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A MODULAR PROOF OF TWO OF RAMANUJAN'S FORMULAE FOR $1/\pi$

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Abstract

In this article, we give new proofs of two of Ramanujan's $1/\pi$ formulae

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{99^2}\sum_{m=0}^{\infty}(26390m+1103)\frac{(4m)!}{396^{4m}(m!)^4}$$

and

$$\frac{1}{\pi}=\frac{2}{84^2}\sum_{m=0}^{\infty}(21460m+1123)\frac{(-1)^m(4m)!}{(84\sqrt{2})^{4m}(m!)^4}$$

using the theory of modular forms. The method can also be used to prove other classical $1/\pi$ formulae.

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1. Introduction

Ramanujan listed 14 formulae of $1/\pi$ in his 1914 paper [8, 11], where all the formulae are of the form

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}(a+bn)\,d_n c^n. \tag{1.1}$$

Two among them are quite impressive:

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{99^2}\sum_{m=0}^{\infty}(26390m+1103)\frac{(4m)!}{396^{4m}(m!)^4} \tag{1.2}$$

and

$$\frac{1}{\pi}=\frac{2}{84^2}\sum_{m=0}^{\infty}(21460m+1123)\frac{(-1)^m(4m)!}{(84\sqrt{2})^{4m}(m!)^4}. \tag{1.3}$$

Although R. Gosper computed 17 million digits of π with formula (1.2) in 1985, extensive discussions of proofs of formulae similar to (1.2) appear only around 1987 [2]. The most mysterious constant in the expression is the number 1103 in (1.2) (or 1123 in (1.3)), whose verification heavily relies on the numerical computation and certain number-theoretical properties of the constant a in (1.1). Motivated by an idea proposed by Newman [7], we prove the formulae of Ramanujan without knowing any number-theoretical properties of the constant a in (1.1).

Following the Borweins [2], we will sketch an outline of the proof of series similar to (1.2). The proof begins with one of the identities in [2, page 181]:

$$(1+k^2)\Big(\frac{2K(k)}{\pi}\Big)^2 = {}_3F_2\Big(\begin{matrix}1/4 & 3/4 & 1/2\\ 1 & 1 & \end{matrix};\frac{16k^2(1-k^2)^2}{(1+k^2)^4}\Big), \tag{1.4}$$

which can be rewritten as

$$\Big(\frac{2K(k)}{\pi}\Big)^2 = a(k)\sum_{n=0}^{\infty} b_n c^n(k), \tag{1.5}$$

where $K(k)$ is the complete elliptic integral of the first kind and $a(k)$ and $c(k)$ are rational functions of k.

Let

$$\theta_2(q) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2},$$
$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2},$$
$$\theta_4(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$$

and

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

It is known that [2, page 69]

$$\Big(\frac{2K(k)}{\pi}\Big)^2 = 2^{4/3}\eta^4(q^2)(kk')^{-2/3} = \theta_3^4(q), \tag{1.6}$$

where

$$k = \frac{\theta_2^2(q)}{\theta_3^2(q)}, \quad k' = \frac{\theta_4^2(q)}{\theta_3^2(q)}.$$

Taking logarithmic differentiation by k on both sides of (1.5) and using formula (2.3.10) in [2],

$$\frac{dq}{dk} = \frac{\pi^2 q}{2kk'^2K^2},$$

we find that

$$\frac{1}{6}P(q) = u(k)\Big(\frac{2K(k)}{\pi}\Big)^2 + v(k)\sum_{n=0}^{\infty} nb_n c^n(k), \tag{1.7}$$

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where

$$P(q) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}};$$

$$u(k) = \frac{k(k')^2}{4}\left(\frac{a'}{a} + \frac{2}{3}\frac{(k')^2-k^2}{k(k')^2}\right) = \frac{1-4k^2+k^4}{6(1+k^2)},$$

$$v(k) = \frac{ac'}{4c}k(k')^2 = \frac{1}{2}\left(1 - \frac{8k^2}{(1+k^2)^2}\right),$$

$$c(k) = \frac{16k^2(1-k^2)^2}{(1+k^2)^4}$$

are all rational functions of k.

