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LOG CANONICAL THRESHOLDS OF COMPLETE INTERSECTION LOG DEL PEZZO SURFACES

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Abstract We compute the global log canonical thresholds of quasi-smooth well-formed complete intersection log del Pezzo surfaces of amplitude 1 in weighted projective spaces. As a corollary we show the existence of orbifold Kähler–Einstein metrics on many of them.

Keywords: global log canonical threshold; weighted complete intersection; α-invariant; Kähler–Einstein metric; exceptional Fano variety; weakly exceptional Fano variety

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1. Introduction

There are several ways to study singularities of log pairs and one such way is to study their discrepancies. The discrepancy of a log pair (X, D) is used to measure how singular the ambient space X and the divisor D on X are. The discrepancy of a non-log canonical pair is always $-\infty$. In such a case, it gives no more information than non-log canonicity of the log pair. Meanwhile, we can measure how singular the divisor D is by considering smaller divisors proportional to D. To be precise, we define the following invariant: the so-called log canonical threshold. Let X be a normal variety with at worst log canonical singularities, let $Z \subseteq X$ be a closed subvariety and let D be an effective $\mathbb{Q}$-Cartier divisor on X. The log canonical threshold of D along Z on X is the number

$$c_Z(X, D) := \sup\{c \in \mathbb{Q} \mid (X, cD) \text{ is log canonical in an open neighbourhood of } Z\}.$$

For simplicity, we put $c(X, D) = c_X(X, D)$. Unlike the discrepancy, the log canonical threshold can work even when the log pair (X, D) is not log canonical, as long as the ambient variety X has at worst log canonical singularities.

Now we suppose that the variety X is a Fano variety with at most log canonical singularities.

Definition 1.1. The global log canonical threshold of the Fano variety X is the number

$$\mathrm{lct}(X) := \inf\left\{ c_X(X,D) \;\middle|\; \begin{array}{c} D \text{ is an effective } \mathbb{Q}\text{-Cartier } \mathbb{Q}\text{-divisor on } X \\ \mathbb{Q}\text{-linearly equivalent to } -K_X \end{array} \right\}.$$

Tian, meanwhile, introduced the α-invariant to study the existence of Kähler–Einstein metrics on Fano manifolds. Let X be a Fano manifold of dimension n and let g be a Kähler metric on X. In local coordinates $(z_1, \dots, z_n)$, we may write

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$$

so that we can obtain a Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{i\bar{j}}\, \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j \in c_1(X)$$

of the Kähler metric g. Let $P(X, g)$ be the set of smooth functions φ in $\mathcal{C}^2(X)$ such that

$$g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \geqslant 0$$

and $\sup_X \varphi = 0$. The α-invariant of the manifold X is defined by the number

$$\alpha(X,g) = \sup\left\{ \alpha > 0 \;\middle|\; \exists C > 0 \text{ such that } \int_X \mathrm{e}^{-\alpha\varphi}\, \mathrm{d}V_X \leqslant C \text{ for all } \varphi \in P(X,g) \right\}$$

where $\mathrm{d}V_X = \omega_g^n$. The number $\alpha(X, g)$ was introduced in [**19**] and [**20**].

The α-invariant of a smooth Fano manifold is proved to coincide with its global log canonical threshold in [**9**]. Thus, the global log canonical threshold is just another name for the α-invariant. However, the global log canonical threshold provides algebraic methods to compute the α-invariant of a given Fano manifold that are relatively easy in comparison to analytic methods.

The global log canonical threshold turns out to play an important role in both birational geometry and complex geometry. We have two significant applications of the global log canonical threshold of a Fano variety X that motivate the present paper.

The first application is for the case in which $\mathrm{lct}(X) \geqslant 1$. For this we give the following definitions.

Definition 1.2 (Prokhorov [18, Definition 4.1.2]). Let (X, D) be a log pair, where D is a boundary. Then a $\mathbb{Q}$-complement of $K_X + D$ is a log divisor $K_X + D'$ such that $D' \geqslant D$, $K_X + D' \sim_{\mathbb{Q}} 0$ and $K_X + D'$ is log canonical.

Definition 1.3 (Prokhorov [18, Definition 4.5.1]). Let $(X/Z \ni P, D)$ be a contraction of varieties such that there is at least one $\mathbb{Q}$-complement of $K_X + D$ near the fibre over P.

- Assume that Z is not a point (local case). Then $(X/Z \ni P, D)$ is said to be exceptional over P if, for any $\mathbb{Q}$-complement of $K_X + D$ near the fibre over P, there exists at most one (not necessarily exceptional) divisor E such that $a(E, D) = -1$.
- Assume that Z is a point (global case). Then (X, D) is said to be exceptional if every $\mathbb{Q}$-complement of $K_X + D$ is Kawamata log terminal.

The following statement shows a strong relationship between local and global exceptional objects.

Proposition 1.4. *Let $(V \ni O)$ be a Kawamata log terminal singularity and let $f\colon (W, E) \to V$ be a purely log terminal blowup of O. The following are then equivalent:*

- *$(V \ni O)$ is exceptional;*
- *$f(E) = O$ and $(E, \mathrm{Diff}_E(0))$ is exceptional;*

where $\mathrm{Diff}_E(0)$ is the effective $\mathbb{Q}$-divisor such that $(K_W + E)|_E = K_E + \mathrm{Diff}_E(0)$.

Proof. See [**18**, Proposition 4.5.5]. □

Similarly, we also define the weakly exceptional singularities.

Definition 1.5 (Kudryavtsev [15, Definition 1.6]). A log canonical singularity $(V \ni O)$ is called weakly exceptional if it admits exactly one purely log terminal blowup up to isomorphism.

Definition 1.6 (Cheltsov *et al.* [7, Definition 1.5]). A Fano variety X is weakly exceptional (strongly exceptional, respectively) if $\mathrm{lct}(X) \geqslant 1$ ($\mathrm{lct}(X) > 1$, respectively).

The following statement also shows a good relationship between local and global weakly exceptional objects.

Proposition 1.7. *Let $V \ni O$ be a Kawamata log terminal singularity and let $f\colon (W, E) \to V$ be a purely log terminal blowup of O. The following are then equivalent:*

- *$(V \ni O)$ is weakly exceptional;*
- *$f(E) = O$ and E is weakly exceptional.*

Proof. See [**5**, Theorem 3.10]. □

We easily see that strong exceptionality implies exceptionality and that exceptionality implies weak exceptionality on a Fano variety.

Let X be a quasi-smooth well-formed complete intersection of a hypersurface of degree d_1 and a hypersurface of degree d_2 (simply, a complete intersection of multidegree $\{d_1, d_2\}$) in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$. Then the surface X is a log del Pezzo surface if and only if $\sum a_i > d_1 + d_2$. Suppose that the surface X is defined by quasi-homogeneous

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polynomial equations $f(x,y,z,t,w)=0$ and $g(x,y,z,t,w)=0$ of degrees d_1 and d_2, respectively. The quasi-homogeneous polynomial equations

$$\left.\begin{array}{r} f(x,y,z,t,w)=0 \\ g(x,y,z,t,w)=0 \end{array}\right\} \subset \mathbb{C}^5 \cong \mathrm{Spec}(\mathbb{C}[x,y,z,t,w])$$

define an isolated quasi-homogeneous singularity $(V \ni O)$, where O is the origin of $\mathbb{C}^5$. If the inequality

$$a_0+a_1+a_2+a_3+a_4-d_1-d_2 \geqslant 1$$

holds, then [**21**, Corollary 2.11] shows that the singularity $(V \ni O)$ is rational. Since V is a complete intersection, the singularity $(V \ni O)$ is Gorenstein, and hence it is canonical (see [**16**, Corollary 5.24] and [**17**, Remark 2.2]). From the weighted blowup of $\mathbb{C}^5$ at the origin O with weight (a_0,a_1,a_2,a_3,a_4), we obtain a purely log terminal blowup of the singularity $(V \ni O)$. It follows from Proposition 1.4 (Proposition 1.7, respectively) that the following conditions are equivalent:

- the singularity $(V \ni O)$ is exceptional (weakly exceptional, respectively);
- the log del Pezzo surface X is exceptional (weakly exceptional, respectively).

We may obtain much information on the exceptionality of V from the global log canonical threshold of X. In [**12**], exceptional log del Pezzo surfaces are called del Pezzo surfaces without tiger. The study of these surfaces is closely related to the uniruledness of affine surfaces (see [**12**]).

The other application of the global log canonical threshold is for the case in which $\mathrm{lct}(X) > \dim(X)/(\dim(X)+1)$. The following result gives the strong connection between the global log canonical threshold and the Kähler–Einstein metric.

Theorem 1.8. *Suppose that X is a Fano variety with at most quotient singularities. Then X admits an orbifold Kähler–Einstein metric if*

$$\mathrm{lct}(X) > \frac{\dim X}{\dim X+1}.$$

Proof. See [**4**, Appendix A]. □

As we have seen so far, from the global log canonical thresholds of Fano varieties, we can obtain various geometrical properties.

In spite of the usefulness of the global log canonical threshold, it is usually difficult to calculate the global log canonical thresholds for arbitrary Fano varieties. However, there are several results that determine the global log canonical thresholds for various Fano varieties. First, the global log canonical thresholds of smooth del Pezzo surfaces were calculated by Cheltsov in [**3**]. Thus, we are concerned with the global log canonical thresholds for singular del Pezzo surfaces. In [**11**], Jonhson and Kollár determined the complete list of del Pezzo hypersurfaces in three-dimensional weighted projective spaces with amplitude 1. They also proved that many of those surfaces admit Kähler–Einstein metrics. Later, Araujo [**1**] also proved that the six surfaces in [**11**] admit Kähler–Einstein metrics. Meanwhile Boyer *et al*. [**2**] extended the results of [**11**] to the case of higher amplitude and used these results to construct a plethora of Sasakian–Einstein metrics in simply connected real five-dimensional manifolds. The completeness of the list in [**2**] was proved by Cheltsov and Shramov in [**6**]. Recently, Cheltsov *et al*. [**7**] calculated the global log canonical thresholds for the examples in [**2**]. They also determined the existence of orbifold Kähler–Einstein metrics and classified exceptional and weakly exceptional quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$. The next stage is to consider complete intersection log del Pezzo surfaces of higher codimensions in weighted projective spaces.

