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ON SUBSEQUENTIAL LIMIT POINTS OF A SEQUENCE OF ITERATES. II

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Abstract

J. B. Diaz and F. T. Metcalf established some results concerning the structure of the set of cluster points of a sequence of iterates of a continuous self-map of a metric space. In this paper it is shown that their conclusions remain valid if the distance function in their inequality is replaced by a continuous function on the product space. Then this idea is extended to some other mappings and to uniform and general topological spaces.

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Diaz and Metcalf [5, 6] have studied the structure of the set of subsequential limit points of a sequence of iterates of a continuous self-map A of a metric space (X, d) satisfying the condition $d(Ax, F(A)) < d(x, F(A))$, where $x \neq Ax$ and $F(A)$, the set of fixed points of A, is nonempty and compact. In this paper it is shown that the conclusions of Diaz and Metcalf [6] may still be derived after replacing d by a continuous function $\varphi: X \times X \to R_0$, where R_0 is the subspace $[0, \infty)$ of the real line with usual topology. Then our analysis is extended to the mappings introduced by Dotson [7], Browder and Petryshyn [2, 3], Singh and Zorzitto [9] and Caristi [4]. Then we show that our results may be carried over to uniform spaces and, further, that some of our conclusions hold in Hausdorff topological spaces.

In what follows $\varphi(x, F(A))$ will be used to denote $\inf_{y \in F(A)}(x, y)$. The orbit of $x \in X$ generated by A will be denoted by $O(x, A)$ and its closure by $\overline{O}(x, A)$. The set of subsequential limit points of the sequence $\{A^n x\}_{n=0}^{\infty}$ will be denoted by $\mathscr{L}(x)$.

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We now establish the generalization of Theorem 2 of Diaz and Metcalf [6].

THEOREM 1. *Let A be a continuous self-map of a metric space (X, d). Suppose that*

(i) *$F(A)$ is nonempty and compact.*

(ii) *there exists a non-negative continuous function φ: $X \times X \to R_0$ such that $\varphi(Ay, F(A)) < \varphi(y, F(A))$ for $y \in X - F(A)$,*

(iii) *$\bar{O}(x, A)$ is compact for some $x \in X$:*

Then $\mathscr{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathscr{L}(x)$ is a singleton or is uncountable. In the case $\mathscr{L}(x)$ is a singleton, $\lim_{n\to\infty} A^n x$ exists and belongs to $F(A)$. In the case $\mathscr{L}(x)$ is uncountable, it is contained in the boundary of $F(A)$.

PROOF. The compactness of $\bar{O}(x, A)$ implies nonemptiness of $\mathscr{L}(x)$. We now show that $\mathscr{L}(x) \subset F(A)$. If some iterate $A^k x \in F(A)$, we have $\mathscr{L}(x) = \{A^k x\} \subset F(A)$ and the theorem is proved. Therefore we assume that $A^k x \notin F(A)$ for $k = 0, 1, 2, \ldots$. Since for any fixed y, $z \to \varphi(z, y)$ is a continuous function from $X \to R_0$, the function $z \to \varphi(z, F(A))$ is an upper semi-continuous function, being the infimum of a family of continuous functions. Since $A^k x \notin F(A)$ for all k, we have $\varphi(A^{k+1}x, F(A)) < \varphi(A^k x, F(A))$ for all k. Therefore $\{\varphi(A^k x, F(A))\}_{k=0}^{\infty}$ is a monotonically decreasing sequence of nonnegative real numbers and so will converge to $r \geqslant 0$, say. Since $\mathscr{L}(x) \neq \varnothing$, for a $\xi \in \mathscr{L}(x)$ there exists a subsequence $\{A^{n_i}x\}_{i=1}^{\infty}$ with $A^{n_i}x \to \xi$ as $i \to \infty$. If $A\xi = \xi$, then we are through. Therefore we assume that $\xi \neq A\xi$. Then $r = \lim \varphi(A^{1+n_i}x, F(A)) = \limsup \varphi(A^{1+n_i}x, F(A)) \leqslant \varphi(\lim A^{1+n_i}x, F(A)) = \varphi(A\xi, F(A)) < \varphi(\xi, F(A))$. Now $y \to \varphi(A^n x, y)$, for fixed $A^n x$, is a continuous function: $X \to R_0$ and so will attain its infimum on $F(A)$. Therefore there exists a $p_n \in F(A)$ such that $\varphi(A^n x, F(A)) = \varphi(A^n x, p_n)$. Corresponding to each $A^{n_i}x$ of the convergent subsequence $\{A^{n_i}x\}_{i=1}^{\infty}$ we have a $p_{n_i} \in F(A)$. Since $F(A)$ is compact, $\{p_{n_i}\}_{i=1}^{\infty}$ will have a convergent subsequence denoted by $\{p_{m_i}\}_{i=1}^{\infty}$ converging to q, say, in $F(A)$. Now $A^{m_i}x \to q$. From $\varphi(A^{n+1}x, p_{n+1}) = \varphi(A^{n+1}x, F(A)) < \varphi(A^n x, F(A)) = \varphi(A^n x, p_n)$, we have, for $m_i > n$, $\varphi(A^{m_i}x, p_{m_i}) < \varphi(A^n x, p_n)$. Letting $m_i \to \infty$, we have, since φ is continuous, $\varphi(\xi, q) < \varphi(A^n x, p_n)$. Since $\varphi(A^n x, p_n) = \varphi(A^n x, F(A)) \to r \geqslant 0$, we have $\varphi(\xi, q) \leqslant r$. Thus $r \leqslant \varphi(A\xi, F(A)) < \varphi(\xi, F(A)) \leqslant \varphi(\xi, q) \leqslant r$. This is absurd. Therefore $\xi = A\xi$, that is, $\xi \in F(A)$. In other words $\mathscr{L}(x) \subset F(A)$. Further, $\mathscr{L}(x)$, being a closed subset of the compact set $F(A)$, is compact.

