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A classification of $\mathbb{Q}$-linear maps from $\overline{\mathbb{Q}}^\times/\overline{\mathbb{Q}}^\times_{\text{tors}}$ to $\mathbb{R}$

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Abstract. **A 2009 article of Allcock and Vaaler explored the $\mathbb{Q}$-vector space $\mathcal{G} := \overline{\mathbb{Q}}^\times/\overline{\mathbb{Q}}^\times_{\text{tors}}$, showing how to represent it as part of a function space on the places of $\overline{\mathbb{Q}}$. We establish a representation theorem for the $\mathbb{R}$-vector space of $\mathbb{Q}$-linear maps from $\mathcal{G}$ to $\mathbb{R}$, enabling us to classify extensions to $\mathcal{G}$ of completely additive arithmetic functions. We further outline a strategy to construct $\mathbb{Q}$-linear maps from $\mathcal{G}$ to $\mathbb{Q}$, i.e., elements of the algebraic dual of $\mathcal{G}$. Our results make heavy use of Dirichlet's S-unit Theorem as well as a measure-like object called a consistent map, first introduced by the author in previous work.**

1 Introduction

1.1 Background

Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$ and let $\overline{\mathbb{Q}}^\times_{\text{tors}}$ denote the group of roots of unity in $\overline{\mathbb{Q}}^\times$. We write $\mathcal{G} = \overline{\mathbb{Q}}^\times/\overline{\mathbb{Q}}^\times_{\text{tors}}$ and note that $\mathcal{G}$ is a vector space over $\mathbb{Q}$ with addition and scalar multiplication given by

$$(\alpha,\beta)\mapsto\alpha\beta \quad \text{and} \quad (r,\alpha)\mapsto\alpha^r.$$

An innovative article of Allcock and Vaaler [2] showed how to interpret $\mathcal{G}$ as a certain function space in the following way.

For each number field K, we write M_K to denote the set of all places of K. If L/K is a finite extension and $w \in M_L$, then w divides a unique place v of K, and in this case, we write K_w to denote the completion of K with respect to v. Additionally, we let p_v be the unique place of $\mathbb{Q}$ such that v divides p_v and let $\|\cdot\|_v$ be the unique extension to K_v of the usual p_v-adic absolute value on $\mathbb{Q}_v$. In this notation, the well-known product formula may be expressed as

$$\sum_{v\in M_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\log\|\alpha\|_v = 0$$

for all non-zero elements $\alpha \in K$.

Letting Y denote the set of all places of $\overline{\mathbb{Q}}$, we define $Y(K,v) = \{y \in Y : y \mid v\}$. Further setting

$$\mathcal{J} = \{(K,v) : [K:\mathbb{Q}] < \infty,\ v \in M_K\},$$

Allcock and Vaaler observed that the collection $\{Y(K,v) : (K,v) \in \mathcal{J}\}$ is a basis for a totally disconnected, Hausdorff topology on Y, and moreover, there is

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a Borel measure λ on Y such that

$$\lambda(Y(K,v)) = \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \quad \text{for all } (K,v) \in \mathcal{J}.$$

Each element $\alpha \in \mathcal{G}$ corresponds to a locally constant function $f_\alpha : Y \to \mathbb{R}$ given by the formula $f_\alpha(y) = \log \|\alpha\|_y$. When $\mathcal{G}$ is equipped with a norm arising from the Weil height, they proved that $\alpha \mapsto f_\alpha$ defines an isometric isomorphism from $\mathcal{G}$ onto a dense $\mathbb{Q}$-linear subspace of

$$\left\{ f \in L^1(Y) : \int_Y f(y) d\lambda(y) = 0 \right\}.$$

More recently, the author [16, 17] began the study of various dual spaces related to $\mathcal{G}$. To that end, we defined a map $c : \mathcal{J} \to \mathbb{R}$ to be *consistent* if

$$c(K,v) = \sum_{w|v} c(L,w) \tag{1.1}$$

for all number fields K, all places v of K, and all finite extensions L/K. The set of all consistent maps forms a vector space over $\mathbb{R}$ with addition and scalar multiplication given by the formulas

$$(c+d)(K,v) = c(K,v) + d(K,v) \quad \text{and} \quad (rc)(K,v) = rc(K,v).$$

We shall write $\mathcal{J}^*$ to denote this space. Every Radon measure μ on Y yields a corresponding consistent map via the formula $c(K,v) := \mu(Y(K,v))$, however, not every consistent map is built in this way (see [1]). The most fundamental consistent map arises from the measure λ appearing in [2] which we shall simply denote by λ, ie.,

$$\lambda(K,v) = \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}.$$

The main result of [16] constructed an isomorphism between $\overline{\mathbb{Q}}^\times/\overline{\mathbb{Z}}^\times$ and a certain $\mathbb{Q}$-linear subspace of $\mathcal{J}^*$. Later in [17], we studied the $\mathbb{R}$-vector space $LC_c(Y)$ of locally constant functions from Y to $\mathbb{R}$ with compact support, and additionally, we examined its co-dimension 1 subspace

$$LC_c^0(Y) = \left\{ f \in LC_c(Y) : \int_Y f(y) d\lambda(y) = 0 \right\}.$$

As a special case of a more general set of theorems, we showed that

$$\mathcal{J}^* \cong LC_c(Y)^* \quad \text{and} \quad \mathcal{J}^*/\mathrm{span}_{\mathbb{R}}\{\lambda\} \cong LC_c^0(Y)^*. \tag{1.2}$$

In both cases, these isomorphisms are defined explicitly, and as such, we regard them as algebraic versions of the Riesz Representation Theorem (see [4, 6, 15], for example).

1.2 Main Results

While we consider [16, 17] to be strong results, they leave open any questions about two important spaces:

(I) The $\mathbb{Q}$-vector space of $\mathbb{Q}$-linear maps from $\mathcal{G}$ to $\mathbb{Q}$, i.e., the algebraic dual $\mathcal{G}^*$ of $\mathcal{G}$.

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(II) The $\mathbb{R}$-vector space of $\mathbb{Q}$-linear maps from $\mathcal{G}$ to $\mathbb{R}$, which we shall denote by $\mathcal{L}(\mathcal{G}, \mathbb{R})$.

For each $c \in \mathcal{J}^*$ and $\alpha \in \overline{\mathbb{Q}}^\times$, we let

$$\Phi_c(\alpha) = \sum_{v \in M_K} c(K, v) \log \|\alpha\|_v,$$

where K is any number field containing α. Because c is assumed to be consistent, this definition does not depend on the choice of K, and moreover, its value is unchanged when α is multiplied by a root of unity. Hence, $\Phi_c : \mathcal{G} \to \mathbb{R}$ is a well-defined $\mathbb{Q}$-linear map, and we may define $\Phi^* : \mathcal{J}^* \to \mathcal{L}(\mathcal{G}, \mathbb{R})$ by $\Phi^*(c) = \Phi_c$. Our main result is the following representation theorem for $\mathcal{L}(\mathcal{G}, \mathbb{R})$.

***Theorem* 1.1** *The map $\Phi^* : \mathcal{J}^* \to \mathcal{L}(\mathcal{G}, \mathbb{R})$ is a surjective $\mathbb{R}$-linear transformation such that* $\ker(\Phi^*) = \operatorname{span}_{\mathbb{R}}\{\lambda\}$.

If F is a number field and q is a place of F, we observe that $\mathcal{J}_q^* := \{c \in \mathcal{J}^* : c(F, q) = 0\}$ is a subspace of $\mathcal{J}^*$. Given an arbitrary consistent map $c \in \mathcal{J}^*$, the coset $c + \ker(\Phi^*)$ contains a unique element $d \in \mathcal{J}_q^*$, namely

$$d(K, v) = c(K, v) - c(F, q)\lambda(K, v).$$

This observation yields the following consequence of Theorem 1.1.