In this paper, we classify all the quasi-smooth well-formed complete intersection log del Pezzo surfaces in weighted projective spaces with amplitude $\alpha = 1$ (see Tables 1 and 2). Note that a quasi-smooth well-formed complete intersection log del Pezzo surface has codimension at most 2 (see Theorem 4.1).

By calculating their global log canonical thresholds of the classified log del Pezzo surfaces, we obtain the following theorem.

Theorem 1.9. *Let X be a quasi-smooth well-formed complete intersection log del Pezzo surface of codimension greater than or equal to 2 in weighted projective space with amplitude $\alpha = 1$, not the intersection of a linear cone with another hypersurface. Suppose that the log del Pezzo surface X is not one of the following:*

- *a complete intersection of multidegree $\{2N, 2N\}$ in $\mathbb{P}(1, 1, N, N, 2N-1)$, where N is a positive integer;*
- *a complete intersection of multidegree $\{6, 8\}$ in $\mathbb{P}(1, 2, 3, 4, 5)$ such that the defining equation of the hypersurface of degree 6 does not contain the monomial yt, where y is the coordinate function of weight 2 and t is the coordinate function of weight 4.*

Then the global log canonical threshold of the log del Pezzo surface X is strictly greater than $\frac{2}{3}$. In particular, the log del Pezzo surface X has an orbifold Kähler–Einstein metric.

Unfortunately, for a quasi-smooth well-formed complete intersection log del Pezzo surface of multidegree $\{4N+2, 4N+3\}$ in weighted projective space $\mathbb{P}(1, 2, 2N+1, 2N+1, 4N+1)$, where N is a positive integer, we cannot determine whether its global log canonical threshold is strictly less than 1 or not. Suppose that X is not such a surface. We can then classify exceptional and weakly exceptional log del Pezzo surfaces by their global log canonical thresholds.

Theorem 1.10. *Let X be a quasi-smooth well-formed complete intersection log del Pezzo surface of multidegree $\{d_1, d_2\}$ in weighted projective space $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$, where $d_1 \leqslant d_2$ and $a_0 \leqslant \cdots \leqslant a_4$, with amplitude $\alpha = 1$, neither the intersection of a linear cone with another hypersurface nor a complete intersection of multidegree $\{4N+2, 4N+3\}$ in $\mathbb{P}(1, 2, 2N+1, 2N+1, 4N+1)$ for any N.*

(1) *The global log canonical threshold of X is strictly greater than* 1 *if and only if the septuple* $(a_0, a_1, a_2, a_3, a_4, d_1, d_2)$ *is one of the following:*

$$\begin{array}{lll}
(2,2,3,3,3,6,6), & (2,3,4,5,5,8,10), & (2,3,5,6,7,10,12), \\
(3,3,5,5,7,10,12), & (3,5,6,8,13,16,18), & (3,5,7,9,11,16,18), \\
(4,5,7,10,13,18,20), & (5,7,10,14,23,28,30), & (5,9,12,20,31,36,40), \\
(5,14,17,21,37,42,51), & (6,7,9,11,14,18,28), & (6,8,9,11,13,22,24), \\
(9,15,23,23,31,46,54), & (9,15,23,23,37,46,60), & (9,23,30,38,67,76,90), \\
(10,17,25,34,43,60,68), & (11,18,27,44,61,72,88), & (11,27,36,62,97,108,124), \\
(11,29,39,49,59,88,98), & (11,29,39,49,67,78,116), & (11,29,38,48,85,96,114), \\
(13,22,55,76,97,110,152), & (13,23,34,56,89,102,112), & (13,23,35,47,57,70,104), \\
(13,23,35,57,79,92,114), & (14,19,25,32,45,64,70). &
\end{array}$$

In which case, the log del Pezzo surface X is exceptional.

(2) *The global log canonical threshold of X is exactly equal to* 1 *if and only if the septuple* $(a_0, a_1, a_2, a_3, a_4, d_1, d_2)$ *is either one of*

$$\begin{array}{lll}
(1,2,2,3,3,4,6), & (1,2,3,4,5,6,8)^*, & (1,3,3,5,5,6,10), \\
(1,4,5,7,11,12,15), & (1,4,7,10,13,14,20), & (1,5,8,12,19,20,24), \\
(1,5,9,13,17,18,26), & (1,7,11,17,27,28,34), & (1,7,12,17,23,24,35), \\
(1,8,13,19,31,32,39), & (1,9,15,23,23,24,46), &
\end{array}$$

where $*$ indicates that the defining equation of the hypersurface of degree 6 *contains the monomial yt, i.e. the product of the coordinate function y of weight* 2 *and the coordinate function t of weight* 4*, or a member of the infinite series*

$$(2, 2N+1, 2N+1, 4N+1, 6N+1, 6N+3, 8N+2),$$

where N runs through the positive integers. In which case, the log del Pezzo surface X is weakly exceptional but not exceptional.

The layout of the paper is as follows. In § 2 we set up the notation that will be used throughout the paper. In § 3 we recall the necessary background on surfaces with quotient singularities. In § 4 we explain how to obtain the complete list of quasi-smooth well-formed complete intersection log del Pezzo surfaces with amplitude 1 in weighted projective spaces. In § 5 we briefly explain the methods that are used to compute the global log canonical thresholds of log del Pezzo surfaces appearing in Tables 1 and 2. In § 6 we provide details of these computations for samples of infinite series of such surfaces and samples of sporadic cases, referring the reader to [**13**] for detailed computations in the remaining cases.

2. Notation

The following notation will be used throughout the paper.

- $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ denotes the weighted projective space $\mathrm{Proj}(\mathbb{C}[x, y, z, t, w])$ with weights $wt(x) = a_0$, $wt(y) = a_1$, $wt(z) = a_2$, $wt(t) = a_3$ and $wt(w) = a_4$, where we always assume that $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$.

- P_x is the point in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ defined by $y = z = t = w = 0$. The points P_y, P_z, P_t and P_w are defined in a similar way.

- X denotes a quasi-smooth well-formed complete intersection log del Pezzo surface defined by quasi-homogeneous polynomials $F_1(x, y, z, t, w)$ and $F_2(x, y, z, t, w)$ of degrees d_1 and d_2, respectively, in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$, where we always assume that $d_1 \leqslant d_2$.

- α is the integer $a_0 + a_1 + a_2 + a_3 + a_4 - d_1 - d_2$. It is called the *amplitude* of X.

- C_x denotes the curve on X cut by the equation $x = 0$. The curves C_y, C_z, C_t and C_w are defined in a similar way.

- L_{xy} is the one-dimensional stratum on $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ defined by $z = t = w = 0$, and the other one-dimensional strata are labelled similarly.

- Let D be a divisor on X and let P be a quotient singular point of type $(1/r)(a, b)$ on X. Then there is an orbifold chart $\pi\colon \tilde{U} \to U$ for some neighbourhood $P \in U \subset X$ such that $\tilde{U}$ is smooth and π is a cyclic cover of degree r unramified over $U \setminus \{P\}$. Note that if $r = 1$, then P is a smooth point on X. Put $D_U = D|_U$ and $D_{\tilde{U}} = \pi^{-1}(D_U)$. Then we write $\mathrm{mult}_P(D_U) = \mathrm{mult}_{\tilde{P}}(D_{\tilde{U}})$, where $\tilde{P}$ is a point on $\tilde{U}$ such that $\pi(\tilde{P}) = P$, and refer to this quantity as the multiplicity of D at P.

- $-K_X$ denotes the anticanonical divisor of X.

3. Preliminaries

Let Y be a singular surface with at most quotient singularities, let D be an effective $\mathbb{Q}$-divisor on Y and let $P \in Y$ be a singular point of type $(1/r)(a, b)$. There is an orbifold chart $\pi\colon \tilde{U} \to U$ for some neighbourhood $U \subset Y$ containing P. Let $\tilde{P}$ be a point in $\tilde{U}$ such that $\pi(\tilde{P}) = P$. Put $D_U = D|_U$ and $D_{\tilde{U}} = \pi^{-1}(D_U)$.

Lemma 3.1. *The log pair (U, D_U) is log canonical at the point P if and only if the log pair $(\tilde{U}, D_{\tilde{U}})$ is log canonical at the point $\tilde{P}$.*

Proof. See [**14**, Proposition 3.16]. □

Let B be an effective $\mathbb{Q}$-divisor on Y such that any component of B is not contained in the support of D. Put $B_U = B|_U$ and $B_{\tilde{U}} = \pi^{-1}(B_U)$. We write $\mathrm{mult}_P(D \cdot B)$ for $\mathrm{mult}_{\tilde{P}}(D_{\tilde{U}} \cdot B_{\tilde{U}})$.

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Lemma 3.2. *The inequality*

$$B \cdot D \geqslant \sum_{Q \in Y} \frac{\mathrm{mult}_Q(B)\,\mathrm{mult}_Q(D)}{r_Q} \geqslant 0$$

holds. Here, the point Q *is of type* $(1/r_Q)(a_Q, b_Q)$.

Proof. It immediately follows from $B \cdot D = \sum_{Q \in Y} \mathrm{mult}_Q(B \cdot D)/r_Q$. □

Suppose that $(Y, \lambda D)$ is not log canonical at a point P for a positive rational number λ.

Lemma 3.3. *The inequality* $\mathrm{mult}_P(D) > 1/\lambda$ *holds.*

Proof. It immediately follows from Lemma 3.1. □

Let C be a reduced and irreducible curve on Y. Write

$$D = mC + \Omega,$$

where m is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve C.

Lemma 3.4. *Suppose that* $P \notin \mathrm{Sing}(Y) \cup \mathrm{Sing}(C)$ *and that* $\lambda m \leqslant 1$. *Then the inequality*

$$(D - mC) \cdot C > \frac{1}{\lambda}$$

holds.

Proof. The log pair $(Y, C + \lambda\Omega)$ is not log canonical at the point P since $\lambda m \leqslant 1$. The inequality

$$\lambda\Omega \cdot C \geqslant \mathrm{mult}_P(\lambda\Omega \cdot C) > 1$$

then follows from [**14**, Theorem 7.5]. Thus, we obtain the inequality

$$(D - mC) \cdot C = \Omega \cdot C > \frac{1}{\lambda}.$$

□

Lemma 3.5. *Suppose that the point* P *is a singular point of type* $(1/r)(a, b)$ *on* Y *and the curve* C *is smooth at* P. *Then,*

$$(D - mC) \cdot C > \frac{1}{\lambda r}.$$

Proof. This immediately follows from Lemmas 3.1, 3.2 and 3.4. □

Now we suppose that $D \equiv_{\mathbb{Q}} -K_Y$.