Taking logarithmic differentiation on both sides of the transformation formula of the η-function

$$\eta(e^{-2\pi/\tau}) = \tau^{1/2}\eta(e^{-2\pi\tau}) \tag{1.8}$$

would lead to

$$\tau^2 P(e^{-\pi\tau}) + P(e^{-\pi/\tau}) = 6\tau/\pi.$$

Let $\tau = \sqrt{n}$, $n \in \mathbb{N}$. Then

$$nP(e^{-\pi\sqrt{n}}) + P(e^{-\pi/\sqrt{n}}) = 6\sqrt{n}/\pi. \tag{1.9}$$

Let

$$G_1 = \frac{nP(q^n) - P(q)}{\theta_3^4(q)} \tag{1.10}$$

and $q = e^{-\pi/\sqrt{n}}$ in (1.10). We can obtain from (1.6) and (1.10) that

$$nP(e^{-\pi\sqrt{n}}) - P(e^{-\pi/\sqrt{n}}) = nG_1\left(\frac{2K(k)}{\pi}\right)^2, \tag{1.11}$$

where $k = k(e^{-\pi\sqrt{n}})$.

Eliminate $P(e^{-\pi/\sqrt{n}})$ from (1.9) and (1.11); we obtain the identity

$$P(e^{-\pi\sqrt{n}}) = \frac{3}{\pi\sqrt{n}} + \frac{G_1}{2}\left(\frac{2K(k)}{\pi}\right)^2. \tag{1.12}$$

Put (1.12) in (1.7) and replace every $(2K(k)/\pi)^2$ by (1.5); then

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}(2\sqrt{n}v(k)m + G_0)b_m c^m(k), \tag{1.13}$$

where $k = k(e^{-\pi\sqrt{n}})$,

$$b_m = \frac{(4m)!}{4^{4m}(m!)^4},$$

$$2v(k) = \left(1 - \frac{2}{((k')^2/(2k))^2 + 1}\right),$$

$$c(k) = \left(\frac{2}{2k/(k')^2 + (k')^2/(2k)}\right)^2$$

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and

$$G_0 = \frac{\sqrt{n}}{3}\Big(1 - \frac{3}{2(((k')^2/(2k))^2+1)} - \frac{1}{1+k^2}\frac{G_1}{2}\Big).$$

REMARK 1.1. We claim that our formula (1.13) coincides with [2, (5.5.16)]. The function $c(k)$ coincides with x_n^2 in [2, (5.5.16)] from the definition of x_n in [2, (5.5.16)] and the formula [2, (5.5.7)]. From (1.10) and [2, (5.2.12) and (5.2.15i)], one can find that the α of the Borweins is related to G_1 by

$$\alpha(n) = \frac{\sqrt{n}}{3}\Big(1 + k^2 - \frac{G_1}{2}\Big). \tag{1.14}$$

From [2, (5.5.16)], (1.14) and (1.13),

$$\begin{aligned} d_m(n)x_n &= \frac{\alpha(n)}{1+k^2} - \frac{\sqrt{n}}{2}\frac{1}{1+((k')^2/(2k))^2} + \sqrt{n}\frac{(k')^2/(2k) - 2k/(k')^2}{(k')^2/(2k) + 2k/(k')^2} \\ &= \frac{\sqrt{n}}{3}\Big(1 - \frac{G_1}{2(1+k^2)} - \frac{3}{2}\frac{1}{1+((k')^2/(2k))^2}\Big) + 2\sqrt{n}v(k) \end{aligned}$$

and our claim follows. A similar argument can be applied to [2, (5.5.17)] and our formula (3.3).

We also note that if $k = k(e^{-\pi\sqrt{n}})$ and $q = e^{-\pi/\sqrt{n}}$, (1.8) implies that

$$\begin{aligned} \frac{2k}{(k')^2} &= 8\frac{\eta^{12}(q^{2n})}{\eta^{12}(q^n)} \\ &= \frac{1}{8}\frac{\eta^{12}(q^2)}{\eta^{12}(q^4)}. \end{aligned} \tag{1.15}$$

From (1.10), (1.6) and (1.15),

$$\begin{aligned} \frac{G_1}{1+k^2} &= \frac{nP(q^n) - P(q)}{4((k')^4 + 4k^2)^{1/2}\eta^4(q^2)(2kk')^{-2/3}} \\ &= \frac{nP(q^n) - P(q)}{\eta^2(q^{2n})\eta^2(q^2)}\Big(\frac{2k}{(k')^2}\Big)^{1/6}\frac{c(k)^{1/4}}{4\sqrt{n}} \\ &= \frac{nP(q^n) - P(q)}{\eta^2(q^{2n})\eta^2(q^4)}\frac{c(k)^{1/4}}{8\sqrt{n}}. \end{aligned} \tag{1.16}$$

The motivation for the identity (1.16) will be stated in the next section.