***Corollary* 1.2** *Let F be a number field and let q be a place of F. Then the map $c \mapsto \Phi_c$ defines an $\mathbb{R}$-vector space isomorphism from $\mathcal{J}_q^*$ to $\mathcal{L}(\mathcal{G}, \mathbb{R})$.*

By using the specific case $F = \mathbb{Q}$ and $q = \infty$, Corollary 1.2 provides a useful framework to classify extensions to $\mathcal{G}$ of completely additive arithmetic functions. Assuming that $c \in \mathcal{J}_\infty^*$, we note the following famous examples:

Natural Logarithm: Φ_c extends the natural logarithm on $\mathbb{N}$ if and only if $c(\mathbb{Q}, p) = -1$ for all $p \neq \infty$.

Prime Omega Function: Let $\Omega(n)$ be the number of prime factors of n, counted with multiplicity (see [5, 9, 11]). Then Φ_c extends Ω if and only if $c(\mathbb{Q}, p) = -1/(\log p)$ for all $p \neq \infty$.

Integer Logarithm: Let $\Psi(n)$ be the sum of the prime factors of n, counted with multiplicity (see [3, 10, 13]). Then Φ_c extends Ψ if and only if $c(\mathbb{Q}, p) = -p/(\log p)$ for all $p \neq \infty$.

It would be interesting to discover a version of Corollary 1.2 that could be applied to all additive functions rather than only to completely additive functions. For now, we are unaware of any way to formulate such a result.

Of the previous work on this subject, we should regard Theorem 1.1 as most analogous to [17, Theorem 1.3], which established the right hand isomorphism of (1.2). It is important to note, however, that our result cannot be proved by directly applying existing work. Although $\mathcal{G}$ appears as a dense subset of $LC_c^0(Y)$ with respect to the L^1-norm, we impose no continuity assumption on elements of $\mathcal{L}(\mathcal{G}, \mathbb{R})$. Hence, prior to proving Theorem 1.1, we cannot be certain that an arbitrary map $\Phi \in \mathcal{L}(\mathcal{G}, \mathbb{R})$ may be extended to a linear map on $LC_c^0(Y)$. While

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our proof is inspired by some ideas of [17], several different methods are needed, including an application of Dirichlet's S-unit Theorem.

Related to these observations, there is a version of Theorem 1.1 which classifies all continuous linear maps from $\mathcal{G}$ to $\mathbb{R}$, where $\mathcal{G}$ is equipped with the Weil height norm as in [2]. Specifically, if we let $\mathcal{J}'$ be the set of consistent maps c for which

$$\sup\left\{\left|\frac{c(K,v)}{\lambda(K,v)}\right| : (K,v) \in \mathcal{J}\right\} < \infty,$$

then $c \mapsto \Phi_c$ is an isomorphism of $\mathcal{J}'$ onto the space of continuous linear maps from $\mathcal{G}$ to $\mathbb{R}$. However, this fact does indeed follow directly from previous work, namely [17, Theorem 1.6]. Because of our observations following Corollary 1.2, both the natural logarithm and the prime Omega function have continuous extensions to $\mathcal{G}$, while the integer logarithm has no such extension.

1.3 Rational Valued Linear Maps

With an eye toward studying $\mathcal{G}^*$, we are particularly interested in identifying consistent maps c for which $\Phi_c(\alpha) \in \mathbb{Q}$ for all $\alpha \in \mathcal{G}$. To facilitate these efforts, we shall write

$$\mathcal{I}^* = \{c \in \mathcal{J}^* : \Phi^*(c) \in \mathcal{G}^*\} = \{c \in \mathcal{J}^* : \Phi_c(\alpha) \in \mathbb{Q} \text{ for all } \alpha \in \mathcal{G}\}$$

and note that $\mathcal{I}^*$ is a $\mathbb{Q}$-linear subspace of $\mathcal{J}^*$. The question now arises to provide a necessary and sufficient condition for the claim that $c \in \mathcal{I}^*$. This problem appears to be quite challenging, however, we may obtain several interesting examples by applying two supplementary results, the first of which is as follows.

***Theorem* 1.3** *Suppose K is a number field, and for each place v of K, let $y_v \in \mathbb{R}$. Then there exists a unique consistent map $c \in \mathcal{J}^*$ such that*

$$c(L,w) = \frac{[L_w : K_v]}{[L:K]} y_v \tag{1.3}$$

for all $v \in M_K$, all finite extensions L/K, and all places w of L dividing v. Moreover, if $\Phi_c(\alpha) \in \mathbb{Q}$ for all $\alpha \in K^\times$ then $c \in \mathcal{I}^*$.

Given a point $\mathbf{y} = (y_v)_{v \in M_K}$, we write $c_{\mathbf{y}}$ to denote the consistent map from Theorem 1.3. We plainly have that $c_{\mathbf{y}}(K,v) = y_v$ for all places v of K, and therefore, if we wish for $c_{\mathbf{y}} \in \mathcal{I}^*$, it is sufficient to find a point $\mathbf{y} = (y_v)_{v \in M_K}$ such that

$$\sum_{v \in M_K} y_v \log \|\alpha\|_v \in \mathbb{Q} \quad \text{for all } \alpha \in K^\times.$$

In the special case where $K = \mathbb{Q}$ or where K an imaginary quadratic extension of $\mathbb{Q}$, we can locate examples of this sort rather easily. Indeed, such number fields have a unique Archimedean place ∞, so we may choose $y_\infty = 0$ and $y_v \log p_v \in \mathbb{Q}$ for all non-Archimedean places v of K. The resulting consistent map $c_{\mathbf{y}}$ certainly belongs to $\mathcal{I}^*$.

The more interesting cases, however, arise from number fields having multiple Archimedean places. The following theorem is somewhat technical, but useful in constructing other examples of consistent maps in $\mathcal{I}^*$.

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***Theorem* 1.4** *Suppose K is a number field. Let $\{v_1, v_2, \ldots, v_n\}$ be the complete list of Archimedean places of K and let $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ be a fundamental set of units in K. For each non-Archimedean place v of K, let $\beta_v \in \mathcal{O}_K$ be such that $\|\beta_v\|_v < 1$ and $\|\beta_v\|_w = 1$ for all non-Archimedean places $w \neq v$. Finally, let $\mathbf{y} = (y_v)_{v \in M_K}$. Then*

$$\sum_{v \in M_K} y_v \log \|\alpha\|_v \in \mathbb{Q} \quad \text{for all } \alpha \in K^\times$$

if and only if the following two conditions hold:

(i) We have

$$\begin{pmatrix} \log\|\alpha_1\|_{v_1} & \log\|\alpha_1\|_{v_2} & \cdots & \log\|\alpha_1\|_{v_n} \\ \log\|\alpha_2\|_{v_1} & \log\|\alpha_2\|_{v_2} & \cdots & \log\|\alpha_2\|_{v_n} \\ \vdots & \vdots & \ddots & \vdots \\ \log\|\alpha_{n-1}\|_{v_1} & \log\|\alpha_{n-1}\|_{v_2} & \cdots & \log\|\alpha_{n-1}\|_{v_n} \end{pmatrix} \begin{pmatrix} y_{v_1} \\ y_{v_2} \\ \vdots \\ y_{v_n} \end{pmatrix} \in \mathbb{Q}^{n-1}$$

(ii) For all non-Archimedean places v of K, we have that

$$y_v \log \|\beta_v\|_v + \sum_{i=1}^{n} y_{v_i} \log \|\beta_v\|_{v_i} \in \mathbb{Q}.$$

The existence of the fundamental set of units $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ is guaranteed by Dirichlet's Unit Theorem, and furthermore, [16, Lemma 3.1] establishes the existence of the points β_v. As a result, it is always possible to select points α_i and β_v that satisfy the assumptions of Theorem 1.4. Now we may use Theorems 1.3 and 1.4 to construct maps $c \in \mathcal{I}^*$ which are distinct from known examples.