Lemma 3.6. *Suppose that there is an effective $\mathbb{Q}$-divisor D_0 on Y such that $D_0 \equiv_{\mathbb{Q}} -K_Y$ and $(Y, \lambda D_0)$ is log canonical at the point P. Then there is an effective $\mathbb{Q}$-divisor D' on Y such that $D' \equiv -K_Y$, at least one irreducible component of D_0 is not contained in the support of D' and $(Y, \lambda D')$ is not log canonical at the point P.*

Proof. See [**3**, Remark 2.1]. □

Lemma 3.7. *Suppose that Y is a quasi-smooth well-formed complete intersection log del Pezzo surface of multidegree $\{d_1, d_2\}$ in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$. Let $\pi\colon Y \dashrightarrow \mathbb{P} = \mathbb{P}(a_0, a_1, a_2)$ be the rational map induced by the projection $\mathbb{P}(a_0, a_1, a_2, a_3, a_4) \dashrightarrow \mathbb{P}(a_0, a_1, a_2)$ centred at L_{tw} and let Q be a smooth point in Y such that $Q \notin C_x$. Suppose that $H^0(\mathbb{P}, O_{\mathbb{P}}(k))$ contains*

- *at least two different monomials of the form $x^{\alpha}y^{\beta}$,*
- *at least two different monomials of the form $x^{\gamma}z^{\delta}$,*

where k is a positive integer and the four constants α, β, γ and δ are non-negative integers. Then,

$$\mathrm{mult}_Q(D) \leqslant \frac{kd_1d_2}{a_0a_1a_2a_3a_4}$$

for the case in which Q is not contained in any curve contracted by π.

Proof. The same argument as in the proof of [**1**, Corollary 3.7] gives the proof. □

We consider a cyclic quotient singularity $W = \mathbb{C}^2/\mathbb{Z}_m(a_1, a_2)$, where the a_i are positive integers and $\gcd(a_1, a_2) = 1$. Let x_1 and x_2 be eigencoordinates in $\mathbb{C}^2$ for $\mathbb{Z}_m$. And let $\phi\colon \bar{W} \to W$ be the weighted blow-up at the origin with respect to weights (a_1, a_2). Then $\bar{W}$ is covered by affine charts U_1 and U_2 such that

$$U_1 = \mathbb{C}^2/\mathbb{Z}_{a_1}(m, -a_2), \qquad U_2 = \mathbb{C}^2/\mathbb{Z}_{a_2}(-a_1, m).$$

The coordinates in W and in U_i are related by

$$x_i = y_i^{a_i/m}, \quad x_j = y_j y_i^{a_j/m}, \quad j \neq i,$$

where y_1 and y_2 are eigencoordinates in U_i, for $\mathbb{Z}_{a_i}$.

Lemma 3.8. *Let $E := \phi^{-1}(P)$ be the exceptional divisor of ϕ. In the above conditions we have*

- $K_{\bar{W}} \equiv_{\mathbb{Q}} \phi^*(K_W) + \left(-1 + \dfrac{a_1}{m} + \dfrac{a_2}{m}\right)E$,
- *if $F = \{x_i = 0\}/\mathbb{Z}_m$ and $\bar{F}$ is the proper transform of F, then $\bar{F} \equiv_{\mathbb{Q}} \phi^*(F) - \dfrac{a_i}{m}E$,*
- $E^2 = -\dfrac{m}{a_1a_2}$.

Proof. See [**18**, Chapter 3.2]. □

4. Classification

In this section, we will classify all the quasi-smooth well-formed complete intersection log del Pezzo surfaces in weighted projective spaces with amplitude $\alpha = 1$. For this purpose, we let Y be a complete intersection of hypersurfaces defined by quasi-homogeneous polynomials F_i of degrees d_i, $i = 1, 2, \ldots, c$, in the weighted projective space $\mathbb{P}(a_0, \ldots, a_n) \cong \mathrm{Proj}(\mathbb{C}[x_0, \ldots, x_n])$, where $wt(x_i) = a_i$, with amplitude α, not the intersection of a linear cone with another hypersurface.

Theorem 4.1. *Suppose that the complete intersection Y is quasi-smooth. Then the codimension c of Y in $\mathbb{P}(a_0, \ldots, a_n)$ satisfies*

$$c \leqslant \begin{cases} \dim Y + \alpha + 1 & \text{if } \alpha \leqslant 0, \\ \dim Y & \text{if } \alpha > 0, \end{cases}$$

where $\alpha = \sum a_i - \sum d_j$.

Proof. See [**8**, Theorem 1.3]. □

Theorem 4.1 implies that the codimension of a quasi-smooth complete intersection log del Pezzo surface in a weighted projective space with amplitude $\alpha = 1$ is either 1 or 2. Reference [**11**] provides the complete list of well-formed quasi-smooth hypersurfaces in three-dimensional weighted projective spaces with amplitude $\alpha = 1$. In the present section, we classify codimension 2 cases. The following are the key theorems in this section. From now on, we assume that $c = 2$ and that Y is general.

Theorem 4.2. *The complete intersection Y is quasi-smooth if and only if for every set $I = \{i\}$, $i = 0, \ldots, n$, one of*

$QS_1(i)$: there exists a monomial $x_i^{d_1}$ in F_1,

$QS_2(i)$: there exists a monomial $x_i^{d_2}$ in F_2,

$QS_3(i)$: there exist monomials $x_i^{d_1-1}x_{e_1}$ in F_1 and $x_i^{d_2-1}x_{e_2}$ in F_2 such that e_1 and e_2 are distinct elements

holds, and for each non-empty subset $I = \{i_0, \ldots, i_{k-1}\}$ of $\{0, \ldots, n\}$, where $k \geqslant 2$, one of

- *there exists a monomial $x_{i_0}^{m_{1,i_0}} \cdots x_{i_{k-1}}^{m_{1,i_{k-1}}}$ of degree d_1 in F_1 and there exists a monomial $x_{i_0}^{m_{2,i_0}} \cdots x_{i_{k-1}}^{m_{2,i_{k-1}}}$ of degree d_2 in F_2,*
- *there exists a monomial $x_{i_0}^{m_{1,i_0}} \cdots x_{i_{k-1}}^{m_{1,i_{k-1}}}$ of degree d_1 in F_1 and there exist monomials $x_{i_0}^{m_{2,i_0}} \cdots x_{i_{k-1}}^{m_{2,i_{k-1}}} x_{e_\mu}$ of degree d_2 in F_2 for $\mu = 1, \ldots, k-1$, where the $\{e_\mu\}$ are $k-1$ distinct elements,*
- *there exists a monomial $x_{i_0}^{m_{2,i_0}} \cdots x_{i_{k-1}}^{m_{2,i_{k-1}}}$ of degree d_2 in F_2 and there exist monomials $x_{i_0}^{m_{1,i_0}} \cdots x_{i_{k-1}}^{m_{1,i_{k-1}}} x_{e_\mu}$ of degree d_1 in F_1 for $\mu = 1, \ldots, k-1$, where the $\{e_\mu\}$ are $k-1$ distinct elements,*

- *for* $\mu = 1, \ldots, k$, *there exist monomials* $x_{i_0}^{m_{1,i_0}} \cdots x_{i_{k-1}}^{m_{1,i_{k-1}}} x_{e_{1,\mu}}$ *of degree* d_1 *in* F_1 *and* $x_{i_0}^{m_{2,i_0}} \cdots x_{i_{k-1}}^{m_{2,i_{k-1}}} x_{e_{2,\mu}}$ *of degree* d_2 *in* F_2 *such that* $\{e_{1,\mu}\}$ *are* k *distinct elements,* $\{e_{2,\mu}\}$ *are* k *distinct elements and* $\{e_{1,\mu}, e_{2,\mu}\}$ *contains at least* $k+1$ *distinct elements*

holds.

Proof. See [**10**, Theorem 8.7]. □

Theorem 4.3. *The complete intersection* Y *is well formed if and only if:*

- *for all distinct* i, j *and* k, *with* $h = \gcd(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, \widehat{a_k}, \ldots, a_n)$, *either* $h|d_1$ *or* $h|d_2$;
- *for all distinct* i *and* j, *with* $h = \gcd(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n)$, *we have* $h|d_1$ *and* $h|d_2$;
- $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ *for all* i.

Proof. See [**10**, Theorem 6.11]. □

Using Theorem 4.2, we classify all four-dimensional weighted projective spaces with two quasi-homogeneous polynomials of degrees d_1 and d_2 that define quasi-smooth well-formed complete intersection log del Pezzo surfaces with amplitude $\alpha = 1$. To this end, we use the septuples $(a_0, a_1, a_2, a_3, a_4, d_1, d_2)$ to represent the weighted projective spaces $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ with two quasi-homogeneous polynomials of degrees d_1 and d_2.

If a septuple $(a_0, a_1, a_2, a_3, a_4, d_1, d_2)$ comes from a quasi-smooth well-formed complete intersection log del Pezzo surface Y with amplitude $\alpha = 1$, then it must satisfy

$$a_0 + a_1 + a_2 + a_3 + a_4 - d_1 - d_2 = 1. \tag{4.1}$$

It also satisfies the equations of the form

$$d_i = m_{i0}a_0 + \cdots + m_{i4}a_4, \tag{4.2}$$

where m_{ij}, $i = 1, 2$, $j = 0, \ldots, 4$, are non-negative integers, if and only if the monomials

$$x_0^{m_{i0}} x_1^{m_{i1}} \cdots x_4^{m_{i4}}$$

appear in the polynomials F_i.

The following theorem shows that one can find upper bounds for m_{ij}, $i = 1, 2$, $j = 1, \ldots, 4$, except for the case of two infinite series.