We now prove that $\mathscr{L}(x)$ is connected. Suppose the contrary. Then there exist two nonempty, disjoint, closed subsets S_1, S_2 of $\mathscr{L}(x)$ such that $\mathscr{L}(x) = S_1 \cup S_2$.

Since S_1 and S_2 are closed subsets of a compact set $\mathscr{L}(x)$, they are themselves compact. Hence $d(S_1, S_2) > 0$. Next, we show that $d(A^m x, F(A)) \to 0$ as $m \to \infty$. If not, there exists an $\varepsilon > 0$ and a subsequence $\{A^{m_i}x\}_{i=1}^\infty$ such that $d(A^{m_i}x, F(A)) \geqslant \varepsilon > 0$ for $i = 1, 2, 3, \ldots$. Since $\overline{O}(x, A)$ is compact, $\{A^{m_i}x\}_{i=1}^\infty$ will have a subsequence $\{A^{n_i}x\}_{i=1}^\infty \to \xi \in F(A)$. Thus $d(A^{n_i}x, F(A)) \leqslant d(A^{n_i}x, \xi) \to 0$ as $i \to \infty$. This is a contradiction. Therefore we must have $d(A^m x, F(A)) \to 0$ as $m \to \infty$. We next prove that $\lim_{m\to\infty} d(A^m x, S_1 \cup S_2) = 0$. If it is not so, then there will exist an $\varepsilon > 0$ and a subsequence $\{A^{m_i}x\}_{i=1}^\infty$ such that $d(A^{m_i}x, S_1 \cup S_2) \geqslant \varepsilon > 0$ for $i = 1, 2, 3, \ldots$. Since $F(A)$ is compact, there exists a $q_{m_i} \in F(A)$ such that $d(A^{m_i}x, F(A)) = d(A^{m_i}x, q_{m_i})$. Because of the compactness of $F(A)$, $\{q_{m_i}\}_{i=1}^\infty$ will have a convergent subsequence $\{q_{n_i}\}_{i=1}^\infty$ with $q_{n_i} \to q \in F(A)$. Now $d(A^{n_i}x, q) \leqslant d(A^{n_i}x, q_{n_i}) + d(q_{n_i}, q) \to 0$. Hence $q \in \mathscr{L}(x) = S_1 \cup S_2$ and $d(A^{n_i}x, S_1 \cup S_2) \leqslant d(A^{n_i}x, q) \to 0$ as $i \to \infty$. This contradiction shows that $\lim_{m\to\infty} d(A^m x, S_1 \cup S_2) = 0$. We further prove that A is asymptotically regular. If not, there will exist an $\varepsilon > 0$ and a subsequence $\{A^{m_i}x\}_{i=1}^\infty$ such that $d(A^{m_i}x, A^{1+m_i}x) \geqslant \varepsilon > 0$. The corresponding sequence $\{q_{m_i}\}_{i=1}^\infty$ in $F(A)$ will have a subsequence $\{q_{n_i}\}_{i=1}^\infty$ converging to $q \in F(A)$. As above $A^{n_i}x \to q \in F(A)$. Since A is continuous at q, we have $A^{1+n_i}x \to A_q = q$. Now $d(A^{n_i}x, A^{1+n_i}x) \leqslant d(A^{n_i}x, q) + d(A^{1+n_i}x, q) \to 0$ as $i \to \infty$. This is contrary to hypothesis. Hence we have proved that $d(A^m x, A^{m+1}x) \to 0$ as $i \to \infty$. Thus from the results found in this paragraph, we can find an integer M such that for $m \geqslant M$, $d(A^m x, A^{m+1}x) < \frac{1}{3}d(S_1, S_2)$ and $d(A^m x, S_1, S_2) < \frac{1}{3}d(S_1, S_2)$. Since $S_1 \cup S_2$ is compact there exists a $q \in S_1 \cup S_2$ such that $d(A^m x, S_1 \cup S_2) = d(A^m x, q)$. If $q \in S_1$, then $d(A^m x, S_1) \leqslant d(A^m x, q) < \frac{1}{3}d(S_1, S_2)$. Therefore for any $m \geqslant M$, either $d(A^m x, S_1) < \frac{1}{3}d(S_1, S_2)$ or, $d(A^m x, S_2) < \frac{1}{3}d(S_1, S_2)$. Both these inequalities cannot hold simultaneously, because in that case $d(S_1, S_2) \leqslant d(S_1, A^m x) + d(S_2, A^m x) < \frac{2}{3}d(S_1, S_2)$ which is absurd. Now it is clear that the set of positive integers $m \geqslant M$ for which $d(A^m x, S_1) < \frac{1}{3}d(S_1, S_2)$ is nonempty, since $\varnothing \neq S_1 \subset \mathscr{L}(x)$. Similarly the set of positive integers $m \geqslant M$ for which $d(A^m x, S_2) < \frac{1}{3}d(S_1, S_2)$ is also nonempty. Let, for $m_1 > M$, $d(A^{m_1}x, S_1) < \frac{1}{3}d(S_1, S_2)$. There exist integers $n > m_1$ such that $d(A^n x, S_2) < \frac{1}{3}d(S_1, S_2)$. Let $k + 1$ be the least such integer. Then $d(A^{k+1}x, S_2) < \frac{1}{3}d(S_1, S_2)$ and $d(A^k x, S_1) < \frac{1}{3}d(S_1, S_2)$. We have