2. Case 1: $n = 58$

Ramanujan [2, 8] took specific values for the integer n in order to produce his 14 formulae in his paper. It is quite astonishing that he succeeded in producing the formulae (1.2) (with $n = 58$) and (1.3) (with $n = 37$) in that those values of n are so large that the modular equations corresponding to these values are extremely complicated. Although Weber had shown in his book [9, Table VI] that

$$\frac{2k}{(k')^2} = \Big(\frac{\sqrt{29} - 5}{2}\Big)^6 \tag{2.1}$$

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when $k = k(e^{-\pi\sqrt{58}})$, there are no tractable modular equations for these large moduli (for example, $n = 58$ or $n = 37$) so that one can figure out the explicit value of G_1.

Our strategy will be as follows.

Definition 2.1. An η-product $[\alpha_1, \dots, \alpha_N](\tau)$ (or $[\alpha_1, \dots, \alpha_N]$ for short) of level N is defined as a finite product of certain η-functions:

$$[\alpha_1, \dots, \alpha_N] = \prod_{\delta|N} \eta^{\alpha_\delta}(e^{2\pi i\delta\tau}),$$

where $\alpha_\delta \in \mathbb{Z}$ and the divisors δ of N are sorted by ascending order.

An η-product $\eta^a(e^{2\pi i\tau})\eta^b(e^{4\pi i\tau})\eta^c(e^{58\pi i\tau})\eta^d(e^{116\pi i\tau})$ of level 58 can be written as $[a, b, c, d]$, since 58 has four divisors.

Theorem 2.2. *If:*

(1) $24 \mid \sum_{\delta|N} \delta\alpha_\delta$;
(2) $24 \mid \sum_{\delta|N} \frac{N}{\delta}\alpha_\delta$;
(3) $\prod_{\delta|N} \delta^{\alpha_\delta}$ *is a rational square;*
(4) $(\sum_{\delta|N} \alpha_\delta)$ *is a multiple of 4,*

then $[\alpha_1, \dots, \alpha_N]$ *is a weakly modular form holomorphic in the upper half plane with level N and weight* $(\sum_{\delta|N} \alpha_\delta)/2$.

Proof. The proof of this important theorem can be found in [1, Theorem 6.2]. □

Corollary 2.3. *If* $n = 58$, *then* $\eta^4(e^{2\pi i n\tau})\eta^4(e^{8\pi i\tau})$ *is a weakly modular form of weight 4 and level 58.*

Proof. $[0, 4, 0, 4]$ satisfies every condition of Theorem 2.2 and the result follows. □

Let

$$H(q) = \frac{nP(q^n) - P(q)}{\eta^2(q^{2n})\eta^2(q^4)}.$$

Note that if $n = 58$, then $nP(e^{\pi i n\tau}) - P(e^{\pi i\tau})$ is a modular form of weight 2 and level 58 [4, page 18, Exercise (1.2.8)]. Since the η-function is zero-free in the upper half plane, we conclude from Corollary 2.3 that $H(e^{\pi i\tau})^2$ is a weakly modular form of weight 0 and level 58 (while $H(e^{\pi i\tau})$ itself is not).

The key observation is that $H(e^{\pi i\tau})^2$ is *a linear combination* of certain η-products of weight 0 and level 58. Newman [7] conjectured in his paper that every weight-0 modular function on $\Gamma_0(N)$ holomorphic at all cusps except ∞ is a linear combination of η-products of weight 0 and level N, where N is a *square-free composite* number.

It is more convenient to work with modular forms holomorphic at every cusp except at ∞. The congruence subgroup $\Gamma_0(58)$ has four cusps: $\infty, 0, 1/2, 1/29$. The modular form $H(e^{\pi i\tau})^2$ is holomorphic at ∞, but not holomorphic at all other cusps of $\Gamma_0(58)$. One needs to find out a level-58 η-product $[a, b, c, d]$ such that $H^2 \times [a, b, c, d]$ is holomorphic at all cusps except at ∞.