Theorem 4.4. *Let* X *be a quasi-smooth well-formed complete intersection log del Pezzo surface defined by two quasi-homogeneous polynomials* F_1 *and* F_2 *of degree* d_1 *and* d_2, *respectively, in the weighted projective space* $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ *with amplitude* $\alpha = 1$. *Suppose that the log del Pezzo surface* X *is not the intersection of a linear cone with another hypersurface. Suppose that the septuple* $(a_0, a_1, a_2, a_3, a_4, d_1, d_2)$ *is neither*

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of the following:

- $(1, 1, M, M, 2M-1, 2M, 2M)$,
- $(1, 2, 2M+1, 2M+1, 4M+1, 4M+2, 4M+3)$,

where M is a positive integer. Then $8a_1 > d_2$.

Proof. Set $S = (a_0, a_1, a_2, a_3, a_4, d_1, d_2)$. In order to prove the statement, we mainly use the first condition in Theorem 4.2. First of all the septuple S must satisfy one of $QS_{q_4}(4)$, $q_4 = 1, 2, 3$.

Case 1 (the septuple S satisfies $QS_1(4)$). In this case, we have $d_1 = n_4a_4$ for some positive integer $n_4 > 1$. Since $d_1 \leqslant d_2$, we have the inequality $a_0 + a_1 + a_2 + a_3 + a_4 - 2n_4a_4 \geqslant 1$. This inequality shows that $d_1 = 2a_4$ and $2a_1 > a_4$. Furthermore, it is easy to see that $d_2 = 2a_4$, and hence $4a_1 > d_2$.

Case 2 (the septuple S satisfies $QS_2(4)$ but not $QS_1(4)$). In this case, $d_2 = m_4a_4$ for some positive integer $m_4 > 1$ but d_1 is not divisible by a_4. By the second condition of Theorem 4.2 for $I = \{3, 4\}$, the quasi-homogeneous polynomial F_1 has monomials of the form $x_4^{m_4}x_3^{m_3}x_e$ such that $m_4 + m_3 \geqslant 1$ and $e \in \{0, 1, 2, 3, 4\}$. Then

$$d_1 = m_4a_4 + m_3a_3 + a_e, \tag{4.3}$$

where $m_3 + m_4 = 1$ due to (4.1), which implies that $d_1 \geqslant a_3 + a_0$. Thus, $d_2 = 2a_4$ due to (4.1).

If $e = 2$, 3 or 4, then $2a_1 > a_4$, and hence $4a_1 > 2a_4 = d_2$.

Now we consider (4.3) with $e = 0$ and 1. Then we obtain $2a_2 > a_4$ from (4.1), and hence $4a_2 > 2a_4 = d_2$.

The septuple S must also satisfy one of $QS_{q_2}(2)$, $q_2 = 1, 2, 3$.

Suppose that the septuple S satisfies $QS_1(2)$. Then $d_1 = 2a_2$ and $2a_1 > a_2$. Therefore, $8a_1 > 4a_2 > d_2$.

Next, suppose that the septuple S satisfies $QS_2(2)$. Then the septuple S satisfies either $d_2 = 2a_2$ or $d_2 = 3a_2$. If the septuple S satisfies $d_2 = 2a_2$, then $a_2 = a_3 = a_4$. However, the well-formed condition contradicts the assumption that the septuple S does not satisfy $QS_1(4)$. Therefore, $d_2 = 3a_2$. Since $d_1 \geqslant a_3+a_0$ and $a_4 = \frac{3}{2}a_2$, (4.1) implies that $2a_1 > a_2$, and hence $8a_1 > d_2$.

Now we suppose that the septuple S satisfies $QS_3(2)$ but none of $QS_1(2)$ and $QS_2(2)$. Then the septuple S satisfies $d_1 = n_2a_2 + a_{i_2}$ and $d_2 = m_2a_2 + a_{j_2}$, where $i_2 \neq 2$, $j_2 \neq 2$, $i_2 \neq j_2$, and n_2 and m_2 are positive integers. Equation (4.1) and $d_2 = 2a_4$ imply that $n_2 = 1$. The inequality $4a_2 > d_2$ implies that m_2 is at most 3. If $m_2 = 1$, then the septuple S is $(1, 1, 1, 1, 1, 2, 2)$, which contradicts our assumption. For the case in which $m_2 = 3$, (4.1) and (4.3) with the inequality $a_4 < 2a_2$ show that $i_2 = 1$, $j_2 = 0$ and $d_1 = a_3 + a_0$. Then,

$$S \in \{(a_0, a_1, 2a_1 - a_0 - 2, 3a_1 - 2a_0 - 2, 3a_1 - a_0 - 3, 3a_1 - a_0 - 2, 6a_1 - 2a_0 - 6) \in \mathbb{Z}^7 \mid 1 \leqslant a_0 \leqslant a_1\}.$$

Thus, we see that $6a_1 > d_2$.

Now we suppose that $m_2 = 2$. Since $a_4 + a_{j_2} < 2a_2 + a_{j_2} = 2a_4$, the index j_2 cannot be 4. If $i_2 = 3$ or 4, then we can easily obtain $a_4 \leqslant 2a_1$, and hence $8a_1 > d_2$. Therefore, we consider only the cases $i_2 = 0$ and 1.

If $d_1 = a_2 + a_0$, then $a_2 = a_3$ since $d_1 \geqslant a_3 + a_0$. Then the well-formed condition contradicts the assumption that the septuple S satisfies neither $QS_1(2)$ nor $QS_2(2)$. Therefore, $d_1 = a_2 + a_1$. From this equation we see that $e = 0$ in (4.3), that is, $d_1 = a_3 + a_0$ or $d_1 = a_4 + a_0$. If $d_1 = a_3 + a_0$, then

$$S \in \{(a_0, a_1, a_2, a_2 + a_1 - a_0, a_2 + a_1 - 1, a_2 + a_1, 2a_2 + 2a_1 - 2) \in \mathbb{Z}^7 \mid 1 \leqslant a_0 \leqslant a_1 \leqslant a_2\}.$$

If $j_2 = 3$, then $a_2 = a_1 + a_0 - 2$. Thus, $2a_1 > a_2$, and hence $8a_1 > 4a_2 > d_2$. If $j_2 = 0$, then $a_0 = 2a_1 - 2$ which implies that $a_1 = a_0 = 2$ and

$$S \in \{(2, 2, a_2, a_2, a_2 + 1, a_2 + 2, 2a_2 + 2) \in \mathbb{Z}^7 \mid 2 \leqslant a_2\}.$$

The well-formed condition implies that a_2 divides either d_1 or d_2, which is a contradiction since S satisfies neither $QS_1(2)$ nor $QS_2(2)$.

By a similar method, we can derive a contradiction from the equality $d_1 = a_4 + a_0$.

Case 3 (the septuple S satisfies $QS_3(4)$ but neither $QS_1(4)$ nor $QS_2(4)$). The septuple S then satisfies two equations $d_1 = n_4 a_4 + a_{i_4}$ and $d_2 = m_4 a_4 + a_{j_4}$, where $i_4 \neq 4$, $j_4 \neq 4$, $i_4 \neq j_4$ and two integers, n_4 and m_4, are positive. By substituting these two equations into (4.1) we see that $n_4 = 1$, $m_4 = 1$ and $2a_3 > a_4$. For each pair (i_4, j_4), we can prove the inequality $8a_1 > d_2$ in essentially the same way except for the case in which $(i_4, j_4) = (0, 1)$. For this reason, we consider only the cases with $(i_4, j_4) = (0, 2)$ and $(0, 1)$.

We first suppose that the septuple S satisfies the two equations $d_1 = a_4 + a_0$ and $d_2 = a_4 + a_2$. The septuple S also satisfies one of $QS_{q_3}(3)$, $q_3 = 1, 2, 3$.

If the septuple S satisfies $QS_1(3)$, then $d_1 = 2a_3$. By substituting $d_2 = a_4 + a_2$ and $d_1 = 2a_3$ into (4.1), we show that $2a_1 > a_3$. Thus, $8a_1 > d_2$.

Next, if the septuple S satisfies $QS_2(3)$, then $d_2 = 2a_3$, which implies that

$$S \in \{(a_0, a_1, a_2, a_1 + a_2 - 1, 2a_1 + a_2 - 2, a_0 + 2a_1 + a_2 - 2, 2a_1 + 2a_2 - 2) \in \mathbb{Z}^7 \mid 1 \leqslant a_0 \leqslant a_1 \leqslant a_2\}.$$

Thus, $4a_2 > d_2$. In this case, we consider $QS_{q_2}(2)$, $q_2 = 1, 2, 3$.

$QS_1(2)$: suppose that the septuple S satisfies $QS_1(2)$. Since $4a_2 > d_2$, the septuple S satisfies either $d_1 = 2a_2$ or $d_1 = 3a_2$. If $d_1 = 2a_2$, then $d_2 = 6a_1 + 2a_0 - 6$. And if $d_1 = 3a_2$, then $d_2 = 4a_1 + a_0 - 4$. Thus, $8a_1 > d_2$.

$QS_2(2)$: suppose that the septuple S satisfies $QS_2(2)$ but not $QS_1(2)$. The septuple S then satisfies either $d_2 = 2a_2$ or $d_2 = 3a_2$. If $d_2 = 2a_2$, then $a_2 = a_3 = a_4$. By the well-formed condition, d_1 is divisible by a_2, which is a contradiction. If $d_2 = 3a_2$, then $a_2 = 2a_1 - 2$. Thus, $6a_1 > d_2$.

$QS_3(2)$: suppose that the septuple S satisfies $QS_3(2)$ but neither $QS_1(2)$ nor $QS_2(2)$. The septuple S then satisfies $d_1 = n_2a_2 + a_e$ for some $n_2 \in \{1, 2\}$ and some $e \in \{0, 1, 3, 4\}$. If $e = 4$, then $a_0 = a_1 = a_2$. The well-formed condition contradicts the assumption that the septuple S does not satisfy $QS_1(2)$. If $e = 3$, then $n_2 = 1$ and $a_2 = a_0 + a_1 - 1$. If $e = 1$, then $n_2 = 2$ and $a_2 = a_0 + a_1 - 2$. Finally, if $e = 0$, then $n_2 = 2$ and $a_2 = 2a_1 - 2$. These imply that $8a_1 > d_2$.