$$d(S_1, S_2) < d\big(S_1, A^k x\big) + d\big(A^k x, A^{k+1}x\big) + d\big(A^{k+1}x, S_2\big)$$
$$< \tfrac{1}{3}d(S_1, S_2) + \tfrac{1}{3}d(S_1, S_2) + \tfrac{1}{3}d(S_1, S_2).$$

This is absurd. Therefore the hypothesis that $\mathscr{L}(x) = S_1 \cup S_2$ with S_1 and S_2 nonempty, disjoint, closed subsets of $\mathscr{L}(x)$ leads to a contradiction. Hence $\mathscr{L}(x)$ is connected.

By Theorem 1 in Berge [1, p. 96] it follows that $\mathscr{L}(x)$ is either a singleton or is uncountable. We have proved above that $\lim_{m\to\infty} d(A^m x, \mathscr{L}(x)) = 0$, so that when $\mathscr{L}(x)$ is a singleton $\{\xi\}$, say, $\lim_{m\to\infty} d(A^m x, \xi) = 0$. Thus $\lim_{m\to\infty} A^m x = \xi \in F(A)$.

To prove that $\mathscr{L}(x)$, if uncountable, lies on the boundary of $F(A)$, we observe that in this case $A^k x \notin F(A)$, $k = 0, 1, 2, \ldots$. If $\xi \in \mathscr{L}(x) \subset F(A)$ is an interior point of $F(A)$, $A^k x \in F(A)$ for some k, as $F(A)$ is a neighborhood of ξ and some subsequence of $\{A^m x\}$ converges to ξ. This is a contradiction.

REMARK 1. We observe here that Theorem 2 of Diaz and Metcalf [6] is a corollary of our theorem if we replace φ by d. For this we have only to show that when $\mathscr{L}(x)$ is nonempty, the assumptions of Diaz and Metcalf imply the compactness of $\overline{O}(x, A)$. This has been shown in [8]. The following example shows that our theorem is indeed a generalization of the theorem of Diaz and Metcalf.

Take $X = \{a, b, c, d, e\}$ with the metric $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Take the mapping $A\colon X \to X$ such that $Aa = b$, $Ab = c$, $Ac = d$, $Ad = e = Ae$. Here $F(A) = \{e\}$ and it is easy to see that for $x \neq Ax$, $d(Ax, F(A)) < d(x, F(A))$ is not satisfied for $x = a, b$ or c. Therefore we cannot invoke the theorem of Diaz and Metcalf to show that $\mathscr{L}(a)$ is a closed, connected subset of $F(A)$. We now define a function $\varphi\colon X \times X \to R_0$ the schematic representation of which is given by

	a	b	c	d	e
a	0	4	5	9	10
b	40	0	3	6	8
c	50	30	0	2	7
d	90	60	20	0	1
e	100	80	70	10	0

where the value of $\varphi(x, y)$ occurs at the intersection of the row containing x with the column containing y. We have

$$\begin{aligned} 1 &= \varphi(d, F(A)) = \varphi(Ac, F(A)) < \varphi(c, F(A)) = 7 \\ &= \varphi(Ab, F(A)) < \varphi(b, F(A)) = 8 = \varphi(Aa, F(A))\varphi(a, F(A)) = 10. \end{aligned}$$

Further φ is continuous on $X \times X$ because it has the discrete topology. Also $F(A)$ is nonempty and compact and so is $\overline{O}(a, A)$. Thus we may invoke our theorem to show that $\mathscr{L}(a)$ is a nonempty, compact and connected subset of $F(A)$.

REMARK 2. Suppose, in addition to the hypotheses of Theorem 1, that $F(A)$ is an at most countable set. In this case $\lim_{m\to\infty} A^m x$ exists and belongs to $F(A)$, because $\mathscr{L}(x)$ is a singleton here and $\lim_{m\to\infty} d(A^m x, \mathscr{L}(x)) = 0$.