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Theorem 2.4. *An η-product $[\alpha_1, \ldots, \alpha_N]$ of level N is holomorphic at a cusp $r = -d/c$ (c, d are relatively prime) if and only if*

$$\frac{1}{24}\sum_{\delta|N}\frac{(\gcd(c,\delta))^2}{\delta}\alpha_\delta \geq 0.$$

Proof. The result follows from [5, Proposition 3.2.8]. □

Corollary 2.5. *$L := H(e^{\pi i\tau})^2 \times [-2, 8, 10, -16]$ is holomorphic at all cusps of $\Gamma_0(58)$ other than ∞.*

Proof. Theorem 2.4 implies that $[-2, 8, 10, -16]/[0, 4, 0, 4]$ is holomorphic at every cusp other than ∞. It suffices to verify that $58P(e^{58\pi i\tau}) - P(e^{\pi i\tau})$ is holomorphic at all cusps of $\Gamma_0(58)$, which follows from [4, page 18, Exercise (1.2.8)]. □

From [7], it suffices to construct a finite number of η-products $[a_i, b_i, c_i, d_i]$ (of level 58, weight 0, holomorphic at all cusps of $\Gamma_0(58)$ except at ∞) and coefficients m_i so that $S_0 = m_0L + \sum_i m_i[a_i, b_i, c_i, d_i]$ has only terms of positive degree in its q-expansion at ∞, which implies that S_0 has to be 0. Since L has a pole of order 36 at ∞, we only need to find out η-products whose orders of poles at ∞ are not larger than 36. Theorems 2.2 and 2.4 suggest that the constraints for such an η-product $[a, b, c, d]$ should satisfy:

- $24 \mid (a + 2b + 29c + 58d)$ and $-24 \times 36 \leq (a + 2b + 29c + 58d) \leq 0$;
- $24 \mid (58a + 29b + 2c + d)$ and $(58a + 29b + 2c + d) \geq 0$;
- $(29a + 58b + c + 2d) \geq 0$ and $(2a + b + 58c + 29d) \geq 0$;
- $2 \mid (b + d)$;
- $2 \mid (c + d)$;
- $a + b + c + d = 0$.

The constraints give exactly 36 η-products of level 58 and weight 0. Those products are listed in Table 1.

In order to calculate m_i, it suffices to work out the null space of a matrix M, whose column comes from the coefficients of terms with degree ≤ 0 in the q-expansions of $[a_i, b_i, c_i, d_i]$ at ∞. We use Mathematica to obtain all η-products and calculate q-expansions of L and $[a_i, b_i, c_i, d_i]$ to get the entries of matrix M. Then we use the **matkerint** command in PARI/GP to get a reduced $\mathbb{Z}$-basis B of the null space of M, so that the entries of B are the corresponding coefficients. The calculation shows that the null space is seven dimensional and we list six vectors of its $\mathbb{Z}$-basis in Table 1, where $m_0 \neq 0$. The numbers in each column of Table 1 are the coefficients $m_0, m_1, \ldots$ in the linear combination.

Theorem 2.6. *If $[a, b, c, d]$ is an η-product of weight 0 and level 58, then*

$$[a, b, c, d]\Big(\frac{i}{\sqrt{58}}\Big) = 2^{-b/2}58^{(a+b)/4}\Big(\frac{\sqrt{2}}{2}\frac{\sqrt{29}-5}{2}\Big)^{(a+d)/2}.$$

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TABLE 1. Coefficients for linear combinations.