We now suppose that the septuple S satisfies $QS_3(3)$ but neither $QS_1(3)$ nor $QS_2(3)$. The septuple S then satisfies two equations $d_1 = a_3 + a_{i_3}$ and $d_2 = m_3a_3 + a_{j_3}$, where $i_3 \neq 3$, $j_3 \neq 3$, $i_3 \neq j_3$ and m_3 is a positive integer. From $d_2 = a_4 + a_2$ and (4.1), we obtain $d_2 = 2a_3 + a_0$. From $d_1 = a_4 + a_0$, we see that i_3 must be either 1 or 2. If $i_3 = 2$, then $d_1 = a_3 + a_2$. By substituting $d_1 = a_3 + a_2$ and $d_2 = a_4 + a_2$ into (4.1) we obtain $a_0 + a_1 - a_2 = 1$. Thus, $6a_1 > d_2$. If $i_3 = 1$, then we obtain

$$S \in \{(1, a_1, a_2, a_1 + a_2 - 2, 2a_1 + a_2 - 3, 2a_1 + a_2 - 2, 2a_1 + 2a_2 - 3) \in \mathbb{Z}^7 \mid 1 \leqslant a_1 \leqslant a_2\}.$$

The septuple S also satisfies one of $QS_{q_2}(2)$, $q_2 = 1, 2, 3$.

$QS_1(2)$: if the septuple S satisfies $QS_1(2)$, then $d_1 = 2a_2$ and $a_2 = 2a_1 - 2$, and hence $6a_1 > d_2$.

$QS_2(2)$: if the septuple S satisfies $QS_2(2)$, then $d_2 = 3a_2$ and $a_2 = 2a_1 - 3$, and hence $6a_1 > d_2$.

$QS_3(2)$: suppose that the septuple S satisfies $QS_3(2)$ but neither $QS_1(2)$ nor $QS_2(2)$. The septuple S then satisfies $d_1 = n_2a_2 + a_e$, where $e \neq 2$. If $e = 4$, then $a_0 = a_1 = a_2$ since $d_1 = a_4 + a_0$. If $e = 3$, then $a_2 = a_1$ since $d_1 = a_3 + a_1$. If $e = 1$, then a_3 is divisible by a_2 since $d_1 = a_3 + a_1$. And if $e = 0$, then a_4 is divisible by a_2 since $d_1 = a_4 + a_0$. These imply that the septuple S satisfies either $QS_1(2)$ or $QS_2(2)$, which is a contradiction.

Finally, we consider the case $(i_4, j_4) = (0, 1)$, that is, the septuple S satisfies two equations $d_1 = a_4 + a_0$ and $d_2 = a_4 + a_1$. Then we can show that the following are the only cases that may have $8a_1 \leqslant d_2$.

Subcase 1 (the septuple S satisfies $QS_1(3)$ and $QS_1(2)$). In this case, we have $d_1 = 2a_3$ and $d_1 = 2a_2$. The septuple S then belongs to the set

$$\{(1, a_1, a_2, a_2, 2a_2 - 1, 2a_2, 2a_2 + a_1 - 1) \in \mathbb{Z}^7 \mid 1 \leqslant a_1 \leqslant a_2\}.$$

From the condition for $I = \{2, 3\}$ in Theorem 4.2 and the equality $a_2 = a_3$, we see that either $d_2 = ma_2$ or $d_2 = ma_2 + a_i$ for some positive integer m and some $i \in \{0, 1, 4\}$. Since $d_2 = 2a_2 + a_1 - 1$, we obtain either $a_1 = 1$ or $a_1 = 2$. Therefore, the septuple S must be of the form

$$(1, 1, M, M, 2M - 1, 2M, 2M) \quad \text{or} \quad (1, 2, 2M + 1, 2M + 1, 4M + 1, 4M + 2, 4M + 3),$$

where M is a positive integer.

Subcase 2 (the septuple S satisfies $QS_1(3)$ and $QS_2(2)$). In this case, $d_1 = 2a_3$ and $d_2 = 2a_2$. We then obtain $a_2 = a_3$ and $a_0 = a_1 = 1$. Therefore, the septuple S is of the form

$$(1, 1, M, M, 2M - 1, 2M, 2M),$$

where M is a positive integer.

Subcase 3 (the septuple S satisfies $QS_2(3)$ and $QS_{i_2}(2)$, $i_2 = 1, 2$). Then $d_2 = 2a_3$ and $d_{i_2} = 2a_2$. From these, we obtain $a_2 = a_3$ and $a_0 = a_1 = 1$ as in the previous case. Therefore, the septuple S is of the form

$$(1, 1, M, M, 2M - 1, 2M, 2M),$$

where M is a positive integer.

Subcase 4 (the septuple S satisfies $QS_3(3)$ and $QS_1(2)$). The septuple S satisfies the equations $d_2 = 2a_3 + a_0$, $d_1 = a_3 + a_2$ for $QS_3(3)$ and $d_1 = 2a_2$ for $QS_1(2)$. Then it must be of the form

$$(1, 2, 2M + 1, 2M + 1, 4M + 1, 4M + 2, 4M + 3),$$

where M is a positive integer.

For the remaining cases, we can show that $8a_1 > d_2$. We omit the details. □

Now we explain how to obtain the complete list of the septuples $(a_0, a_1, a_2, a_3, a_4, d_1, d_2)$ that represent quasi-smooth well-formed complete intersection log del Pezzo surfaces defined by two quasi-homogeneous polynomials of degrees d_1 and d_2 in the weighted projective spaces $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ with amplitude $\alpha = 1$. We only consider the septuples with $8a_1 > d_2$ because the other cases are already described in Theorem 4.4.

Let L be a system consisting of the linear equation (4.1) and the linear equations of the form (4.2) derived from the condition in Theorem 4.2 for each non-empty subset I of $\{0, 1, 2, 3, 4\}$. We fix the coefficients m_{ij} of all the linear equations in the system L. We may then obtain the septuples $(a_0, a_1, a_2, a_3, a_4, d_1, d_2)$ that are positive integral solutions of the system L, if any. By Theorem 4.2, these represent quasi-smooth complete intersection log del Pezzo surfaces defined by two quasi-homogeneous polynomials in weighted projective spaces.

Therefore, if we find all such systems of linear equations, then we obtain the complete list of such septuples. However, we have infinitely many possibilities for these systems of linear equations because, for the linear equations of the system L derived from the condition in Theorem 4.2, for the subsets I containing 0 the coefficients m_{i0} of the unknown a_0 run through all positive integers. So it is hard to perform the procedure that fixes the coefficients m_{ij} and solves the system L using a computer program. For this reason, instead of L, we consider a reduced system of linear equations consisting of some linear equations corresponding to some of the index sets I with $|I| \leqslant 2$, $0 \notin I$, and plus the linear equation (4.1).

To obtain the reduced system, we first consider the reduced systems $\tilde{L}$ that consist of the linear equation (4.1) and one of the following:

- $d_1 = m_{1i}a_i$;
- $d_2 = m_{2i}a_i$;
- $d_1 = m_{1i}a_i + a_{e_1}$ and $d_2 = m_{2i}a_i + a_{e_2}$

for each $i \in \{2,3,4\}$, where m_{1i} and m_{2i} are positive integers and $e_1 \neq i$, $e_2 \neq i$, $e_1 \neq e_2$. If a reduced system $\tilde{L}$ has only four linear equations, then it must contain the linear equations $d_q = m_{qk_1}a_{k_1}$ and $d_q = m_{qk_2}a_{k_2}$ for some $q \in \{1,2\}$ and some distinct $k_1, k_2 \in \{2,3,4\}$. In such a case, we replace the reduced system $\tilde{L}$ of four linear equations with linear systems of five types that are obtained by, for each $l = 0,1,\ldots,4$, adding one linear equation

$$d_{\hat{q}} = n_{\hat{q}k_1}a_{k_1} + n_{\hat{q}k_2}a_{k_2} + a_l, \tag{4.4}$$

derived from the condition in Theorem 4.2 for the index set $I = \{k_1, k_2\}$, where $\hat{q} \in \{1,2\} \setminus \{q\}$, $n_{\hat{q}k_1}$ and $n_{\hat{q}k_2}$ are non-negative integers, to the original system $\tilde{L}$.

The number of the linear equations of each reduced system $\tilde{L}$ is at least five. Since we only consider the septuples S with $8a_1 > d_2$, we only have to consider the systems $\tilde{L}$ with $1 \leqslant m_{ji} < 8$ and $0 \leqslant n_{ji} < 8$. Then the number of the reduced systems in our consideration is finite. Furthermore, one can check that the ranks of the reduced systems $\tilde{L}$ are at least five.

Next, we consider the reduced systems $\hat{L}$ constructed by adding the linear equations derived from condition $QS_{i_1}(1)$, $i_1 \in \{1,2,3\}$, in Theorem 4.2 with coefficients $1 \leqslant m_{ji} < 8$ to the reduced systems $\tilde{L}$.

If a reduced system $\hat{L}$ is of rank seven, then we solve the system of linear equations. If it has a positive integral solution with $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ and $d_1 \leqslant d_2$, then we keep it.

If a reduced system $\hat{L}$ is of rank six, then the solutions are the septuples of the form

$$(p_0N + q_0, p_1N + q_1, p_2N + q_2, p_3N + q_3, p_4N + q_4, p_5N + q_5, p_6N + q_6, p_7N + q_7),$$

where p_i, q_j are fixed rational numbers. Among the one-dimensional solutions, we find positive integral solutions with $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ and $d_1 \leqslant d_2$, and keep them.

Finally, there are not so many reduced systems $\hat{L}$ of rank five among the systems in our consideration. In such cases, one can easily find positive integral solutions with $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ and $d_1 \leqslant d_2$. For example, there is a reduced system $\hat{L}$ consisting of the linear equations $d_2 = a_4 + a_1$, $d_1 = a_4 + a_0$, $d_2 = 2a_3$ and $d_2 = 2a_2$ after removing dependent linear equations. Then the solutions are

$$(1, 1, a_2, a_2, 2a_2 - 1, 2a_2, 2a_2).$$

Since we solve the reduced systems $\hat{L}$ instead of the systems L, their positive integral solutions with $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ and $d_1 \leqslant d_2$ are not necessarily septuples that represent quasi-smooth well-formed complete intersection surfaces. Therefore, in order to obtain the complete list of the septuples that represent quasi-smooth complete intersection log del Pezzo surfaces defined by two quasi-homogeneous polynomials in weighted projective spaces, we must check whether the obtained positive solutions satisfy both quasi-smoothness and well-formedness. This can easily be done with the aid of a computer.