Example 1.5 We consider $K = \mathbb{Q}(\sqrt{2})$ and fix an embedding of K into $\mathbb{R}$ with $|\cdot|$ denoting the usual absolute value on $\mathbb{R}$. We know that K has two Archimedean places which we shall denote by v_1 and v_2. We may assume that

$$\|a + b\sqrt{2}\|_{v_1} = |a + b\sqrt{2}| \quad \text{and} \quad \|a + b\sqrt{2}\|_{v_2} = |a - b\sqrt{2}|$$

for all $a, b \in \mathbb{Z}$. We observe that K has class number equal to 1, its ring of integers is $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$, and $1 + \sqrt{2}$ is the fundamental unit in K. Additionally, a prime $p \in \mathbb{Q}$ splits in K if and only if $p \equiv 1, 7 \mod 8$.

We now select the points y_v for use in Theorem 1.4.

(i) We define

$$y_{v_1} = \frac{1}{\log(1+\sqrt{2})} \quad \text{and} \quad y_{v_2} = -\frac{1}{\log(1+\sqrt{2})}.$$

(ii) If $p \not\equiv 1, 7 \mod 8$ and v divides p, we let $y_v = 0$.

(iii) If $p \equiv 1, 7 \mod 8$ then we let v and w be distinct places of K dividing p. In this case, p has the form $p = \beta_v \beta_w$, where β_v and β_w are generators of the prime ideals

$$\{\alpha \in \mathcal{O}_K : \|\alpha\|_v < 1\} \quad \text{and} \quad \{\alpha \in \mathcal{O}_K : \|\alpha\|_w < 1\},$$

respectively. In this situation, we define

$$y_v = \frac{\log\|\beta_v\|_{v_1} - \log\|\beta_v\|_{v_2}}{(\log p)(\log(1+\sqrt{2}))} \qquad y_w = \frac{\log\|\beta_w\|_{v_1} - \log\|\beta_w\|_{v_2}}{(\log p)(\log(1+\sqrt{2}))}.$$

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It is straightforward to verify that the points y_v satisfy the two properties of Theorem 1.4, and then by Theorem 1.3, $c = c_{\mathbf{y}} \in \mathcal{I}^*$. In other words, Φ_c is a (rational-valued) linear functional on $\mathcal{G}$.

Since β_v and β_w are conjugates over $\mathbb{Q}$, the values in (iii) satsify $y_v = -y_w$. Therefore, we have $c(K, v) = -c(K, w)$ whenever v and w divide the same place of $\mathbb{Q}$, or equivalently, we have $c(\mathbb{Q}, p) = 0$ for all $p \in M_{\mathbb{Q}}$. The first few non-zero values of $c(K, v)$ are approximated in the following table.

p	Factorization of p in $\mathbb{Z}[\sqrt{2}]$	$c(K, v)$ for $v \mid p$ (approx.)
∞	NA	± 1.13459
7	$(3+\sqrt{2})(3-\sqrt{2})$	± 0.596913
17	$(5+2\sqrt{2})(5-2\sqrt{2})$	± 0.513516
23	$(5+\sqrt{2})(5-\sqrt{2})$	$\pm 0.0.513516$
31	$(7+3\sqrt{2})(7-3\sqrt{2})$	± 0.464359
41	$(7+2\sqrt{2})(7-2\sqrt{2})$	± 0.261831
47	$(7+\sqrt{2})(7-\sqrt{2})$	± 0.120733
71	$(11+5\sqrt{2})(11-5\sqrt{2})$	± 0.406159

1.4 Organizational Summary

We shall structure the remainder of this article by separating the proof of Theorem 1.1 into two components. First, in Section 2, we show that Φ^* is a linear transformation such that $\ker(\Phi^*) = \operatorname{span}_{\mathbb{R}}\{\lambda\}$. The surjectivity component of the proof requires applying Theorem 1.3, and hence, we use Section 3 to prove that result. Finally, in Section 4, we complete the proof of Theorem 1.1 by proving that Φ^* is surjective. The proof of Theorem 1.4 is included in that section as well.

2 The Kernel of Φ^*

***Theorem* 2.1** *The map $\Phi^* : \mathcal{J}^* \to \mathcal{L}(\mathcal{G}, \mathbb{R})$ is an $\mathbb{R}$-linear transformation such that $\ker(\Phi^*) = \operatorname{span}_{\mathbb{R}}\{\lambda\}$.*

Before proceeding with the proof of Theorem 2.1, we remind the reader of the relevant features of Dirichlet's Unit Theorem [12, Theorem 7.31]. If K is a number field, then O_K denotes its ring of integers and

$$O_K^\times = \{\alpha \in K : \|\alpha\|_v = 1 \text{ for all } v \nmid \infty\}$$

is called its *group of units*. If K has n Archimedean places, then Dirchlet's Unit Theorem asserts that the $O_K^\times$ has rank equal to $n-1$. If ζ is a root of unity and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in O_K$ are such that $\{\zeta, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ generates $O_K^\times$, then the collection $\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ is called a *fundamental set of units* in K.

The proof of Theorem 2.1 begins with the following lemma.

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Lemma 2.2 *Let K be a number field having Archimedean places $\{v_1, v_2, \ldots, v_n\}$, let $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ be a fundamental set of units in K, and define*

$$A = \begin{pmatrix} \log\|\alpha_1\|_{v_1} & \log\|\alpha_1\|_{v_2} & \cdots & \log\|\alpha_1\|_{v_n} \\ \log\|\alpha_2\|_{v_1} & \log\|\alpha_2\|_{v_2} & \cdots & \log\|\alpha_2\|_{v_n} \\ \vdots & \vdots & \ddots & \vdots \\ \log\|\alpha_{n-1}\|_{v_1} & \log\|\alpha_{n-1}\|_{v_2} & \cdots & \log\|\alpha_{n-1}\|_{v_n} \end{pmatrix}.$$

Then $\operatorname{rank}(A) = n - 1$ *and* $\dim(\ker A) = 1$.

Proof For simplicity, we write $D_i = [K_{v_i} : \mathbb{Q}_{v_i}]$ and define the following additional matrices:

$$D = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_n \end{pmatrix}$$

and

$$B = AD = \begin{pmatrix} D_1 \log\|\alpha_1\|_{v_1} & D_2 \log\|\alpha_1\|_{v_2} & \cdots & D_n \log\|\alpha_1\|_{v_n} \\ D_1 \log\|\alpha_2\|_{v_1} & D_2 \log\|\alpha_2\|_{v_2} & \cdots & D_n \log\|\alpha_2\|_{v_n} \\ \vdots & \vdots & \ddots & \vdots \\ D_1 \log\|\alpha_{n-1}\|_{v_1} & D_2 \log\|\alpha_{n-1}\|_{v_2} & \cdots & D_n \log\|\alpha_{n-1}\|_{v_n} \end{pmatrix}.$$

Clearly $\det(D) \neq 0$ so that A and B must have the same rank. However, if we let B_i denote the matrix obtained by removing column i from B, then it is well-known that $|\det(B_i)|$ is non-zero and independent of i. This value is called the *regulator of* K and is thoroughly studied throughout the literature on algebraic number theory (see [12, Def. 10.8], for example). In any case, it now follows that the rows of B are linearly independent so that

$$\operatorname{rank}(A) = \operatorname{rank}(B) = n - 1.$$

Now applying the rank-nullity theorem, we conclude that $\dim(\ker(A)) = 1$. ∎

Our next result is the primary ingredient in identifying the kernel of Φ^*. As we shall find, it also plays a crucial role in showing that Φ^* is surjective.