$L(q)$	2	1	3	2	−4	−7
[−2, 4, 26, −28]	1769	−696	232	58	638	21025
[2, 2, 22, −26]	−6964	−3911	−11 668	−8127	17 730	38 414
[6, 0, 18, −24]	−1087	1741	2099	2079	−6240	−36 907
[10, −2, 14, −22]	−174	−399	−448	−581	970	1638
[14, −4, 10, −20]	−42	16	38	73	−102	−1427
[−10, 20, 10, −20]	−512	−280	−808	−512	1088	2088
[−1, 1, 29, −29]	−40 488	−15 176	−53 375	−35 630	69 125	58 603
[3, −1, 25, −27]	−4786	8241	11 908	10 363	−31 852	−194 229
[7, −3, 21, −25]	1057	−1878	−589	−1462	3764	7141
[11, −5, 17, −23]	−56	111	8	91	−176	828
[15, −7, 13, −21]	1	−2	0	−2	3	−56
[19, −9, 9, −19]	0	0	0	0	0	1
[−4, 12, 12, −20]	1008	−3680	−960	2672	1888	12 192
[−3, 9, 15, −21]	−7352	22 936	16 608	−24 032	−46 032	−52 512
[1, 7, 11, −19]	−1496	−6904	−7488	−144	8880	7840
[−2, 6, 18, −22]	29 284	−74 054	−72 386	82 004	170 808	306 698
[2, 4, 14, −20]	−6144	12 496	48	−12 032	−7328	157 328
[6, 2, 10, −18]	−1044	−618	−1134	508	2184	−33 786
[−1, 3, 21, −23]	−123 816	27 132	−37 952	−250 468	−33 384	293 008
[3, 1, 17, −21]	−1712	1968	32 924	21 572	−73 864	−688 620
[7, −1, 13, −19]	−656	−5984	−5292	−11 764	11 832	80 812
[11, −3, 9, −17]	−584	−444	1120	484	−1320	−28 096
[−8, 16, 8, −16]	1936	1896	4680	1408	−6624	−17 224
[−2, 8, 10, −16]	528	−45 488	−59 568	7312	103 584	149 200
[−1, 5, 13, −17]	−69 008	64 552	63 752	−33 344	−280 320	62 520
[3, 3, 9, −15]	720	−19 992	3624	−10 304	27 776	−405 352
[0, 2, 16, −18]	41 820	−79 122	−33 270	−133 188	136	−352 434
[4, 0, 12, −16]	−21 456	−35 560	−93 656	−68 720	87 552	728 568
[8, −2, 8, −14]	−4476	−12 502	12 030	388	−5096	−267 222
[−6, 12, 6, −12]	−15 232	−16 688	−47 888	704	65 024	170 640
[0, 4, 8, −12]	49 296	−80 832	140 000	−118 256	136 352	−1 267 904
[1, 1, 11, −13]	94472	−22 976	−228 456	109 024	−187 440	−463 000
[5, −1, 7, −11]	−5560	−125 840	58 680	−25 968	44 368	−808 536
[−4, 8, 4, −8]	34 848	−17 776	115 600	−88 640	−78 656	−283 408
[2, 0, 6, −8]	−42 640	−317 616	149 008	230 128	−18 080	43 9248
[−2, 4, 2, −4]	−165 504	49 160	−203 272	153 536	65 472	−1 058 168
[0, 0, 0, 0]	51 152	−62 968	−8280	−159 808	140 320	1 529 048

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Proof. Equation (1.8) and the constraint $a+b+c+d=0$ for the weight imply that

$$\begin{aligned}[a,b,c,d]\Big(\frac{i}{\sqrt{58}}\Big) &= \eta^a(e^{-2\pi/\sqrt{58}})\eta^b(e^{-4\pi/\sqrt{58}})\eta^c(e^{-\pi\sqrt{58}})\eta^d(e^{-2\pi\sqrt{58}})\\ &= 2^{-b/2}58^{(a+b)/4}\eta^{b+c}(e^{-\pi\sqrt{58}})\eta^{a+d}(e^{-2\pi\sqrt{58}})\\ &= 2^{-b/2}58^{(a+b)/4}\Big(\frac{\eta(e^{-2\pi\sqrt{58}})}{\eta(e^{-\pi\sqrt{58}})}\Big)^{a+d}.\end{aligned}$$

Then the result follows from (1.15) and (2.1). □

Theorem 2.7. $(H(e^{-\pi/\sqrt{58}}))^2 = 65\,870\,496 + 8\,439\,552\sqrt{29}$.

Proof. We have stated that the q-series L and $[a_i,b_i,c_i,d_i]$ span a linear space of dimension 7. The entries of each column in Table 1 are the coefficients in a linear combination $m_0L+\sum_i m_i[a_i,b_i,c_i,d_i]=0$. Then our theorem follows from coefficients in any column[1] of Table 1 and Theorem 2.6. □

Theorem 2.7 implies that $H(e^{-\pi/\sqrt{58}}) = 36\sqrt{2}(148+11\sqrt{29})$. So,

$$\begin{aligned}G_0 &= \frac{\sqrt{58}}{3}\Big(1-\frac{3}{4\times 99^2}\Big(\frac{\sqrt{29}-5}{2}\Big)^6 - \frac{36\sqrt{2}(148+11\sqrt{29})}{99\times 16\sqrt{58}}\Big)\\ &= \frac{2\sqrt{2}\times 1103}{99^2}.\end{aligned}\tag{2.2}$$

The following results follow from the computation of Weber:

$$c(k) = \frac{1}{9801^2} = \frac{1}{99^4},\tag{2.3}$$

$$2v(k) = \frac{1820\sqrt{29}}{99^2}.\tag{2.4}$$

Combining (2.2)–(2.4) and (1.13),

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2}\sum_{m=0}^{\infty}(26390m+1103)\frac{(4m)!}{396^{4m}(m!)^4}$$

and we complete the proof of (1.2).