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COROLLARY 1. *Let $A: X \to X$ be such that A^k is continuous for some k. Suppose*
(i) *$F(A^k)$ is nonempty and compact,*
(ii) *for each $x \in X$, $\overline{O}(x, A^k)$ is compact,*
(iii) *there exists a continuous real-valued function $\varphi: X \times X \to R_0$ such that for $x \in X - F(A^k)$, $\varphi(A^k x, F(A^k)) < \varphi(x, F(A^k))$.*

Then, for $x \in X$, the set $\mathscr{L}_k(x)$ of subsequential limit points of the sequence of iterates $\{A^{mk}x\}_{m=1}^{\infty}$ is a nonempty, compact and connected subset of $F(A^k)$. Further the set $\mathscr{L}_1(x)$ of subsequential limit points of the sequence of iterates $\{A^m x\}_{m=1}^{\infty}$ is the union of the k nonempty, compact, connected subsets $\mathscr{L}_k(A^j x)$, $j = 0, 1, 2, \dots, k - 1$.

The proof is omitted because it is a minor modification of the proof of Theorem 2^k of Diaz and Metcalf [6].

COROLLARY 2. *Suppose, in addition to the hypotheses of Corollary 1, that $F(A^k)$ is an at most countable set. Then for $x \in X$, $\mathscr{L}_1(x)$ contains at most k points. This is because each $\mathscr{L}_k(A^j x)$ is a singleton.*

Dotson [7] calls a mapping A quasi-nonexpansive if $F(A) \neq \varnothing$ and for each $x \in X - F(A)$, $p \in F(A)$, $d(Ax, p) \leqslant d(x, p)$. We call A *quasi-contractive* if the strict inequality sign holds. The concept of quasi-contractiveness has been discussed by Diaz and Metcalf [6]. We define the mapping A to be φ-quasi-nonexpansive if $F(A) \neq \varnothing$ and for $x \in X - F(A)$, $p \in F(A)$ we have $\varphi(Ax, p) \leqslant \varphi(x, p)$ where $\varphi: X \times X \to R_0$. We say that A is φ-quasi-contractive if the strict inequality sign holds. In this connection we now prove

THEOREM 2. *Let $A: X \to X$ be a continuous self-map of a metric space (X, d). Suppose that A is φ-quasi-contractive, where φ is a continuous function from $X \times X \to R_0$. Then $\mathscr{L}(x) \subset F(A)$. If $\varphi(x, y) = 0 \Leftrightarrow x = y$, then $\mathscr{L}(x)$ consists of at most one point. If $\overline{O}(x, A)$ is compact in addition, then $\lim_{m\to\infty} A^m x$ exists and belongs to $F(A)$.*

PROOF. If $\mathscr{L}(x)$ is empty there is nothing to prove. Therefore we shall assume that $\mathscr{L}(x) \neq \varnothing$ and that $A^k x \notin F(A)$, $k = 0, 1, 2, \dots$, as in Theorem 1. Then for any $p \in F(A)$, the sequence of positive numbers $\{\varphi(A^n x, p)\}$ is monotonically decreasing, because $\varphi(A^{n+1}x, p) < \varphi(A^n x, p)$ by hypothesis. Hence $\lim_{n\to\infty} \varphi(A^n x, p)$ exists and is $r \geqslant 0$. Let $\xi \in \mathscr{L}(x)$ and let the subsequence $\{A^{n_i}x\}_{i=1}^{\infty}$ converge to ξ. If possible, let $\xi \neq A\xi$. Now

$$r = \lim_{i\to\infty} \varphi(A^{1+n_i}x, p) = \varphi\Big(\lim_{i\to\infty} A^{1+n_i}x, p\Big) = \varphi(A\xi, p)$$
$$< \varphi(\xi, p) = \varphi\Big(\lim_{i\to\infty} A^{n_i}x, p\Big) = \lim_{i\to\infty} \varphi(A^{n_i}x, p) = \lim_{i\to\infty} \varphi(A^n x, p) = r.$$

This contradiction proves that $\xi = A\xi$ and so $\xi \in F(A)$. We have thus proved that $\mathscr{L}(x) \subset F(A)$. Obviously $\mathscr{L}(x)$ is closed.

Assume now that $\varphi(x, y) = 0$ if and only if $x = y$. Let $p, q \in \mathscr{L}(x) \subset F(A)$. Obviously $\varphi(A^m x, p) \to 0$ as $m \to \infty$. If the subsequence $\{A^{m_i}x\}_{i=1}^{\infty}$ converges to q, then $\varphi(A^{m_i}x, p) \to \varphi(p, q)$. Hence $\varphi(q, p) = 0$ so that $p = q$. Therefore $\mathscr{L}(x)$ can consist of at most one point. If $\overline{O}(x, A)$ is compact, then $\mathscr{L}(x)$ is obviously nonempty and so is a singleton. Let $\mathscr{L}(x) = \{p\}$. If $A^n x \nrightarrow p$ as $n \to \infty$ then for some $\varepsilon > 0$ there exists a subsequence $\{A^{n_i}x\}_{i=1}^{\infty}$ with $d(A^{n_i}x, p) \geqslant \varepsilon > 0$. The compactness of $\overline{O}(x, A)$ implies the existence of a subsequence of $\{A^{n_i}x\}_{i=1}^{\infty}$ converging to p as $\mathscr{L}(x) = \{p\}$. This contradicts our hypothesis that $d(A^{n_i}x, p) \geqslant \varepsilon$. Hence $d(A^n x, p) \to 0$ as $n \to \infty$, implying $A^n x \to p$.