Lemma 2.3 *Suppose that $c : \mathcal{J} \to \mathbb{R}$ is a consistent map and K is a number field. If $\Phi_c(\alpha) = 0$ for all $\alpha \in K^\times$ then*

$$c(K, v) = c(\mathbb{Q}, \infty)\lambda(K, v) \tag{2.1}$$

for all $v \in M_K$.

Proof We first establish (2.1) in the case that v is Archimedean. To this end, we let $v_1, v_2, \ldots, v_n$ be the complete list of Archimedean places of K so that the consistency property (1.1) implies that

$$c(\mathbb{Q}, \infty) = \sum_{i=1}^{n} c(K, v_i). \tag{2.2}$$

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If $n = 1$ then $\lambda(K, v_1) = 1$, and the required property follows immediately from (2.2). Therefore, we shall assume that $n \geq 2$.

According to Dirichlet's Unit Theorem, we may let $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ be a set of fundamental units in K and let A be the $(n-1) \times n$ matrix A given by Lemma 2.2. Additionally, we let

$$C = \begin{pmatrix} c(K, v_1) \\ c(K, v_2) \\ \vdots \\ c(K, v_n) \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} \lambda(K, v_1) \\ \lambda(K, v_2) \\ \vdots \\ \lambda(K, v_n) \end{pmatrix}.$$

As we have assumed that $\Phi_c(\alpha) = 0$ for all $\alpha \in K$, we have

$$AC = \begin{pmatrix} \Phi_c(\alpha_1) \\ \Phi_c(\alpha_2) \\ \vdots \\ \Phi_c(\alpha_{n-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so that $C \in \ker(A)$. However, the product formula implies that Λ also belongs to $\ker(A)$, and since Λ is clearly non-zero, Lemma 2.2 establishes that $\ker(A) = \{r\Lambda : r \in \mathbb{R}\}$. We now obtain a real number r such that

$$c(K, v_i) = r\lambda(K, v_i) \quad \text{for all } 1 \leq i \leq n.$$

Finally, property (2.2) shows that

$$c(\mathbb{Q}, \infty) = \sum_{i=1}^{n} c(K, v_i) = r \sum_{i=1}^{n} \lambda(K, v_i) = r$$

and we have established (2.1) for all Archimedean places v of K.

We now establish (2.1) when v is non-Archimedean. Because of what we have already shown, we may assume for the remainder of this proof that

$$c(K, u) = c(\mathbb{Q}, \infty) \frac{[K_u : \mathbb{Q}_\infty]}{[K : \mathbb{Q}]} \tag{2.3}$$

for all Archimedean places u of K. According to [16, Theorem 3.1], there exists $\beta \in K$ such that $\|\beta\|_v < 1$ and $\|\beta\|_w = 1$ for all other non-Archimedean places w of K. Now applying (2.3), we obtain

$$\begin{aligned} \Phi_c(\beta) &= c(K, v) \log \|\beta\|_v + \sum_{u|\infty} c(K, u) \log \|\beta\|_u \\ &= c(K, v) \log \|\beta\|_v + c(\mathbb{Q}, \infty) \sum_{u|\infty} \frac{[K_u : \mathbb{Q}_\infty]}{[K : \mathbb{Q}]} \log \|\beta\|_u. \end{aligned}$$

According to the product formula on K, the summation on the right hand side may be simplified so that

$$\Phi_c(\beta) = c(K, v) \log \|\beta\|_v - c(\mathbb{Q}, \infty) \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|\beta\|_v.$$

From our assumptions we have that $\Phi_c(\beta) = 0$ and $\|\beta\|_v \neq 1$, so it follows that

$$c(K, v) = c(\mathbb{Q}, \infty) \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}$$

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as required. ∎

Proof Suppose that $c, d \in \mathcal{J}^*$ and $r \in \mathbb{R}$. For each $\alpha \in \overline{\mathbb{Q}}^\times$, we assume that K is a number field containing α and observe that

$$\begin{aligned}\Phi_{c+d}(\alpha) &= \sum_{v \in M_K} [c(K,v) + d(K,v)] \log \|\alpha\|_v \\ &= \sum_{v \in M_K} c(K,v) \log \|\alpha\|_v + \sum_{v \in M_K} d(K,v) \log \|\alpha\|_v \\ &= \Phi_c(\alpha) + \Phi_d(\alpha)\end{aligned}$$

which proves that $\Phi^*(c+d) = \Phi^*(c) + \Phi^*(d)$. Also, we have

$$\Phi_{rc}(\alpha) = r \sum_{v \in M_K} c(K,v) \log \|\alpha\|_v = r\Phi_c(\alpha)$$

establishing that $\Phi^*(rc) = r\Phi^*(c)$ and showing that Φ^* is a linear transformation.

Assuming that $c \in \operatorname{span}_{\mathbb{R}}\{\lambda\}$, there exists $r \in \mathbb{R}$ such that

$$c(K,v) = r \cdot \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}$$

for all number fields K and all places v of K. For each non-zero point $\alpha \in \overline{\mathbb{Q}}$, the product formula now implies that

$$\Phi_c(\alpha) = \sum_{v \in M_K} c(K,v) \log \|\alpha\|_v = r \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|\alpha\|_v = 0$$

proving that $\Phi_c \equiv 0$ and $\operatorname{span}_{\mathbb{R}}\{\lambda\} \subseteq \ker(\Phi^*)$.

Now assuming that $\Phi_c \equiv 0$ and $(K,v) \in \mathcal{J}$, we certainly have that $\Phi_c(\alpha) = 0$ for all $\alpha \in K^\times$. Hence, Lemma 2.3 applies to yield

$$c(K,v) = c(\mathbb{Q}, \infty)\lambda(K,v)$$

for all $v \in M_K$ establishing that $c \in \operatorname{span}_{\mathbb{R}}\{\lambda\}$ and $\ker(\Phi^*) \subseteq \operatorname{span}_{\mathbb{R}}\{\lambda\}$, as required. ∎

3 Extensions of Consistent Maps

Before we continue with the proof of Theorem 1.1, we will provide our proof of Theorem 1.3. As we shall find, Theorem 1.3 is required in the proof of Theorem 1.1, and as such, it makes sense to provide its proof first.

If K is a number field and L/K is a finite extension, we note the two well-known identities

$$[L : K] = \sum_{w \mid v} [L_w : K_v] \quad \text{and} \quad \operatorname{Norm}_{L/K}(\alpha) = \prod_{w \mid v} \operatorname{Norm}_{L_w/K_v}(\alpha) \tag{3.1}$$

for all $\alpha \in L$ (see [8, Eq. (2) and Prop. 4], for example).

Proof If L is any number field, let F be a finite extension of L containing K. Further, if t is a place of F dividing the place v of K, then we write $y_t = y_v$. For

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each $w \in M_L$ we define

$$d_F(L,w) = \sum_{t|w} \frac{[F_t : K_t]}{[F : K]} y_t, \tag{3.2}$$

where the summation on the right hand side of (3.2) runs over places t of F dividing w. We claim that $d_F(L,w)$ is independent of F.

To see this, we suppose that E is a finite extension of F. In the ensuing calculations, we shall adopt the convention of writing t for places of F and s for places of E. Clearly if $s \mid t$ then $y_t = y_s$ and $K_s = K_t$, so we obtain

$$\begin{aligned} d_E(L,w) &= \sum_{s|w} \frac{[E_s : K_s]}{[E : K]} y_s \\ &= \sum_{t|w} \sum_{s|t} \frac{[E_s : F_t] \cdot [F_t : K_t]}{[E : F] \cdot [F : K]} y_t \\ &= \sum_{t|w} \frac{[F_t : K_t]}{[F : K]} y_t \sum_{s|t} \frac{[E_s : F_t]}{[E : F]}. \end{aligned}$$

Now applying the left hand equality of (3.1), we obtain that

$$d_E(L,w) = \sum_{t|w} \frac{[F_t : K_t]}{[F : K]} y_t = d_F(L,w)$$

showing that $d_F(L,w)$ is indeed independent of F. Hence, we may define $c : \mathcal{J} \to \mathbb{R}$ by $c(L,w) = d_F(L,w)$, where F is any number field containing both K and L.