In the way described so far, we can obtain the complete list of the septuples that represent quasi-smooth complete intersection log del Pezzo surfaces defined by two quasi-homogeneous polynomials in weighted projective spaces. Tables 1 and 2 in § 6 show the result.

5. The scheme of the proof

We have 41 families of quasi-smooth well-formed complete intersection log del Pezzo surfaces in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ in Tables 1 and 2. There are too many of these families for us to demonstrate all the computations for the global log canonical thresholds. Moreover, these computations are based on the same methods. In this section, we explain the steps to evaluate the global log canonical thresholds of these families.

Step 1. Using Lemma 3.1, we compute the log canonical thresholds $c(X, (1/a_0)C_x)$, $c(X, (1/a_1)C_y)$, $c(X, (1/a_2)C_z)$, $c(X, (1/a_3)C_t)$ and $c(X, (1/a_4)C_w)$. We then set

$$\mu = \min\left\{c\left(X, \frac{1}{a_0}C_x\right), c\left(X, \frac{1}{a_1}C_y\right), c\left(X, \frac{1}{a_2}C_z\right), c\left(X, \frac{1}{a_3}C_t\right), c\left(X, \frac{1}{a_4}C_w\right)\right\}.$$

It follows that the global log canonical threshold $\operatorname{lct}(X)$ is at most μ.

We claim that the global log canonical threshold $\operatorname{lct}(X)$ is bounded below by some number $\lambda \leqslant \mu$. In many cases, however, we will use μ for the lower bound λ, so that we could show $\operatorname{lct}(X) = \mu$.

First, we put $\lambda = \mu$. Next, we proceed with Steps 2 and 3. If we cannot see that λ is a lower bound for $\operatorname{lct}(X)$ using Steps 2 and 3, then we proceed with Steps 2 and 3 for unknown λ. Then, from Step 3 we obtain inequalities containing λ and can determine an optimal λ for which we can obtain a contradiction to the assumption of Step 2. This number λ will be our lower bound for $\operatorname{lct}(X)$.

Step 2. We assume that $\operatorname{lct}(X) < \lambda$. Then there exists an effective $\mathbb{Q}$-divisor

$$D \equiv_{\mathbb{Q}} -K_X$$

such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$. In particular, we obtain $\operatorname{mult}_P(D) > 1/\lambda$ from Lemma 3.3.

Step 3. We show that the point P cannot be any point of X, so that the assumption in Step 2 should not hold.

To do so, we first show that the point P cannot be a smooth point of X. For this purpose, apply Lemma 3.7. However, this method does not always work. If the method fails, then we try to find an appropriate linear system on X such that it has a member F passing through the point P and the log pair $(X, \lambda F)$ is log canonical at the point P.

Following Lemma 3.6, we then assume that the support of F is not contained in the support of D. Using Methods 5.1 and 5.2 below, we exclude smooth points.

We then show that the point P cannot be a singular point of X, using Methods 5.1–5.3. For Methods 5.1 and 5.2, we consider a suitable irreducible curve C. Usually, it will be taken from the irreducible components of C_x, C_y, C_z, C_t or C_w. In some cases, it is taken from the irreducible components of a member of an appropriate linear system on X.

Method 5.1. We consider an appropriate irreducible curve C passing through the point P. We then obtain the inequality

$$C \cdot D \geqslant \frac{\mathrm{mult}_P(C)\,\mathrm{mult}_P(D)}{r} > \frac{\mathrm{mult}_P(C)}{\lambda r}$$

from Lemma 3.2, where r is the index of the quotient singular point P. If this inequality derives a contradiction, then the point P is excluded. This method can be applied to exclude a smooth point.

Method 5.2. We consider a suitable irreducible curve C smooth at P. We write $D = mC + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve C. We check that $\lambda m \leqslant 1$. We then obtain the inequality

$$(D - mC) \cdot C = \Omega \cdot C > \frac{1}{\lambda r}$$

from Lemma 3.5. If we can derive a contradiction from the inequality then the point P is excluded. This method can also be applied to exclude a smooth point.

Method 5.3. Sometimes we cannot obtain a contradiction solely by using Methods 5.1 and 5.2. In such a case, we take a suitable weighted blow-up $\pi\colon Y \to X$ at the point P. We can write

$$K_Y + D^Y \equiv_{\mathbb{Q}} \pi^*(K_X + \lambda D),$$

where D^Y is the log pull-back of λD by π. We then apply Methods 5.1 and 5.2 to the log pair (Y, D^Y), or repeat this method until we get a contradictory inequality.

6. Demonstrations of the methods

In the previous section we explained the methods used to compute the global log canonical thresholds of the 41 families of quasi-smooth well-formed complete intersection log del Pezzo surfaces in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ in Tables 1 and 2. In the present section we provide details of these computations for samples of infinite series of such surfaces and samples of sporadic cases.

6.1. Infinite series

We have three infinite series of families of quasi-smooth well-formed complete intersection log del Pezzo surfaces in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$, which are listed in Table 1. Here, we fully describe the computations for the global log canonical thresholds of two infinite series. For the remaining infinite series we can compute the global log canonical threshold in a similar way.

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Table 1. *Infinite series.*

Weight	Multidegree	lct(X)	Singular points
$(1,1,n+1,n+1,2n+1)$	$\{2n+2,2n+2\}$	$\dfrac{2n+5}{4n+6}$	P_w
$(1,2,2n+1,2n+1,4n+1)$	$\{4n+2,4n+3\}$	$\geqslant \dfrac{20n+4}{24n+5}$	$P_z,\ P_t,\ P_w$
$(2,2n+1,2n+1,4n+1,6n+1)$	$\{6n+3,8n+2\}$	1	$P_w,\ P_{yz}=3\times\dfrac{1}{2n+1}(2,-1)$

Here, n is a positive integer.

Lemma 6.1. *Let X be a quasi-smooth complete intersection log del Pezzo surface defined by two quasi-homogeneous polynomials of degree $2n$ in the weighted projective space $\mathbb{P}(1,1,n,n,2n-1)$, where n is a positive integer bigger than 1. Then $\mathrm{lct}(X)=(2n+3)/(4n+2)$.*

Proof. The surface X can be assumed to be defined by the quasi-homogeneous equations

$$\begin{aligned} wx+z(a_1z+b_1t)+f_n(x,y)z+\hat{f}_n(x,y)t+f_{2n}(x,y)=0,\\ wy+t(a_2z+b_2t)+g_n(x,y)z+\hat{g}_n(x,y)t+g_{2n}(x,y)=0,\end{aligned}$$

where f_d, $\hat{f}_d$, g_d and $\hat{g}_d$ are homogeneous polynomials of degree d. For the surface X to be quasi-smooth, the polynomials a_1z+b_1t and a_2z+b_2t must not be proportional and $a_1\neq 0$, $b_2\neq 0$. The surface X is singular at the point P_w.

Let $\mathcal{L}$ be the linear system on the surface X cut by the equation $\lambda x+\mu y=0$, where $[\lambda:\mu]\in\mathbb{P}^1$. Let C_μ be the member of the pencil $\mathcal{L}$ cut by the equation $x-\mu y=0$ on the surface X. Then C_μ can be defined by the quasi-homogeneous equations

$$\begin{aligned} \mu wy+z(a_1z+b_1t)+f_n(\mu,1)y^nz+\hat{f}_n(\mu,1)y^nt+f_{2n}(\mu,1)y^{2n}=0,\\ wy+t(a_2z+b_2t)+g_n(\mu,1)y^nz+\hat{g}_n(\mu,1)y^nt+g_{2n}(\mu,1)y^{2n}=0\end{aligned}$$

in $\mathrm{Proj}(\mathbb{C}[y,z,t,w])$. Consider the affine piece of the curve C_μ defined by $y\neq 0$. It is the affine curve defined by the equations

$$\begin{aligned} \mu w+z(a_1z+b_1t)+f_n(\mu,1)z+\hat{f}_n(\mu,1)t+f_{2n}(\mu,1)=0,\\ w+t(a_2z+b_2t)+g_n(\mu,1)z+\hat{g}_n(\mu,1)t+g_{2n}(\mu,1)=0\end{aligned}$$

in $\mathrm{Spec}(\mathbb{C}[z,t,w])$. Furthermore, it is isomorphic to the affine curve defined by the equation

$$\begin{aligned} z(a_1z+b_1t)-\mu t(a_2z+b_2t)+(f_n(\mu,1)-\mu g_n(\mu,1))z\\ +(\hat{f}_n(\mu,1)-\mu\hat{g}_n(\mu,1))t+(f_{2n}(\mu,1)-\mu g_{2n}(\mu,1))=0\end{aligned}$$

in $\mathrm{Spec}(\mathbb{C}[z,t])$. There is a constant μ_1 such that the affine curve $C_{\mu_1}\backslash C_y$ is defined by

$$(\alpha_1 z+\beta_1 t+\gamma_1)(\alpha_2 z+\beta_2 t+\gamma_2)=0,$$

where α_i, β_i and γ_i are constants. Note that the polynomials $(\alpha_1 z+\beta_1 t+\gamma_1)$ and $(\alpha_2 z+\beta_2 t+\gamma_2)$ must not be proportional; otherwise X would not be quasi-smooth. Therefore, C_{μ_1} can be defined by the quasi-homogeneous equations

$$\begin{aligned}(\alpha_1 z+\beta_1 t+\gamma_1 y^n)(\alpha_2 z+\beta_2 t+\gamma_2 y^n)&=0,\\ wy+t(a_2z+b_2t)+g_n(\mu_1,1)y^nz+\hat{g}_n(\mu_1,1)y^nt+g_{2n}(\mu_1,1)y^{2n}&=0.\end{aligned}$$

Then the curves C_{μ_1} consists of two irreducible and reduced curves L_1 and L_2, where L_i is defined by the equations

$$\begin{aligned}x-\mu_1 y&=0,\\ \alpha_i z+\beta_i t+\gamma_i y^n&=0,\\ wy+t(a_2z+b_2t)+g_n(\mu_1,1)y^nz+\hat{g}_n(\mu_1,1)y^nt+g_{2n}(\mu_1,1)y^{2n}&=0.\end{aligned}$$

A member of the pencil $\mathcal{L}$ is always one of the following:

- an irreducible and reduced quasi-smooth curve;
- the sum of two quasi-smooth curves.

