21)). From the results displayed in Figs. 8.6, 8.9 and 10.1

https://doi.org/10.1017/9781009401920.020 Published online by Cambridge University Press

one can write

$$\frac{1}{2}G_b\Omega_b = 0.5\Delta_{\text{exp}}, \tag{H.31}$$

$$\frac{1}{2}G_b\Omega_d = 1.4\Delta_{\text{exp}}, \tag{H.32}$$

and

$$\frac{1}{2}g_{\text{p-v}}\Omega_d \approx 0.8\Delta_{\text{exp}}, \tag{H.33}$$

where Ω_d is the effective (dressed) degeneracy arising from the coupling of single-particle motion to collective vibrations (ω-effective mass, see Section 9.2), while $g_{\text{p-v}}$ is the induced pairing interaction due to the exchange of vibrations between pairs of nucleons moving in time-reversal states close to the Fermi energy (see Section 9.3).

In keeping with the results displayed in Fig. 10.16, one can also write

$$\frac{1}{2}G_d\Omega_d = \Delta_{\text{exp}}, \tag{H.34}$$

where G_d is the dressed pairing interaction. Because the density of levels is proportional to the ω-mass (see discussion end of Section 9.1.1 as well as equation (9.23)), one can write

$$\Omega_d \approx \frac{\Omega_b}{Z_\omega}, \tag{H.35}$$

where $Z_\omega = (m_\omega/m)^{-1}$ (see also Section 9.3).

Due to the coupling to vibrations, nucleons spend part of the time in more complicated configurations than pure single-particle states (see Fig. 9.2). The factor Z_ω measures the content of single-particle strength present in levels around the Fermi energy available to nucleons to interact through a (pairing) force and correlate, eventually giving rise to a superfluid system. In the case of the dressed pairing coupling constant, one then can write the expression

$$G_d = Z_\omega^2(G_b + g_{\text{p-v}}). \tag{H.36}$$

Making use of this relation and of equation (H.34) one can write

$$\frac{1}{2}G_d\Omega_d = Z_\omega\frac{1}{2}G_b\Omega_b + Z_\omega\frac{1}{2}g_{\text{p-v}}\Omega_b. \tag{H.37}$$

The above relation implies that, without considering the contribution of the induced pairing interaction to the dressed pairing gap, the increase of the density of levels arising from the coupling of nucleons to collective vibrations is overcompensated by the reduction in the single-particle content of these levels, the net result being a decrease of the pairing gap (from the minimum value it can have in the static mean-field approximation, i.e. $\frac{1}{2}G_b\Omega_b$). On the other hand, relations (H.32) and (H.33) imply

$$g_{\text{p-v}} \approx 0.6G_b. \tag{H.38}$$

Summing up, taking into account the renormalization effects leading to an ω-mass, and thus to an increase of the density of levels, one has to consider, at the same time, the actual single-particle strength in the levels lying close to the Fermi energy.

https://doi.org/10.1017/9781009401920.020 Published online by Cambridge University Press

Note that a proper treatment of the dressing of single-particle states not only involves the Z_ω-coefficients (arising from $\Delta E = \mathrm{Re}\Sigma$), but also the splitting of the single-particle strength (arising from $-2\mathrm{Im}\Sigma$, see Section 9.1, equation (9.11), see also (9.14)). This last effect leads to a further reduction of the ability of time reversal single-particle states to participate in Cooper pair formation (see equation (9.41)).