To prove that c is consistent, we assume that M is a finite extension of L and w is a place of L. We select a number field F containing both M and K. Then using x to denote places of M, we obtain from (3.2) that

$$\begin{aligned} \sum_{x|w} c(M,x) &= \sum_{x|w} d_F(M,x) \\ &= \sum_{x|w} \sum_{t|x} \frac{[F_t : K_t]}{[F : K]} y_t \\ &= \sum_{t|w} \frac{[F_t : K_t]}{[F : K]} y_t = d_F(L,w) = c(L,w), \end{aligned}$$

proving that c is consistent.

To establish (1.3), we assume that L is a finite extension of K, v is a place of K, and w is a a place of L dividing v. Now apply (3.2) with $F = L$ to obtain

$$c(L,w) = \frac{[L_w : K_w]}{[L : K]} y_w = \frac{[L_w : K_v]}{[L : K]} y_v$$

which is the required property (1.3).

To prove that this consistent map is unique, we suppose that $c, d \in \mathcal{J}^*$ both satisfy (1.3). This means $c(L,w) = d(L,w)$ for all finite extensions L/K and all places w of L. Now if L' is an arbitrary number field and w' is a place of L', we may choose a number field L containing both K and L'. Then applying the

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consistency of c and d, we obtain

$$c(L', w') = \sum_{w \mid w'} c(L, w) = \sum_{w \mid w'} d(L, w) = d(L', w')$$

proving that $c = d$.

Finally, we assume that $\Phi_c(\alpha) \in \mathbb{Q}$ for all $\alpha \in K^\times$. If $\beta \in \overline{\mathbb{Q}}^\times$ we may let L be a number field containing both β and K. Now applying (1.3) we obtain

$$\begin{aligned}\Phi_c(\beta) &= \sum_{w \in M_L} c(L, w) \log \|\beta\|_w \\ &= \sum_{v \in M_K} \sum_{w \mid v} c(L, w) \log \|\beta\|_w = \sum_{v \in M_K} y_v \sum_{w \mid v} \frac{[L_w : K_v]}{[L : K]} \log \|\beta\|_w .\end{aligned}$$

Then using the right hand equality of (3.1), we find that

$$\begin{aligned}\Phi_c(\beta) &= \sum_{v \in M_K} y_v \sum_{w \mid v} \frac{[L_w : K_v]}{[L : K]} \log \|\mathrm{Norm}_{L_w/K_v}(\beta)\|_v^{1/[L_w : K_v]} \\ &= \frac{1}{[L : K]} \sum_{v \in M_K} y_v \sum_{w \mid v} \log \|\mathrm{Norm}_{L_w/K_v}(\beta)\|_v \\ &= \frac{1}{[L : K]} \sum_{v \in M_K} y_v \log \|\mathrm{Norm}_{L/K}(\beta)\|_v \\ &= \frac{1}{[L : K]} \Phi_c(\mathrm{Norm}_{L/K}(\beta)).\end{aligned}$$

which is clearly rational. ∎

4 Surjectivity of Φ^*

In order to complete the proof of Theorem 1.1, we must establish the following result.

***Theorem* 4.1** *The map $\Phi^* : \mathcal{J}^* \to \mathcal{L}(\mathcal{G}, \mathbb{R})$ is surjective.*

The proof of Theorem 4.1 requires some background notation as well as three lemmas. Let K be a number field and let S be a finite subset of M_K containing the Archimedean places of K. The set

$$U_{K,S} = \left\{\alpha \in K^\times : \|\alpha\|_v = 1 \text{ for all } v \in M_K \setminus S\right\}$$

is a subgroup of $K^\times$ called the *group of S-units in K*, which according to Dirichlet's S-unit Theorem (see [14, Thm. III.3.5] or [7, §1.1]), is finitely generated of rank $\#S - 1$. If ζ is a root of unity and $\alpha_1, \alpha_2, \ldots, \alpha_{\#S-1} \in U_{K,S}$ are such that $\{\zeta, \alpha_1, \alpha_2, \cdots, \alpha_{\#S-1}\}$ generates $U_{K,S}$, the then collection $\{\alpha_1, \alpha_2, \cdots, \alpha_{\#S-1}\}$ is called a *fundamental set of S-units* in K. Of course, this terminology generalizes our definitions from the beginning of Section 2.

For our proof that Φ^* is surjective, we require a reinterpretation of Dirichlet's S-unit Theorem in the language of linear algebra. To this end, we write

$$K_{\mathrm{div}} = \left\{\alpha \in \overline{\mathbb{Q}}^\times : \alpha^n \in K \text{ for some } n \in \mathbb{N}\right\},$$

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and note that K_{div} is a subgroup of $\overline{\mathbb{Q}}^{\times}$ containing $\overline{\mathbb{Q}}^{\times}_{\text{tors}}$. We further write

$$\mathcal{G}_K = K_{\text{div}}/\overline{\mathbb{Q}}^{\times}_{\text{tors}}$$

and observe that $\mathcal{G}_K$ is a subspace of $\mathcal{G}$. If $\alpha \in K_{\text{div}}$, we shall adopt the convention of writing $\bar{\alpha}$ to denote its image under the canonical homomorphism $K_{\text{div}} \to K_{\text{div}}/\overline{\mathbb{Q}}^{\times}_{\text{tors}}$. For each place v of K and $\alpha \in K_{\text{div}}$, we may assume that $\ell \in \mathbb{N}$ is such that $\alpha^{\ell} \in K$ and define

$$\|\alpha\|_v = \|\alpha^{\ell}\|_v^{1/\ell}.$$

The right hand side of this equality does not depend on the choice of ℓ, and hence, $\|\cdot\|_v$ is a well-defined map on $\mathcal{G}_K$ which satisfies

(A1) $\|\alpha\beta\|_v = \|\alpha\|_v\|\beta\|_v$ for all $\alpha, \beta \in K_{\text{div}}$
(A2) $\|\alpha^r\|_v = \|\alpha\|_v^r$ for all $\alpha \in K_{\text{div}}$ and all $r \in \mathbb{Q}$.

Equivalently, $\alpha \mapsto \log\|\alpha\|_v$ defines a linear transformation from $\mathcal{G}_K$ to $\mathbb{R}$ when viewed as $\mathbb{Q}$-vector spaces. Finally, we let

$$\mathcal{G}_{K,S} = \{\alpha \in \mathcal{G}_K : \|\alpha\|_v = 1 \text{ for all } v \in M_K \setminus S\}$$

and note the following manner of identifying a basis for $\mathcal{G}_{K,S}$ over $\mathbb{Q}$.

***Lemma* 4.2** *Suppose that K is a number field having n Archimedean places. Assume that S_∞ is the complete set of Archimedean places of K, $S_0 = \{w_1, w_2, \ldots, w_m\}$ is a finite (possibly empty) set of non-Archimedean places of K, and $S = S_\infty \cup S_0$. Further assume the following:*

(I) $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ is a fundamental set of units in K
(II) For every $1 \le i \le m$, $\beta_i \in K$ is such that $\|\beta_i\|_{w_i} < 1$ and $\|\beta_i\|_w = 1$ for all non-Archimedean places $w \ne w_i$.

Then $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n-1}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\}$ is a basis for $\mathcal{G}_{K,S}$ over $\mathbb{Q}$. In particular, $\mathcal{G}_{K,S}$ is a finite dimensional subspace of $\mathcal{G}_K$ with $\dim(\mathcal{G}_{K,S}) = \#S - 1$.