Browder and Petryshyn [2, 3] define a self-map A of a Banach space to be asymptotically regular if $A^{n+1}x - A^n x \to 0$ strongly as $n \to \infty$. We shall say that a mapping A is φ-asymptotically regular if $\varphi(A^n x, A^{n+1}x) \to 0$ as $n \to \infty$. We are now in a position to give our

THEOREM 3. *Let A be a continuous self-map of a metric space (X, d). Suppose*

(i) *$F(A)$ is nonempty and compact,*

(ii) *there exists a continuous function φ: $X \times X \to R_0$ such that $\varphi(y, z) = 0$ if and only if $y = z$,*

(iii) *A is φ-asymptotically regular,*

(iv) *$\overline{O}(x, A)$ is compact.*

Then $\mathscr{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathscr{L}(x)$ is singleton or uncountable. In the case $\mathscr{L}(x)$ is a singleton $\lim_{n\to\infty} A^n x$ exists and belongs to $F(A)$. In the case $\mathscr{L}(x)$ is uncountable it is contained in the boundary of $F(A)$.

PROOF. If $A^k x \in F(A)$ for some k, then the proof is trivial. Therefore, assume $A^k x \notin F(A)$ for all k. The sequence $\{\varphi(A^m x, F(A))\}_{m=1}^{\infty}$ is non-increasing and bounded below by zero and so converges to $r \geqslant 0$. Since $\overline{O}(x, A)$ is compact, $\mathscr{L}(x) \neq \varnothing$. Let $\xi \in \mathscr{L}(x)$ with $A^{m_i}x \to \xi$ as $i \to \infty$. Then $\varphi(A^{m_i}x, A^{1+m_i}x) \to \varphi(\xi, A\xi) = 0$, since A is φ-asymptotically regular. Hence $\xi = A\xi$ and $\xi \in F(A)$. Therefore $\mathscr{L}(x) \subset F(A)$. Obviously $\mathscr{L}(x)$ is closed. Since $F(A)$ is compact and $\mathscr{L}(x)$ is closed, $\mathscr{L}(x)$ itself is compact. In view of the proof of Theorem 1, to

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prove that $\mathscr{L}(x)$ is connected we need prove only $d(A^m x, F(A)) \to 0$ as $m \to \infty$ and this follows from the compactness of $\overline{O}(x, A)$ and the fact that $\mathscr{L}(x) \subset F(A)$. The remaining part of the proof is as in Theorem 1.

We may relax the compactness conditions on $F(A)$ by assuming A to be φ-quasi-nonexpansive. This we state as

THEOREM 4. *Let A be a continuous self-map of a metric space (X, d). Suppose*

(i) *$F(A)$ is nonempty,*

(ii) *A is φ-asymptotically regular where φ is a continuous function: $X \times X \to R_0$ and $\varphi(y, z) = 0$ if and only if $y = z$.*

Then $\mathscr{L}(x) \subset F(A)$. If, in addition, A is φ-quasi-nonexpansive, then $\mathscr{L}(x)$ consists of at most one point. If $\overline{O}(x, A)$ is compact, then $\lim_{m\to\infty} A^n x = p$, where $\mathscr{L}(x) = \{p\}$

PROOF. The fact that A is φ-asymptotically regular and vanishes only on the diagonal shows that $\mathscr{L}(x) \subset F(A)$. If A is φ-quasi-nonexpansive and $p, q \in \mathscr{L}(x)$ with $p \neq q$, then $\varphi(A^n x, q) \to r \geqslant 0$. Also there exist subsequences $\{A^{m_i}x\}$, $\{A^{n_j}x\}$ such that $A^{m_i}x \to p$, $A^{n_j} \to q$. Hence $\varphi(A^{m_i}x, A^{n_j}x) \to \varphi(p, q)$. Keeping i fixed and letting $j \to \infty$, we have $\varphi(A^{m_i}x, A^{n_j}x) \to \varphi(A^{m_i}x, q)$. We can extract a subsequence $\{m'_j\}$ from $\{m_i\}$ such that $m'_j > n_j$. Since $\lim_{i\to\infty} \varphi(A^{m_i}x, q) = r \geqslant 0$, we have,

$$r = \lim_{i\to\infty} \varphi(A^{m_i}x, q) = \lim_{m'_j\to\infty} \varphi(A^{m'_j}x, q) \leqslant \lim_{n_j\to\infty} \varphi(A^{n_j}x, q) = \varphi(q, q) = 0.$$

But $\varphi(A^{m_i}x, q) \to \varphi(p, q)$. Therefore $\varphi(p, q) = 0$ whence $p = q$. Thus $\mathscr{L}(x)$ consists of at most one point. If $\overline{O}(x, A)$ is compact, then $\mathscr{L}(x)$ is nonempty and so $\mathscr{L}(x) = \{p\}$, say. Now, proceeding as in Theorem 2 we can show that $\lim_{n\to\infty} A^n x = p$.