Remark 2.1. The conjecture of Newman for weakly holomorphic modular forms of weight 0 suggests that the technique developed in the current section can be applied to other composite square-free n in [8] or other Ramanujan–Sato series of other levels in [3]. We have rigorously proved formulae (40)–(44) in [8] and a few other Ramanujan-type formulae in which n are square-free idoneal numbers (for example,

[1] Although one column suffices, we can use other columns to cross-check our calculations. We also note that these linear combinations can be treated as modular equations (with comparatively small coefficients) that might deserve further investigations.

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$n = 30, 42$ etc) associated to imaginary quadratic number fields of class number 4. Among all these idoneal numbers, $n = 190$ and $n = 253$ give two extremely rapidly convergent series:

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}(a_1 m + b_1)\frac{(4m)!}{4^{4m}(m!)^4}\Big(\frac{1}{3\sqrt{19}(481+340\sqrt{2})}\Big)^{4m},$$
$$a_1 = \frac{20(693\,121 + 5457\sqrt{2})}{233\,289\sqrt{19}},$$
$$b_1 = \frac{1\,877\,581 - 869\,892\sqrt{2}}{466\,578\sqrt{19}}$$

and

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}(a_2 m + b_2)\frac{(-1)^m(4m)!}{4^{4m}(m!)^4}\Big(\frac{1}{21\sqrt{2}(1121+338\sqrt{11})}\Big)^{4m},$$
$$a_2 = \frac{1495(11\,092\sqrt{11} - 19\,437)}{1\,630\,818},$$
$$b_2 = \frac{-37\,515\,813 + 11\,937\,508\sqrt{11}}{6\,523\,272}.$$

Each term adds 16 or 19 decimal digits, respectively.

3. Case 2: $n = 37$

Formula (1.4) is not the only hypergeometric identity that Ramanujan dealt with in the 1914 paper. Following what we have done in the introduction, we can start with another formula from [2, page 181, Theorem 5.7]:

$$(1-2k^2)\Big(\frac{2K(k)}{\pi}\Big)^2 = {}_3F_2\Big(\begin{matrix}1/4 & 3/4 & 1/2 \\ 1 & 1 & \end{matrix}; -\frac{16k^2(1-k^2)}{(1-2k^2)^4}\Big). \tag{3.1}$$

One can use the same method as the derivation of (1.7) to obtain the formula below:

$$\frac{1}{6}P(q) = \bar{u}(k)\Big(\frac{2K(k)}{\pi}\Big)^2 + \bar{v}(k)\sum_{m=0}^{\infty}(-1)^m m b_m \bar{c}^m(k), \tag{3.2}$$

where

$$\bar{u}(k) = \frac{1+2k^2-2k^4}{6(1-2k^2)},$$
$$\bar{v}(k) = \frac{1}{2}\Big(\frac{1+4k^2-4k^4}{(1-2k^2)^2}\Big),$$
$$\bar{c}(k) = \frac{16k^2(1-k^2)}{(1-2k^2)^4}.$$

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Combining (1.12), (3.2) and replacing every $(2K(k)/\pi)^2$ by (3.1), we obtain a formula similar to (1.13):

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}(2\sqrt{n}\bar{v}(k)m + \bar{G}_0)(-1)^m b_m \bar{c}^m(k), \tag{3.3}$$

where $k = k(e^{-\pi\sqrt{n}}), q = e^{-\pi/\sqrt{n}}$,

$$\begin{aligned} b_m &= \frac{(4m)!}{4^{4m}(m!)^4}, \\ 2\bar{v}(k) &= \Big(1 - \frac{2}{1-(2kk')^{-2}}\Big), \\ \bar{c}(k) &= \Big(\frac{2}{(2kk')^{-1} - 2kk'}\Big)^2, \\ \bar{G}_0 &= \frac{\sqrt{n}}{3}\Big(1 - \frac{3}{2(1-(2kk')^{-2})} - \frac{1}{(k')^2 - k^2}\frac{G_1}{2}\Big) \end{aligned}$$

and G_1 is still defined by (1.10).

Although Weber [9, Table VI] had already shown that

$$2kk' = (\sqrt{37} - 6)^3, \tag{3.4}$$

where $k = k(e^{-\pi\sqrt{37}})$, the method working for $n = 58$ fails for the case $n = 37$. This is because 37 is a prime number which does not offer us enough prime factors to construct sufficiently many η-products of level 37. Fortunately, Mazur and Swinnerton-Dyer constructed certain modular forms on $\Gamma_0(37)$ in [6], which is just the last piece that we need for the computation of G_1.