Proof For each $0 \le k \le m$, we define $T_k = S_\infty \cup \{w_1, w_2, \ldots, w_k\}$ so that $T_m = S$. We shall prove by induction on k that

$$\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n-1}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_k\}$$

is a basis for $\mathcal{G}_{K,T_k}$ for all $0 \le k \le m$. The lemma would then follow by taking the special case $k = m$.

Base Case: Since we have assumed that $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ is a fundamental set of units in K, we obtain that

(i) For every $\alpha \in U_{K,T_0}$, there exist $r_1, r_2, \ldots, r_{n-1} \in \mathbb{Z}$ such that $\alpha = \alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_{n-1}^{r_{n-1}}$.
(ii) If $r_1, r_2, \ldots, r_{n-1} \in \mathbb{Z}$ are such that $\alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_{n-1}^{r_{n-1}} = 1$ then $r_i = 0$ for all i.

Let $\bar{\alpha} \in \mathcal{G}_{K,T_0}$ and assume that α is some representative of $\bar{\alpha}$ in K_{div}. By definition of K_{div}, there exists $\ell \in \mathbb{N}$ such that $\alpha^{\ell} \in K^{\times}$. But clearly we also have that $\|\alpha^{\ell}\|_w = 1$ for all non-Archimedean places w of K, and hence, $\alpha^{\ell} \in U_{K,T_0} = U_{K,S_\infty}$. Then according to (i), there exist $r_1, r_2, \ldots, r_{n-1} \in \mathbb{Z}$ such that $\alpha^{\ell} =$

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$\alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_{n-1}^{r_{n-1}}$. We now conclude that

$$\bar{\alpha}^{\ell} = \bar{\alpha}_1^{r_1}\bar{\alpha}_2^{r_2}\cdots\bar{\alpha}_{n-1}^{r_{n-1}}$$

Since ℓ is a positive integer, we get that

$$\bar{\alpha} = \bar{\alpha}_1^{r_1/\ell}\bar{\alpha}_2^{r_2/\ell}\cdots\bar{\alpha}_{n-1}^{r_{n-1}/\ell},$$

proving that $\bar{\alpha} \in \operatorname{span}\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n-1}\}$.

Now assume that $a_i \in \mathbb{Z}$ and $b_i \in \mathbb{N}$ are such that

$$\bar{\alpha}_1^{a_1/b_1}\bar{\alpha}_2^{a_2/b_2}\cdots\bar{\alpha}_{n-1}^{a_{n-1}/b_{n-1}} = 1.$$

After raising both sides to the $b_1 b_2 \cdots b_{n-1}$ power, we obtain

$$1 = \bar{\alpha}_1^{r_1}\bar{\alpha}_2^{r_2}\cdots\bar{\alpha}_{n-1}^{r_{n-1}} = \overline{\alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_{n-1}^{r_{n-1}}}, \quad \text{where } r_i = a_i\prod_{j\neq i} b_i$$

As a result, there must exist a root of unity ζ such that $\zeta = \alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_{n-1}^{r_{n-1}}$. We certainly have that $\zeta^d = 1$ for some $d \in \mathbb{N}$, and hence,

$$1 = \alpha_1^{dr_1}\alpha_2^{dr_2}\cdots\alpha_{n-1}^{dr_{n-1}}.$$

By applying (ii), we conclude that $dr_i = 0$ for all i, and since d is certainly non-zero, we obtain that $r_i = 0$. It now follows that $a_i = 0$, as required.

Inductive Step: We now let

$$\mathcal{B}_k = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n-1}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_k\}$$

and proceed with the inductive step assuming that $\mathcal{B}_{k-1}$ is a basis for $\mathcal{G}_{K,T_{k-1}}$. We shall first show that $\mathcal{B}_k$ is linearly independent. To this end, we assume that $r_1, r_2, \ldots, r_{n-1}, s_1, s_2, \ldots, s_k \in \mathbb{Q}$ are such that

$$1 = \prod_{i=1}^{n-1}\bar{\alpha}_i^{r_i}\prod_{j=1}^{k}\bar{\beta}_j^{s_j}. \tag{4.1}$$

We remind the reader that $\|\cdot\|_{w_k}$ is indeed a well-defined map on $\mathcal{G}_K$ which satisfies (A1) and (A2). Moreover, by our assumption (II), we have $\|\beta_j\|_{w_k} = 1$ for all $1 \leq j \leq k-1$. Additionally, the points α_i are ordinary units in K, meaning that $\|\alpha_i\|_{w_k} = 1$ for all $1 \leq i \leq n-1$. Now applying $\|\cdot\|_{w_k}$ to both sides of (4.1), we conclude that

$$1 = \|\bar{\beta}_k\|_{w_k}^{s_k}.$$

Again by our assumption (II), we know that $\|\bar{\beta}_k\|_{w_k} < 1$, and hence, we conclude that $s_k = 0$. Plugging this value into (4.1), we obtain

$$1 = \prod_{i=1}^{n-1}\bar{\alpha}_i^{r_i}\prod_{j=1}^{k-1}\bar{\beta}_j^{s_j}$$

so it follows from the inductive hypothesis that $r_i = 0$ for all $1 \leq i \leq n-1$ and $s_j = 0$ for all $1 \leq j \leq k-1$.

Since $\mathcal{B}_k$ contains $n + k - 1$ elements, in order to complete the proof, it is sufficient to show that $\dim(\mathcal{G}_{K,T_k}) \leq n + k - 1$. To this end, we apply Dirichlet's S-unit Theorem with T_k in place of S to obtain a fundamental set of T_k-units $\{\gamma_1, \gamma_2, \ldots, \gamma_{n+k-1}\}$. We claim that

$$\{\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_{n+k-1}\}$$

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spans $\mathcal{G}_{K,T_k}$. To see this, let $\bar{\gamma} \in \mathcal{G}_{K,T_k}$ and let γ be a representative of $\bar{\gamma}$ in K_{div}. Then there exists $\ell \in \mathbb{N}$ such that $\gamma^\ell \in K^\times$, and note that γ^ℓ must belong to U_{K,T_k}. As a result, we obtain integers $r_1, \ldots, r_{n+k-1}$ such that $\gamma^\ell = \prod_{i=1}^{n+k-1} \gamma_i^{r_i}$, and therefore,

$$\bar{\gamma} = \prod_{i=1}^{n+k-1} \bar{\gamma}_i^{r_i/\ell}$$

proving that $\bar{\gamma} \in \mathrm{span}\{\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_{n+k-1}\}$. It now follows that $\dim(\mathcal{G}_{K,T_k}) \leq n + k - 1$, establishing the lemma. ∎

We will also need the following basic linear algebra lemma.

***Lemma* 4.3** *Suppose V is vector space over a field F containing subspaces A and B. Let $x, y \in V$.*

(i) If $A + B = V$ then $(x + A) \cap (y + B) \neq \emptyset$.
(ii) If $A \cap B = \{0\}$ then $(x + A) \cap (y + B)$ contains at most one point.
(iii) If $A \oplus B = V$ then $(x + A) \cap (y + B)$ contains a unique point.

Proof For (i), we observe that $x = a_1 + b_1$ and $y = a_2 + b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and therefore, we obtain

$$x + a_2 = a_1 + a_2 + b_1 \quad \text{and} \quad y + b_1 = a_2 + b_1 + b_2.$$

Hence

$$a_2 + b_1 = x + a_2 - a_1 \in x + A \quad \text{and} \quad a_2 + b_1 = y + b_1 - b_2 \in y + B,$$

showing that $a_2 + b_1 \in (x + A) \cap (y + B)$.

To prove (ii), suppose that $a, b \in (x+A) \cap (y+B)$. It follows that $a - b$ belongs to both A and B, forcing $a - b = 0$, or equivalently, $a = b$. Now (iii) follows from (i) and (ii).