We now take the range of A to be compact and derive

THEOREM 5. *Let A: $X \to X$ be continuous. Suppose*

(i) *$A(X)$ is compact,*

(ii) *A is φ-asymptotically regular where φ is a continuous function: $X \times X \to R_0$ and $\varphi(x, y) = 0$ if and only if $x = y$.*

Then, for $x \in X$, the set $\mathscr{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathscr{L}(x)$ contains exactly one point or is uncountable. In the case $\mathscr{L}(x)$ is a singleton, $\lim_{m\to\infty} A^m x$ exists and belongs to $F(A)$. In the case $\mathscr{L}(x)$ is uncountable, it is contained in the boundary of $F(A)$.

PROOF. Since A is continuous, $F(A)$ is closed and so is compact as $F(A) \subset A(X)$, which is compact. Since $\overline{O}(Ax, A) \subset A(X)$, we have $\overline{O}(Ax, A)$ is compact.

Condition (ii) now implies that $\mathscr{L}(Ax) \subset F(A)$. But $\mathscr{L}(x) = \mathscr{L}(Ax)$. Hence $\mathscr{L}(x) \subset F(A)$. The compactness of $\overline{O}(Ax, A)$ implies that $d(A^n x, F(A)) \to 0$ as $n \to \infty$. The remaining conclusions can be derived as in Theorem 1.

Following the idea of Singh and Zorzitto [9] we have

THEOREM 6. *Let A be a continuous self-map of a metric space (X, d). Suppose*

(i) *$F(A)$ is nonempty and compact*

(ii) *there exists a continuous function $\varphi\colon X \times X \to R_0$ such that $\varphi(y, z) = 0$ if and only if $y = z$ and for $y \in X - F(A)$, $\varphi(Ay, F(A)) \leqslant \varphi(y, F(A))$ and $\varphi(A^m y, F(A)) < \varphi(y, F(A))$ for an integer $m = m(y)$,*

(iii) *$\overline{O}(x, A)$ is compact.*

Then $\mathscr{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathscr{L}(x)$ is a singleton or is uncountable. In the case $\mathscr{L}(x)$ is a singleton, $\lim_{m\to\infty} A^m x$ exists and belongs to $F(A)$. In the case $\mathscr{L}(x)$ is uncountable it is contained in the boundary of $F(A)$.

PROOF. Since $\overline{O}(x, A)$ is compact, $\mathscr{L}(x)$ is nonempty. It is enough to prove that $\mathscr{L}(x) \subset F(A)$. The remaining portion of the proof can be derived as in Theorem 1. Assume $A^k x \notin F(A)$ for all k. Let $\xi \in (x)$. If possible let $\xi \neq A\xi$. Hence there exists a subsequence $\{A^{n_i}x\}$ of $\{A^n x\}$ such that $A^{n_i}x \to \xi$. Obviously, $\lim_{n\to\infty} \varphi(A^n x, F(A))$ exists and is equal to $r \geqslant 0$. Now $r = \lim_{i\to\infty} \varphi(A^{m(\xi)+n_i}x, F(A)) \leqslant \varphi(\lim_{i\to\infty} A^{m(\xi)+n_i}x, F(A)) = \varphi(A_\xi^{m(\xi)}, F(A)) < \varphi(\xi, F(A))$. Proceeding as in Theorem 1, we can show that $\varphi(\xi, F(A)) \leqslant r$. Thus $r \leqslant \varphi(A^{m(\xi)}\xi, F(A)) < \varphi(\xi, F(A)) \leqslant r$, which is absurd. Thus $\xi = A\xi$ and $\mathscr{L}(x) \subset F(A)$.

Corresponding to Theorem 2 we state the following theorem without proof as it can be derived by combining the methods of Theorem 2 and Theorem 6.

THEOREM 7. *Let $A\colon X \to X$ be a continuous self-map of a metric space (X, d). Suppose that $F(A) \neq \varnothing$ and A is φ-quasi-inonexpansive where $\varphi\colon X \times X \to R_0$ is continuous. Assume further that for $y \in X - F(A)$ and $p \in F(A)$ there exists an integer $m = m(y, p)$ such that $\varphi(A^m y, p) < \varphi(y, p)$. Then for $x \in X$, $\mathscr{L}(x) \subset F(A)$. If $\varphi(y, z) = 0 \Leftrightarrow y = z$, then, for any $x \in X$, $\mathscr{L}(x)$ consists of at most one point. If, in addition, $\overline{O}(x, A)$ is compact then $\lim_{m\to\infty} A^m x$ exists and belongs to $F(A)$.*

Now we shall use conditions similar to those of Caristi [4] to derive the same conclusions as those of Diaz and Metcalf [6].

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THEOREM 8. *Let A be a self-map of a metric space (X, d). Suppose*

(i) *A is continuous at each pont of $F(A)$,*

(ii) *$F(A)$ is nonempty and compact,*

(iii) *there exists a function $\psi\colon X \to R_0$ such that for $y \in X$, $d(Ay, F(A)) < \psi(y) - \psi(Ay)$.*

Then, for $x \in X$, $\mathscr{L}(x)$ is a nonempty, compact and connected subset of $F(A)$. Either $\mathscr{L}(x)$ is a singleton or is uncountable. In the case $\mathscr{L}(x)$ is a singleton, $\lim_{m\to\infty} A^m x$ exists and belongs to $F(A)$. In the case $\mathscr{L}(x)$ is uncountable, it is contained in the boundary of $F(A)$.