The key observation in our proof is that the function

$$M(q) = \frac{1}{\theta_2^8(q)} + \frac{1}{\theta_3^8(q)} + \frac{1}{\theta_4^8(q)}, \quad q = \exp(\pi i\tau),$$

is an $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic modular form of weight -4 in the upper half plane. Notice that

$$M(q) = \frac{1}{\theta_3^8(q)}\frac{(1-(kk')^2)^2}{(kk')^4},$$

so

$$G_1^2 = (37P(q^{37}) - P(q))^2 M(q)\Big(\frac{(1-(kk')^2)^2}{(kk')^4}\Big)^{-1}.$$

The function

$$S(q) = (37P(q^{37}) - P(q))^2 M(q)$$

is a modular function of weight 0 on $\Gamma_0(37)$. Instead of working with the modular curve $X_0(37)$, we prefer to work on a similar Riemann surface with smaller genus, that is, $X_0^+(37) := X_0(37)/w$, where $w : z \mapsto -1/(37z)$ is the Fricke involution. The genus of the Riemann surface $X_0^+(37)$ is 1.

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Mazur and Swinnerton-Dyer [6] constructed a basis of meromorphic functions on the Riemann surface $X_0^+(37)$. Here we summarize their construction briefly. Following Theorems 2.2 and 2.4 in Section 2, one obtains that $f_1(z) = \eta^2(e^{74\pi i\tau})/\eta^2(e^{2\pi i\tau})$ is a modular function of weight 0 on $\Gamma_0(37)$. The transformation formula (1.8) of the η-function suggests that $f_2(\tau) = 37f_1(\tau) + (f_1(\tau))^{-1}$ is a modular function on $X_0^+(37)$ with a pole of order 3 at infinity. We need a modular function with a double pole at infinity to form part of a basis of meromorphic functions on $X_0^+(37)$. Mazur and Swinnerton-Dyer remarked that [6, page 20] $\phi(\tau) = (37P(e^{37\pi i\tau}) - P(e^{\pi i\tau}))/12$ is a weight-2 modular form on $\Gamma_0(37)$ with $\phi(-1/(37\tau)) = -37\tau^2\phi(\tau)$. It is also known that [6, page 20], given a quadratic form

$$Q(x) := x^T \begin{pmatrix} 4 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 20 & 1 \\ 1 & 1 & 1 & 10 \end{pmatrix} x,$$

the associated θ-function

$$\theta(\tau) = \sum_{x\in\mathbb{Z}^4} \exp(\pi i\tau Q(x))$$

is also a weight-2 modular form on $\Gamma_0(37)$ with $\theta(-1/(37\tau)) = -37\tau^2\theta(\tau)$. So, $\varphi = (3\theta - \phi)/4$ is a cusp form of weight 2 and $f_3 = \theta/\varphi$ is a meromorphic function on $X_0^+(37)$. Mazur and Swinnerton-Dyer noted that [6, page 21]

$$x = \frac{f_2 - f_3 - 5}{f_3 - 6} = e^{-4\pi i\tau} + 2e^{-2\pi i\tau} + 5 + \cdots \tag{3.5}$$

and

$$y = \frac{(f_2 - 11)(f_3 - 1)}{f_3 - 6} = e^{-6\pi i\tau} + 3e^{-4\pi i\tau} + 9e^{-2\pi i\tau} + 21 + \cdots \tag{3.6}$$

are modular functions on $X_0^+(37)$ with a pole of orders 2 and 3 at infinity, respectively, and they are the explicit modular parameterization of the elliptic curve $E : y^2 - y = x^3 - x$ with conductor 37.

THEOREM 3.1. *$x(i/\sqrt{37}) = 15 + 2\sqrt{37}$ and $y(i/\sqrt{37}) = 69 + 12\sqrt{37}$.*

PROOF. In [6, page 21], it is shown that the functions f_2 and f_3 are related by the algebraic equation

$$f_2^2 - (f_3^3 - 8f_3^2 + 16f_3 - 2)f_2 + 12f_3^3 - 99f_3^2 + 200f_3 - 107 = 0.$$

From the formula (1.8) of the η-function, $f_2(i/\sqrt{37}) = 2\sqrt{37}$, which leads to $f_3(i/\sqrt{37}) = (\sqrt{37} + 6)/2$ or $-2 - \sqrt{37}$. We can discard the second solution and the result follows directly from (3.5) and (3.6). □