∎

In order to prove Theorem 4.1, we will need to identify a consistent map associated to an arbitrary linear map $\Phi : \mathcal{G} \to \mathbb{R}$. Our next lemma is the key ingredient in constructing such a map. For a linear map $\Phi : \mathcal{G} \to \mathbb{R}$ and $\alpha \in \overline{\mathbb{Q}}^\times$, we shall adopt the convention that $\Phi(\alpha) = \Phi(\bar{\alpha})$.

***Lemma* 4.4** *Suppose $\Phi : \mathcal{G} \to \mathbb{R}$ is a $\mathbb{Q}$-linear map and $r \in \mathbb{R}$. Then for every number field K, there exists a consistent map $c_K : \mathcal{J} \to \mathbb{R}$ such that*

$$c_K(\mathbb{Q}, \infty) = r \quad \text{and} \quad \Phi(\alpha) = \Phi_{c_K}(\alpha) \text{ for all } \alpha \in K^\times. \tag{4.2}$$

Moreover, if L is a finite extension of K and c_L satisfies (4.2) with L in place of K, then $c_L(K, v) = c_K(K, v)$ for all places v of K.

Proof We shall select values y_v for each $v \in M_K$ and then apply Theorem 1.3. As in the proof of Lemma 2.3, we begin by considering the case where v is Archimedean. We suppose that $\{v_1, v_2, \ldots, v_n\}$ are the Archimedean places of K, assume that $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ is a fundamental set of units in K, and let A be the matrix from the statement of Lemma 2.2. In this situation, $\ker(A)$ is a

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one-dimensional subspace of $\mathbb{R}^n$. Writing

$$\Lambda = (\lambda(K, v_1), \lambda(K, v_2), \ldots, \lambda(K, v_n))^T,$$

we see clearly that $\ker(A) = \{r\Lambda : r \in \mathbb{R}\}$. We shall further write

$$\mathbf{b} = (\Phi(\alpha_1), \Phi(\alpha_2), \ldots, \Phi(\alpha_{n-1}))^T$$

and note that $A^{-1}(\mathbf{b})$ is a coset of $\ker(A)$ in $\mathbb{R}^n$.

Now define the $n-1$ dimensional subspace Δ of $\mathbb{R}^n$ by

$$\Delta = \left\{ (x_1, x_2, \ldots, x_n)^T : \sum_{i=1}^{n} x_i = 0 \right\}.$$

Plainly we have that $\Delta \cap \ker(A) = \{0\}$ so that $\mathbb{R}^n = \Delta \oplus \ker(A)$. According to Lemma 4.3(iii), each coset of Δ shares a unique point with each coset of $\ker(A)$. Therefore, there exists a unique point $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ such that

$$\mathbf{x} \in [(r, 0, 0, \ldots, 0) + \Delta] \cap A^{-1}(b). \tag{4.3}$$

We let $y_{v_i} = x_i$ for all i, and as a result, we have defined y_v for each Archimedean place v of K and note that

$$\sum_{v|\infty} y_v = r. \tag{4.4}$$

Now let $\{w_1, w_2, w_3, \ldots\}$ be the complete list of non-Archimedean places of K. For each j, we apply [16, Lemma 3.1] to obtain a point $\beta_j \in \mathcal{O}_K$ such that $\|\beta_j\|_{w_j} < 1$ and $\|\beta_j\|_{w_i} = 1$ for all $i \neq j$. For $m \in \mathbb{N} \cup \{0\}$, we now define

$$S_m = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m\}.$$

Now for each $j \in \mathbb{N}$, we define

$$y_{w_j} = \frac{\Phi(\beta_j) - \sum_{v|\infty} y_v \log \|\beta_j\|_v}{\log \|\beta_j\|_{w_j}} \tag{4.5}$$

so that we have defined y_v for all non-Archimedean places v of K. According to Theorem 1.3, there exists a consistent map $c_K : \mathcal{J} \to \mathbb{R}$ such that

$$c_K(K, v) = y_v \quad \text{for all } v \in M_K. \tag{4.6}$$

Because of (4.4), c_K clearly satisfies the first equality of (4.2) so it remains only to show the second equality.

To see this, we obtain from (4.3) that

$$\begin{pmatrix} \Phi(\alpha_1) \\ \Phi(\alpha_2) \\ \vdots \\ \Phi(\alpha_{n-1}) \end{pmatrix} = b = A\mathbf{x} = A \begin{pmatrix} y_{v_1} \\ y_{v_2} \\ \vdots \\ y_{v_n} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_{v_i} \log \|\alpha_1\|_{v_i} \\ \sum_{i=1}^n y_{v_i} \log \|\alpha_2\|_{v_i} \\ \vdots \\ \sum_{i=1}^n y_{v_i} \log \|\alpha_{n-1}\|_{v_i} \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} \Phi(\alpha_1) \\ \Phi(\alpha_2) \\ \vdots \\ \Phi(\alpha_{n-1}) \end{pmatrix} = \begin{pmatrix} \sum_{v|\infty} c_K(K, v) \log \|\alpha_1\|_v \\ \sum_{v|\infty} c_K(K, v) \log \|\alpha_2\|_v \\ \vdots \\ \sum_{v|\infty} c_K(K, v) \log \|\alpha_{n-1}\|_v \end{pmatrix}.$$

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Each point α_i is a unit, so we have $\|\alpha_i\|_v = 1$ for all non-Archimedean places v of K. Therefore, it follows that

$$\begin{pmatrix} \Phi(\alpha_1) \\ \Phi(\alpha_2) \\ \vdots \\ \Phi(\alpha_{n-1}) \end{pmatrix} = \begin{pmatrix} \Phi_{c_K}(\alpha_1) \\ \Phi_{c_K}(\alpha_2) \\ \vdots \\ \Phi_{c_K}(\alpha_{n-1}) \end{pmatrix},$$

or in other words,

$$\Phi(\alpha_i) = \Phi_{c_K}(\alpha_i) \quad \text{for all } 1 \leq i \leq n-1.$$

Next, we let $j \in \mathbb{N}$ and observe that

$$\begin{aligned} \Phi_{c_K}(\beta_j) &= \sum_{v \in M_K} c(K,v) \log \|\beta_j\|_v \\ &= \sum_{i=1}^{\infty} c(K,w_i) \log \|\beta_j\|_{w_i} + \sum_{v|\infty} c(K,v) \log \|\beta_j\|_v \\ &= c(K,w_j) \log \|\beta_j\|_{w_j} + \sum_{v|\infty} c(K,v) \log \|\beta_j\|_v, \end{aligned}$$

where the last equality follows because of our assumption that $\|\beta_j\|_{w_i} = 1$ for all $i \neq j$. Now using (4.6), we obtain

$$\Phi_{c_K}(\beta_j) = y_{w_j} \log \|\beta_j\|_{w_j} + \sum_{v|\infty} y_v \log \|\beta_j\|_v,$$

and using (4.5) to replace y_{w_j}, we quickly obtain

$$\Phi(\beta_j) = \Phi_{c_K}(\beta_j) \quad \text{for all } j \in \mathbb{N}.$$

Assuming now that $\alpha \in K^\times$, there exists $m \in \mathbb{N}$ such that α is an S_m-unit. In other words, we have $\bar{\alpha} \in \mathcal{G}_{K,S_m}$. According to Lemma 4.2, $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n-1}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\}$ is a basis for $\mathcal{G}_{K,S_m}$, so there exist $r_1, \ldots, r_{n-1}, s_1, \ldots s_m \in \mathbb{Q}$ such that

$$\bar{\alpha} = \prod_{i=1}^{n-1} \bar{\alpha}_i^{r_i} \prod_{i=1}^{m} \bar{\beta}_j^{s_j}.$$

Since Φ and Φ_{c_K} are both linear maps, we conclude that $\Phi(\bar{\alpha}) = \Phi_{c_K}(\bar{\alpha})$, as required.