PROOF. We assume at the outset $A^k x \notin F(A)$ for all k, because otherwise the proof is trivial. This implies that $\psi(A^{k+1}x) < \psi(A^k x)$ for all k. Thus $\{\psi(A^k x)\}_{k=1}^{\infty}$ is a monotonically decreasing sequence of reals bounded below by zero and so converges to $r \geqslant 0$. We have

$$\begin{gathered} d(A^m x, F(A)) \leqslant \psi(A^{m-1}x) - \psi(A^m x), \\ d\big(A^{m+1}x, F(A)\big) \leqslant \psi(A^m x) - \psi(A^{m+1}x), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ d(A^n x, F(A)) \leqslant \psi(A^{n-1}x) - \psi(A^n x), \end{gathered}$$

whence, by adding, we get

$$d(A^m x, F(A)) + d\big(A^{m+1}x, F(A)\big) + \cdots + d(A^n x, F(A)) \leqslant \psi(A^{m-1}x) - \psi(A^n x).$$

For m, n sufficiently large, the right hand side can be made less than any preassigned $\varepsilon > 0$, since $\{\psi(A^k x)\}_{k=1}^{\infty}$ is a convergent sequence of reals. Hence $d(A^m x, F(A)) \to 0$ as $m \to \infty$. Now, since $F(A)$ is compact we can find a $p_m \in F(A)$ such that $d(A^m x, F(A)) = d(A^m x, p_m)$. The sequence $\{p_m\}_{m=1}^{\infty}$ will have a convergent subsequence $\{p_{m_i}\}_{i=1}^{\infty}$ converging to $p \in F(A)$. Now

$$\begin{aligned} d(p, A^{m_i}x) &\leqslant d(p, p_{m_i}) + d\big(p_{m_i}, A^{m_i}x\big) \\ &= d(p, p_{m_i}) + d\big(A^{m_i}x, F(A)\big) \to 0 \quad \text{as } i \to \infty. \end{aligned}$$

Thus $A^{m_i}x \to p$ as $i \to \infty$ and so $p \in \mathscr{L}(x)$ and hence $\mathscr{L}(x)$ is nonempty. If $\xi \in \mathscr{L}(x)$, then there is a subsequence $\{A^{n_i}x\}_{i=1}^{\infty}$ converging to ξ. Now $d(\xi, F(A)) = d(\lim A^{n_i}x, F(A)) = \lim d(A^{n_i}x, F(A)) = 0$ and as $F(A)$ is closed (being compact), $\xi \in F(A)$. Thus $\mathscr{L}(x) \subset F(A)$. Obviously $\mathscr{L}(x)$ is a closed subset of the compact set $F(A)$ and hence is itself compact. Now we can proceed as in proving Theorem 1 to establish the remaining conclusions.

COROLLARY 3. *If A^k satisfies the conditions of Theorem 8, then the set $\mathscr{L}_k(x)$ of subsequential limit points of the sequence $\{A^{mk_x}x\}_{m=1}^{\infty}$ is a nonempty, compact, and* *connected subset of* $F(A^k)$. *The set* $\mathscr{L}_1(x)$ *of the subsequential limit points of* $\{A^m x\}_{m=1}^{\infty}$ *is the union of the k closed and connected sets* $\mathscr{L}_k(A^j x)$, $j = 0, 1, 2, \ldots, k - 1$. *If* $F(A^k)$ *is at most countable, then* $\mathscr{L}_1(x)$ *consists of at most k points.*

COROLLARY 4. *The conclusions of Theorem* 8 *remain valid if condition* (iii) *is replaced by*

(iii)′ *there exists a monotonically decreasing sequence* $\{r_n\}$ *of positive reals such that* $d(A^{n+1}x, F(A)) \leqslant r_n - r_{n+1}$, *or*

(iii)″ *there exists a sequence* $\{s_n\}$ *of positive reals converging to zero such that* $d(A^n x, F(A)) \leqslant s_n$.

Tarafdar [10] has extended some results of Diaz and Metcalf [2] to uniform spaces. We shall show that the results of Tarafdar still hold when our condition replaces his inequality. Our notations will conform to those of Thron [11].

Let (X, h) be a uniform space, h being the uniformity. The uniform topology induced by h will be denoted by $\mathscr{T}_h$. A family $\{\rho_\alpha: \alpha \in I\}$ of pseudometrics on X is called an associated family for the uniformity h on X if the family $\{H(\alpha, \varepsilon) | \alpha \in I, \varepsilon > 0\}$ where $H(\alpha, \varepsilon) = \{(x, y) | \rho_\alpha(x, y) < \varepsilon\}$ is a subbase for h. A family $\{\rho_\alpha | \alpha \in I\}$ of pseudometrics on X is called an augmented associated family for h if $\{\rho_\alpha | \alpha \in I\}$ is an associated family for h and has the additional property that given $\alpha, \beta \in I$, there is $\gamma \in I$ such that $\rho_\gamma(x, y) \geqslant \max(\rho_\alpha(x, y), \rho_\beta(x, y))$ for all $x, y \in X$. An associated family and an augmented associated family for h will be denoted respectively by $\mathscr{F}(h)$ and $\mathscr{F}^*(h)$.