Following the construction in [6], the function

$$T(q) = 37^2(37P(q^{37}) - P(q))^2M(q) + (37P(q^{37}) - P(q))^2M(q^{37})$$

is a modular function on $X_0^+(37)$. The only pole of T is at infinity, so T can be expressed as a polynomial of x, y. With a Gaussian elimination process, a polynomial expansion for $16T$ can be given below:

$$\begin{aligned}16T = &-76\,950 + 592740x + 19285650x^2 - 127471338x^3 + 92127717x^4\\ &+ 723037734x^5 - 1173733308x^6 - 1018583271x^7 + 2790914580x^8\\ &- 94187160x^9 - 2475353880x^{10} + 999364437x^{11} + 720381546x^{12}\\ &- 518203053x^{13} + 30859947x^{14} + 35407512x^{15} - 4823613x^{16}\\ &- 378801x^{17} - 2889x^{18} - 844650y - 11372238xy\\ &+ 117580086x^2y - 208224792x^3y - 467267283x^4y\\ &+ 1515431403x^5y - 215958168x^6y - 2499996123x^7y\\ &+ 1722900159x^8y + 1177433235x^9y - 1430836020x^{10}y\\ &+ 95279805x^{11}y + 299139750x^{12}y - 94708881x^{13}y\\ &+ 361422x^{14}y + 1880190x^{15}y + 44424x^{16}y + 81x^{17}y.\end{aligned}$$

THEOREM 3.2.

$$S(e^{-\pi/\sqrt{37}}) = 4050(2\,011\,673\,312\,873 + 330\,717\,057\,625\sqrt{37})/37.$$

PROOF. A straightforward calculation from Theorem 3.1 and the polynomial expansion of T shows that

$$T(e^{-\pi/\sqrt{37}}) = 299\,700(2\,011\,673\,312\,873 + 330\,717\,057\,625\sqrt{37}).$$

Recall that $M(q)$ is a modular form of weight -4 over $\mathrm{SL}_2(\mathbb{Z})$, which implies that $M(e^{-\pi\sqrt{37}}) = 37^2 M(e^{-\pi/\sqrt{37}})$. So,

$$S(e^{-\pi/\sqrt{37}}) = 4050(2\,011\,673\,312\,873 + 330\,717\,057\,625\sqrt{37})/37. \qquad \square$$

Recall that

$$\frac{1}{((k')^2 - k^2)^2} G_1^2 = \frac{1}{1 - (2kk')^2}(nP(q^n) - P(q))^2 M(q)\Big(\frac{(1 - (kk')^2)^2}{(kk')^4}\Big)^{-1}.$$

With Theorems 3.1, 3.2 and (3.4),

$$\frac{1}{(k')^2 - k^2} G_1 = \frac{1}{2} + \frac{101}{14\sqrt{37}}.$$

Hence

$$\begin{aligned}\bar{G}_0 &= \frac{\sqrt{37}}{3}\Big(1 + \frac{3}{4\times 882}(\sqrt{37} - 6)^3 - \frac{1}{2}\Big(\frac{1}{2} + \frac{101}{14\sqrt{37}}\Big)\Big)\\ &= \frac{1123}{4\times 882}.\end{aligned} \tag{3.7}$$

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Weber's computation (3.4) implies that

$$\bar{c}(k) = \frac{1}{882^2} = \frac{1}{(21\sqrt{2})^4} \tag{3.8}$$

and

$$2\bar{v}(k) = \frac{145\sqrt{37}}{882}. \tag{3.9}$$

Combining (3.7)–(3.9) and (3.3),

$$\frac{1}{\pi} = \frac{2}{84^2}\sum_{m=0}^{\infty}(21460m + 1123)\frac{(-1)^m(4m)!}{(84\sqrt{2})^{4m}(m!)^4}$$

and the proof of (1.3) is complete.

REMARK 3.1. Similar constructions are much easier for (35)–(38) in [8] (all related modular curves have genus 0, so theta functions are unnecessary for the basis construction of modular functions). In the case $n = 37$, the construction of theta functions is by no means *ad hoc*. It is rooted in the basis problem of modular forms (which can be systematically constructed from quaternion algebra) on congruence subgroups which can be used to give a rigorous proof for other Ramanujan–Sato series where n are prime numbers (for example, Chudnovsky's formula in [3, 11]). We will give more details in a subsequent paper [10].

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