On the other hand, we consider the open subset $C_\mu\backslash C_w$ of the curve C_μ, which is the affine curve defined by the equations

$$\begin{aligned}\mu y+z(a_1z+b_1t)+f_n(\mu,1)y^nz+\hat{f}_n(\mu,1)y^nt+f_{2n}(\mu,1)y^{2n}&=0,\\ y+t(a_2z+b_2t)+g_n(\mu,1)y^nz+\hat{g}_n(\mu,1)y^nt+g_{2n}(\mu,1)y^{2n}&=0\end{aligned}$$

in $\mathrm{Spec}(\mathbb{C}[y,z,t])$. We can see that there exists μ_2 such that

$$z(a_1z+b_1t)-\mu_2 t(a_2z+b_2t)=(\alpha_3 z+\beta_3 t)^2,$$

where α_3 and β_3 are constants. The affine curve $C_{\mu_2}\setminus C_w$ is then analytically isomorphic to the equation

$$z^2+\psi_{\geqslant 2n+1}(z,t)=0,$$

where $t^{2n+1}\in\psi_{2n+1}$. We then obtain $c(X,C_{\mu_2})=(2n+3)/(4n+2)$ from [**14**, Proposition 8.14]. Furthermore, we can see that $c(X,C)\geqslant(2n+3)/(4n+2)$ for every member C of the linear system $\mathcal{L}$.

Suppose that $\mathrm{lct}(X)<(2n+3)/(4n+2)$. Then there is an effective $\mathbb{Q}$-divisor

$$D\equiv -K_X$$

such that the log pair $(X,((2n+3)/(4n+2))D)$ is not log canonical at some point $P\in X$.

We have the following intersection numbers:

$$-K_X \cdot C_\mu = \frac{4}{2n-1}, \qquad -K_X \cdot L_1 = -K_X \cdot L_2 = \frac{2}{2n-1},$$
$$L_1 \cdot L_2 = \frac{2n}{2n-1}, \qquad L_1^2 = L_2^2 = \frac{2-2n}{2n-1}.$$

Suppose that P is a smooth point in the surface X. Let C be the unique member in $\mathcal{L}$ passing through the point P. Suppose that the curve C is irreducible. By Lemma 3.6, we may assume that C is not contained in the support of D. Then the inequality

$$\frac{4n+2}{2n+3} < \mathrm{mult}_P(D) \leqslant D \cdot C = \frac{4}{2n-1}$$

implies a contradiction. Thus, the curve C must be reducible. Put $C = L_1 + L_2$, where L_1 and L_2 are irreducible and reduced curves on the surface X. By Lemma 3.6, we may assume that the support of C is not contained in the support of D. Without loss of generality we may assume that L_2 is not contained in the support of D. Suppose that the point P lies on the curve L_2. Then the inequality

$$\frac{4n+2}{2n+3} \leqslant \mathrm{mult}_P(D) \leqslant D \cdot L_2 = \frac{2}{2n-1}$$

implies a contradiction. Thus, the point P must belong to $L_1 \setminus L_2$. Put $D = \lambda L_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on the surface X with $L_1 \not\subset \mathrm{Supp}(\Omega)$. The inequality

$$\lambda \frac{2n}{2n-1} = \lambda L_1 \cdot L_2 \leqslant D \cdot L_2 = \frac{2}{2n-1}$$

implies that $0 \leqslant \lambda \leqslant 1/n$. Since $\lambda(2n+3)/(4n+2) \leqslant 1$, the log pair $(X, L_1 + (2n+3)/(4n+2)\Omega)$ is not log canonical at the point P. Then the inequality

$$\frac{4n+2}{2n+3} < (D - \lambda L_1) \cdot L_1 = \frac{2}{2n-1} + \lambda \cdot \frac{2n-2}{2n-1} \leqslant \frac{2}{n}$$

implies a contradiction. Thus, the point P must be the point P_w.

Let $\pi\colon \bar{X} \to X$ be the weighted blow-up at the point P_w with weights $(1,1)$. Let E be the exceptional divisor of π. Then we have

$$K_{\bar{X}} \equiv_{\mathbb{Q}} \pi^*(K_X) - \frac{2n-3}{2n-1}E, \qquad \pi^* D \equiv_{\mathbb{Q}} \bar{D} + \frac{\alpha}{2n-1}E,$$

where $\bar{D}$ is the proper transform of D on $\bar{X}$ and α is a positive rational number.

For simplicity, put $\phi = (2n+3)/(4n+2)$. The log pull-back of the log pair $(X, \phi D)$ is the log pair

$$(\bar{X}, \phi\bar{D} + \theta_\alpha E),$$

where

$$\theta_\alpha = \frac{2n-3}{2n-1} + \frac{2n+3}{4n+2}\frac{\alpha}{2n-1}.$$

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Let C be an irreducible member in the linear system $\mathcal{L}$. We then have

$$\pi^*C \equiv_{\mathbb{Q}} \bar{C} + \frac{2}{2n-1}E,$$

where $\bar{C}$ is the proper transform of C. By Lemma 3.6, we may assume that C is not contained in the support of D. Then we have

$$D \cdot C = \bar{D} \cdot \bar{C} + \frac{2\alpha}{2n-1}.$$

It follows that

$$0 \leqslant \bar{D} \cdot \bar{C} = \frac{4-2\alpha}{2n-1}.$$

Thus, we have $\theta_\alpha \leqslant 1$.

The log pair $(\bar{X}, \phi\bar{D} + \theta_\alpha E)$ must not be log canonical at some point $\bar{p} \in E$. For the point $\bar{P} \in E$, there is a curve $L \in \mathcal{L}$ whose proper transform $\bar{L}$ passes through the point $\bar{P}$. Suppose that L is irreducible. Then,

$$\pi^*L \equiv_{\mathbb{Q}} \bar{L} + \frac{2}{2n-1}E.$$

We have

$$D \cdot L = \bar{D} \cdot \bar{L} + \frac{2\alpha}{2n-1}.$$

Thus,

$$\bar{D} \cdot \bar{L} = \frac{4-2\alpha}{2n-1}.$$

Since the log pair $(\bar{X}, \phi\bar{D} + \theta_\alpha E)$ is not log canonical at the point $\bar{P}$, we have

$$\mathrm{mult}_{\bar{P}}(\phi\bar{D} + \theta_\alpha E) > 1.$$

Thus,

$$\mathrm{mult}_{\bar{P}}\, \bar{D} > \frac{2}{\phi(2n-1)} - \frac{\alpha}{2n-1}.$$

The inequality

$$\bar{D} \cdot \bar{L} \geqslant \mathrm{mult}_{\bar{P}}\, \bar{D}\, \mathrm{mult}_{\bar{P}}\, \bar{L} > \frac{2}{\phi(2n-1)} - \frac{\alpha}{2n-1}$$

implies that

$$\frac{4-2\alpha}{2n-1} > \frac{4n+2}{2n+3}\frac{2}{2n-1} - \frac{\alpha}{2n-1}.$$

Thus, $\alpha < 8/(2n+3)$. By Lemma 3.5,

$$1 < \phi\bar{D} \cdot E = \phi\alpha < \frac{2n+3}{4n+2}\frac{8}{2n+3} = \frac{4}{2n+1} < 1$$

for all $n \geqslant 2$. This is a contradiction. Therefore, the curve L must be reducible. Write $L = L_1 + L_2$ and assume that $\bar{P}$ lies on the proper transform $\bar{L}_1$ of the curve L_1. Also, we may assume that the support of D does not contain one of the curves L_1 and L_2. Put $D = \mu L_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on the surface X whose support does not contain the curve L_1. Then,

$$\pi^*\Omega \equiv_{\mathbb{Q}} \bar{\Omega} + \frac{\beta}{2n-1}E, \qquad \pi^*L_1 \equiv_{\mathbb{Q}} \bar{L}_1 + \frac{1}{2n-1}E, \qquad \pi^*L_2 \equiv_{\mathbb{Q}} \bar{L}_2 + \frac{1}{2n-1}E,$$

where $\bar{\Omega}$ and $\bar{L}_2$ are the proper transforms of Ω and L_2, respectively.

The log pull-back of the log pair $(X, \phi D)$ is the log pair

$$(\bar{X}, \phi\mu\bar{L}_1 + \phi\bar{\Omega} + \theta_\beta E),$$

where

$$\theta_\beta = \frac{2n-3}{2n-1} + \frac{\mu\phi}{2n-1} + \frac{\phi\beta}{2n-1}.$$

If $\mu = 0$, we obtain an absurd inequality

$$\frac{2-\beta}{2n-1} > \frac{4n+2}{2n+3}\frac{2}{2n-1} - \frac{\beta}{2n-1}$$

since

$$\bar{D} \cdot \bar{L}_1 = \frac{2-\beta}{2n-1}$$

and

$$\bar{D} \cdot \bar{L}_1 \geqslant \operatorname{mult}_{\bar{P}} \bar{D} \operatorname{mult}_{\bar{P}} \bar{L}_1 > \frac{2}{\phi(2n-1)} - \frac{\beta}{2n-1}.$$

Thus, $\mu > 0$.

Since

$$\Omega \cdot L_2 = (D - \mu L_1) \cdot L_2 = \frac{2}{2n-1} - \mu\frac{2n}{2n-1},$$

we have

$$\frac{2}{2n-1} - \mu\frac{2n}{2n-1} = \Omega \cdot L_2 = \bar{\Omega} \cdot \bar{L}_2 + \frac{\beta}{2n-1} \geqslant \frac{\beta}{2n-1}.$$

Thus, $2 \geqslant 2n\mu + \beta$. Furthermore, we have $\theta_\beta \leqslant 1$ and $\phi\mu \leqslant 1$.

By Lemma 3.5, the log pair

$$(E, (\phi\mu\bar{L}_1 + \phi\bar{\Omega})|_E)$$

is not log canonical at the point $\bar{P}$. We then have

$$1 < (\phi\mu\bar{L}_1 + \phi\bar{\Omega}) \cdot E = \phi(\mu + \beta).$$

Thus, $\mu + \beta > 1/\phi$. The log pair

$$(\bar{L}_1, (\phi\bar{\Omega} + \theta E)|_{\bar{L}_1})$$

is not log canonical at the point $\bar{P}$ either. We then have

$$1 < \phi\bar{\Omega} \cdot \bar{L}_1 + \frac{2n-3}{2n-1} + \frac{\mu\phi}{2n-1} + \frac{\phi\beta}{2n-1}.$$

Since

$$\bar{\Omega} \cdot \bar{L}_1 = \frac{2}{2n-1} + \mu \frac{2n-2}{2n-1} - \frac{\beta}{2n-1},$$

we obtain $\mu > 2/(2n+3)$. However, there are no μ and β satisfying the following inequalities

$$2 \geqslant 2n\mu + \beta, \qquad \mu + \beta > \frac{4n+2}{2n+3}, \qquad \mu > \frac{2}{2n+3}.$$

The obtained contradiction completes the proof. □

In the following case, the global log canonical threshold is bigger than $\frac{2}{3}$. Unfortunately, however, we are not able to determine whether it is strictly less than 1 or not.