We are now in a position to give our

THEOREM 9. *Let* (X, h) *be a Hausdorff uniform space and* $\mathscr{F}^*(h) = \{\rho_\alpha | \alpha \in I\}$. *Let* $A: X \to X$ *be* $\mathscr{T}_h$*-continuous. Suppose*

(i) $A(X)$ *is* $\mathscr{T}_h$*-compact,*

(ii) A *is* φ*-asymptotically regular where* φ *is a* $\mathscr{T}_h \times \mathscr{T}_h$ *continuous function on* $X \times X \to R_0$ *such that* $\varphi(x, y) = 0$ *if and only if* $x = y$.

Then, for each $x \in X$, *the* $\mathscr{T}_h$*-cluster set* $\mathscr{L}(x)$ *is a nonempty* $\mathscr{T}_h$*-closed and* $\mathscr{T}_h$*-connected subset of* $F(A)$. *In the case* $\mathscr{L}(x)$ *is just one point then* $\mathscr{T}_h$-lim $A^n x$ *exists and belongs to* $F(A)$. *In the case* $\mathscr{L}(x)$ *contains more than one point then it is contained in the* $\mathscr{T}_h$*-boundary of* $F(A)$.

PROOF. The sequence $\{A^n x\}_{n=1}^{\infty}$ being a net in $A(X)$, which is compact, $\mathscr{L}(x)$ is nonempty. If $y \in \mathscr{L}(x)$, then there is a subnet $\{A^{n_j}x\}_{j \in J}$ of the net $\{A^n x\}_{n=1}^{\infty}$ such that $A^{n_j} \to y$ in the $\mathscr{T}_h$-topology. Since A is $\mathscr{T}_h$-continuous, $A^{1+n_j}x \to Ay$ in the $\mathscr{T}_h$-topology. Hence $\varphi(A^{n_j}x, A^{1+n_j}x) \to \varphi(y, Ay)$ as φ is $\mathscr{T}_h \times \mathscr{T}_h$ continuous. Since A is φ-asymptotically regular, $\varphi(y, Ay) = 0$ and hence by condition (ii),

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$y = Ay$. Therefore $\mathscr{L}(x) \subset F(A)$. Obviously $\mathscr{L}(x)$ is closed. Now, we can proceed as in Tarafdar [10, Theorem 2.1] to prove that $\mathscr{L}(x)$ is $\mathscr{T}_h$-connected. The other parts of the conclusion are to be established likewise.

Corresponding to Theorem 2.2 of Tarafdar [10] we have

THEOREM 10. *Let* (X, h) *be a Hausdorff uniform space and let* $\{\rho_\alpha | \alpha \in I\} = \mathscr{F}^*(h)$. *Let* $A: X \to X$ *be* $\mathscr{T}_h$*-continuous. Suppose*

(i) $F(A)$ *is nonempty and compact,*

(ii) *there exists a* $\mathscr{T}_h \times \mathscr{T}_h$ *continuous function* $\varphi: X \times X \to R_0$ *such that for* $y \neq Ay$, $\varphi(Ay, F(A)) < \varphi(y, F(A))$,

(iii) $\bar{O}(x, A)$ *is compact.*

Then $\mathscr{L}(x)$ *is a closed subset of* $F(A)$. *If* $\mathscr{L}(x)$ *consists of more than one point, then* $\mathscr{L}(x)$ *is contained in the* $\mathscr{T}_h$*-boundary of* $F(A)$.

The proof is omitted. A careful perusal of the proof of Theorem 1 shows that no metric properties of the space have been used in proving that $\mathscr{L}(x)$ is a subset of $F(A)$. Therefore we have the following theorem for Hausdorff topological spaces.

THEOREM 11. *Let* $(X, \mathscr{T})$ *be a Hausdorff topological space and* A, *a continuous self-map. Suppose*

(i) $F(A)$ *is nonempty and compact,*

(ii) *there eixsts a continuous function* $\varphi: X \times X \to R_0$ *such that for* $y \neq Ay$, $\varphi(Ay, F(A)) < \varphi(y, F(A))$.

Then $\mathscr{L}(x)$, *the set cluster points of* $\{A^n x\}_{n=1}^{\infty}$ *is a closed subset of* $F(A)$. *If* $\mathscr{L}(x)$ *consists of more than one point then it is contained in the boundary of* $F(A)$. *If we further assume that* $\varphi(x, y) = 0$ *if and only if* $x = y$, *then* $\mathscr{L}(x)$ *is at most a singleton. If* $\mathscr{L}(x)$ *is a singleton and* $\bar{O}(x, A)$ *is compact, then* $\lim_{n \to \infty} A^n x$ *exists and belongs to* $F(A)$.

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