For the second statement of the Lemma, we have assumed that $\Phi_{c_L}(\alpha) = \Phi(\alpha)$ for all $\alpha \in L^\times$, so in particular, this equality holds for all $\alpha \in K^\times$. As a result, we have

$$\Phi_{c_L}(\alpha) = \Phi_{c_K}(\alpha) \quad \text{for all } \alpha \in K^\times.$$

For each such point α, Theorem 2.1 implies that $0 = \Phi_{c_L}(\alpha) - \Phi_{c_K}(\alpha) = \Phi_{c_L - c_K}(\alpha)$. If v is a place of K, we may apply Lemma 2.3 with $c = c_K - c_L$ to conclude that

$$[c_L - c_K](K,v) = [c_L(\mathbb{Q},\infty) - c_K(\mathbb{Q},\infty)]\lambda(K,v) = (r-r)\lambda(K,v) = 0,$$

so the result follows immediately. ∎

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We now complete the proof of Theorem 1.1 by proving that Φ^* is surjective.

Proof We suppose that $\Phi : \mathcal{G} \to \mathbb{R}$ is a $\mathbb{Q}$-linear map. For each number field K, we apply Lemma 4.4 with $r = 0$ to obtain a consistent map $c_K : \mathcal{J} \to \mathbb{R}$ such that

$$c_K(\mathbb{Q}, \infty) = 0 \quad \text{and} \quad \Phi(\alpha) = \Phi_{c_K}(\alpha) \text{ for all } \alpha \in K^\times.$$

Now we let $c : \mathcal{J} \to \mathbb{R}$ be given by

$$c(K, v) = c_K(K, v)$$

and claim that c is consistent. To this end, we assume that K is a number field, v is a place of K, and L is a finite extension of K. Using the fact that c_L is consistent, we obtain that

$$\sum_{w \mid v} c(L, w) = \sum_{w \mid v} c_L(L, w) = c_L(K, v),$$

and then applying the second statement of Lemma 4.4, we conclude that

$$\sum_{w \mid v} c(L, w) = c_K(K, v) = c(K, v).$$

Since c is now known to be consistent, the function Φ_c is well-defined. Moreover, if $\alpha \in \overline{\mathbb{Q}}^\times$, we let K be a number field containing α and apply the first statement of Lemma 4.4 to obtain

$$\Phi_c(\alpha) = \sum_{v \in M_K} c(K, v) \log \|\alpha\|_v = \sum_{v \in M_K} c_K(K, v) \log \|\alpha\|_v = \Phi_{c_K}(\alpha) = \Phi(\alpha).$$

As this equality holds for all $\alpha \in \overline{\mathbb{Q}}^\times$, we have shown that $\Phi^*(c) = \Phi$ implying that Φ^* is surjective. ∎

Our only remaining task is to prove Theorem 1.4.

Proof For simplicity, we let $\Phi : K^\times \to \mathbb{R}$ be given by

$$\Phi(\alpha) = \sum_{v \in M_K} y_v \log \|\alpha\|_v$$

and first assume that $\Phi(\alpha) \in \mathbb{Q}$ for all $\alpha \in K^\times$. Each point α_i is a unit in K, and hence, $\|\alpha_i\|_w = 1$ for all non-Archimedean places w of K. For all $1 \le i \le n$, we now have that

$$\sum_{i=1}^{n} y_i \log \|\alpha_i\|_{v_i} = \Phi(\alpha_i) \in \mathbb{Q},$$

and (i) follows immediately. Similarly, we have assumed that $\|\beta_v\|_w = 1$ for all non-Archimedean places $w \neq v$, so we conclude immediately that (ii) holds.

Now suppose that (i) and (ii) hold and let $\alpha \in K^\times$. Let S_∞ be the set of Archimedean places of K. There exists a finite set S of places of K containing S_∞ such that $\alpha \in U_{K,S}$, and hence, $\bar{\alpha} \in \mathcal{G}_{K,S}$. Let $S_0 = S \setminus S_\infty$. Now according to Lemma 4.2, the set

$$\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n-1}\} \cup \{\beta_v : v \in S_0\}$$

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is a basis for $\mathcal{G}_{K,S}$. Therefore, there exist $r_1, r_2, \ldots, r_{n-1} \in \mathbb{Q}$ and $s_v \in \mathbb{Q}$ for all $v \in S_0$ such that

$$\bar{\alpha} = \prod_{i=1}^{n-1} \bar{\alpha}_i^{r_i} \prod_{v \in S_0} \bar{\beta}_v^{s_v}.$$

Now using our definitions presented prior to Lemma 4.2, Φ is a well-defined linear map on $\mathcal{G}_{K,S}$, and hence

$$\Phi(\alpha) = \Phi(\bar{\alpha}) = \sum_{i=1}^{n-1} r_i \Phi(\bar{\alpha}_i) + \sum_{v \in S_0} s_v \Phi(\bar{\beta}_v) = \sum_{i=1}^{n-1} r_i \Phi(\alpha_i) + \sum_{v \in S_0} s_v \Phi(\beta_v)$$

It follows from (i) that $\Phi(\alpha_i) \in \mathbb{Q}$ for all i, and (ii) implies that $\Phi(\beta_v) \in \mathbb{Q}$ for all $v \in S_0$. We now obtain $\Phi(\alpha) \in \mathbb{Q}$, as required. ∎

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References

[1] L. Aberg and C. L. Samuels. The connections between consistent maps and measures. *PUMP J. Undergrad. Res.*, 7:69–78, 2024.
[2] D. Allcock and J. D. Vaaler. A Banach space determined by the Weil height. *Acta Arith.*, 136(3):279–298, 2009.
[3] K. Atanassov. New integer functions, related to ψ and σ functions. IV. *Bull. Number Theory Related Topics*, 12:31–35, 1988.
[4] V. I. Bogachev. *Measure theory. Vol. I, II*. Springer-Verlag, Berlin, 2007.
[5] R. E. Dressler and J. van de Lune. Some remarks concerning the number theoretic functions $\omega(n)$ and $\Omega(n)$. *Proc. Amer. Math. Soc.*, 41:403–406, 1973.
[6] N. Dunford and J. T. Schwartz. *Linear operators. Part I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.
[7] P. Fili and Z. Miner. A generalization of Dirichlet's unit theorem. *Acta Arith.*, 162(4):355–368, 2014.
[8] A. Fröhlich. Local fields. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 1–41. Academic Press, London, 1967.
[9] G. H. Hardy and S. Ramanujan. The normal number of prime factors of a number n [Quart. J. Math. **48** (1917), 76–92]. In *Collected papers of Srinivasa Ramanujan*, pages 262–275. AMS Chelsea Publ., Providence, RI, 2000.
[10] R. Jakimczuk. Sum of prime factors in the prime factorization of an integer. *Int. Math. Forum*, 7(53-56):2617–2621, 2012.
[11] R. Jakimczuk and M. Lalín. The number of prime factors on average in certain integer sequences. *J. Integer Seq.*, 25(2):Art. 22.2.3, 15, 2022.
[12] F. Jarvis. *Algebraic number theory*. Springer Undergraduate Mathematics Series. Springer, Cham, 2014.
[13] M. Lal. Iterates of a number-theoretic function. *Math. Comp.*, 23:181–183, 1969.
[14] W. Narkiewicz. *Elementary and analytic theory of algebraic numbers*. Springer-Verlag, Berlin; PWN—Polish Scientific Publishers, Warsaw, second edition, 1990.
[15] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[16] C. L. Samuels. A classification of $\mathbb{Q}$-valued linear functionals on $\overline{\mathbb{Q}}^{\times}$ modulo units. *Acta Arith.*, 205(4):341–370, 2022.
[17] C. L. Samuels. Consistent maps and their associated dual representation theorems. *Colloq. Math.*, 178(1):11–30, 2025.

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