Lemma 6.2. *Let X be a quasi-smooth complete intersection log del Pezzo surface defined by two quasi-homogeneous polynomials of degrees $4n+2$ and $4n+3$ in the weighted projective space $\mathbb{P} = \mathbb{P}(1, 2, 2n+1, 2n+1, 4n+1)$, where n is a positive integer. Then $\operatorname{lct}(X) \geqslant (20n+4)/(24n+5)$.*

Proof. The surface X can be assumed to be defined by the quasi-homogeneous equations

$$x^2 f_{4n}(x,y) + x f_{2n}(x,y)z + x\hat{f}_{2n}(x,y)t + a_1 y^{2n+1} + zt + xw = 0,$$
$$x g_{4n+2}(x,y) + g_{2n+2}(x,y)z + \hat{g}_{2n+2}(x,y)t + x(a_2 z^2 + a_3 t^2) + cxzt + yw = 0,$$

where the a_i are non-zero constants, c is a constant and f_j, $\hat{f}_j$, g_j and $\hat{g}_j$ are quasi-homogeneous polynomials of degree j. The surface X is singular at the points P_z, P_t and P_w.

The curve C_x is defined by the quasi-homogeneous equations

$$a_1 y^{2n+1} + zt = 0,$$
$$b_1 y^{n+1} z + b_2 y^{n+1} t + yw = 0$$

in $\operatorname{Proj}(\mathbb{C}[y,z,t,w])$, where $b_1 = g_{2n+2}(0,1)$ and $b_2 = \hat{g}_{2n+2}(0,1)$. Then $C_x = L_1 + L_2 + R_x$, where L_1, L_2 and R_x are irreducible and reduced quasi-smooth curves defined by

$$L_1 = \{x = y = z = 0\},$$
$$L_2 = \{x = y = t = 0\},$$
$$R_x = \{x = a_1 y^{2n+1} + zt = b_1 y^n z + b_2 y^n t + w = 0\}.$$

Then we have $L_1 \cap L_2 = \{P_w\}$, $L_1 \cap R_x = \{P_t\}$, $L_2 \cap R_x = \{P_z\}$ and $L_1 \cap L_2 \cap R_x = \varnothing$. We also have $c(X, C_x) = 1$.

Suppose that $\operatorname{lct}(X) < (20n+4)/(24n+5)$. There is then an effective $\mathbb{Q}$-divisor

$$D \equiv_{\mathbb{Q}} -K_X$$

such that the log pair $(X, ((20n+4)/(24n+5))D)$ is not log canonical at some point $P \in X$.

Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(2n+1))$ contains the monomials x^{2n+1}, z and t, from Lemma 3.7, we obtain the inequality

$$\mathrm{mult}_Q(D)\leqslant\frac{(2n+1)(4n+2)(4n+3)}{1\times 2(2n+1)(2n+1)(4n+1)}=\frac{4n+3}{4n+1}<\frac{24n+5}{20n+4}$$

for the point $Q\in X\setminus C_x$. It follows that the point P belongs to C_x.

We have the following intersection numbers:

$$-K_X\cdot L_1=-K_X\cdot L_2=\frac{1}{(2n+1)(4n+1)},\quad -K_X\cdot R_x=\frac{1}{2n+1},\quad L_1\cdot L_2=\frac{1}{4n+1},$$
$$L_1\cdot R_x=L_2\cdot R_x=\frac{1}{2n+1},\quad L_1^2=L_2^2=-\frac{6n+1}{(2n+1)(4n+1)},\quad R_x^2=-\frac{1}{2n+1}.$$

We may assume that the support of the curve C_x is not contained in the support of D. Suppose that the curve L_i is not contained in the support of D. Put $D=\beta_jL_j+\beta R_x+\Omega$, where $i\neq j$ and Ω is an effective $\mathbb{Q}$-divisor such that L_j and R_x are not contained in the support of Ω. We have

$$\frac{1}{(2n+1)(4n+1)}=D\cdot L_i\geqslant\beta_jL_j\cdot L_i+\beta R_x\cdot L_i=\frac{\beta_j}{4n+1}+\frac{\beta}{2n+1}.$$

The inequalities

$$(4n+1)(D\cdot L_i)=\frac{1}{2n+1}<\frac{24n+5}{20n+4},$$
$$(2n+1)(D-\beta_jL_j)\cdot L_j\leqslant(2n+1)\left(\frac{1}{(2n+1)(4n+1)}+\frac{1}{2n+1}\frac{6n+1}{(2n+1)(4n+1)}\right)$$
$$=\frac{8n+2}{(2n+1)(4n+1)}<\frac{24n+5}{20n+4},$$
$$(D-\beta R_x)\cdot R_x\leqslant\frac{1}{2n+1}+\frac{1}{4n+1}\frac{1}{2n+1}<\frac{24n+5}{20n+4}$$

imply the contradiction that the point P cannot lie on C_x. Therefore, the support of D must contain the curves L_1 and L_2. Put $D=\alpha_1L_1+\alpha_2L_2+\Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L_1 nor L_2. The inequality

$$\frac{1}{2n+1}=D\cdot R_x\geqslant\alpha_1L_1\cdot R_x+\alpha_2L_2\cdot R_x=\alpha_1\frac{1}{2n+1}+\alpha_2\frac{1}{2n+1}$$

implies that $\alpha_1+\alpha_2\leqslant 1$. We have

$$(2n+1)(D\cdot R_x)=1<\frac{24n+5}{20n+4}.$$

Thus, $P\notin R_x$. Moreover, we have

$$(D-\alpha_1L_k)\cdot L_k\leqslant\frac{1}{(2n+1)(4n+1)}+\frac{6n+1}{(2n+1)(4n+1)}=\frac{6n+2}{(2n+1)(4n+1)}<\frac{24n+5}{20n+4}$$

for $k=1,2$. Thus, the point P must be the point P_w.

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Without loss of generality, we assume that $\alpha_1 \leqslant \alpha_2$. From

$$\frac{24n+5}{20n+4} < (4n+1)(D - \alpha_1 L_1) \cdot L_1 = \frac{1}{2n+1} + \alpha_1 \frac{6n+1}{2n+1}$$

we obtain $(8n+1)/(20n+4) < \alpha_1 \leqslant 1/2$.

Let $\pi \colon \bar{X} \to X$ be the weighted blow-up at the point P_w with weights $(1,1)$. Let E be the exceptional divisor of π. Then,

$$K_{\bar{X}} \equiv \pi^*(K_X) + \frac{1-4n}{4n+1}E, \qquad \pi^* L_1 \equiv \bar{L}_1 + \frac{1}{4n+1}E,$$
$$\pi^* L_2 \equiv \bar{L}_2 + \frac{1}{4n+1}E, \qquad \pi^* \Delta \equiv \bar{\Delta} + \frac{\alpha}{4n+1}E,$$

where α is a positive rational number and $\bar{L}_1$, $\bar{L}_2$ and $\bar{\Delta}$ are the proper transforms of L_1, L_2 and Δ, respectively. Let O_i be the intersection point of E and L_i for $i = 1, 2$. Put $\varepsilon = (20n+4)/(24n+5)$. The log pull-back of the log pair $(X, \varepsilon D)$ is the log pair

$$(\bar{X}, \varepsilon(\alpha_1 \bar{L}_1 + \alpha_2 \bar{L}_2 + \bar{\Delta}) + \theta E),$$

where

$$\theta = \frac{4n-1}{4n+1} + \varepsilon \frac{\alpha_1}{4n+1} + \varepsilon \frac{\alpha_2}{4n+1} + \varepsilon \frac{\alpha}{4n+1}.$$

This is not log canonical at some point $O \in E$. The inequalities

$$\begin{aligned}
0 \leqslant \bar{\Delta} \cdot \bar{L}_1 &= \Delta \cdot L_1 + \frac{\alpha}{(4n+1)^2} E^2 \\
&= \frac{1}{(2n+1)(4n+1)} + \alpha_1 \frac{6n+1}{(2n+1)(4n+1)} - \alpha_2 \frac{1}{4n+1} - \alpha \frac{1}{4n+1}, \\
0 \leqslant \bar{\Delta} \cdot \bar{L}_2 &= \Delta \cdot L_2 + \frac{\alpha}{(4n+1)^2} E^2 \\
&= \frac{1}{(2n+1)(4n+1)} + \alpha_2 \frac{6n+1}{(2n+1)(4n+1)} - \alpha_1 \frac{1}{4n+1} - \alpha \frac{1}{4n+1}
\end{aligned}$$

imply that $\alpha \leqslant 1$. Then, $\theta < 1$. Suppose that the point O is contained in the set $E \setminus (\bar{L}_1 \cup \bar{L}_2)$. We have

$$\bar{\Delta} \cdot E = \alpha < 1 < \frac{1}{\varepsilon},$$

which is a contradiction. Thus, the point O is either O_1 or O_2.

Suppose that the point O is the point O_1. Then the log pair $(\bar{X}, \varepsilon\alpha_1 \bar{L}_1 + \varepsilon\bar{\Delta} + \theta E)$ is not log canonical at the point O_1. Since $\varepsilon\alpha_1 \leqslant 1$, we have

$$\begin{aligned}
\varepsilon \bar{\Delta} \cdot \bar{L}_1 + \theta E \cdot \bar{L}_1 = \varepsilon \bigg(&\frac{1}{(2n+1)(4n+1)} + \alpha_1 \frac{6n+1}{(2n+1)(4n+1)} - \alpha_2 \frac{1}{4n+1} - \alpha \frac{1}{4n+1} \bigg) \\
&+ \left(\frac{4n-1}{4n+1} + \varepsilon\alpha_1 \frac{1}{4n+1} + \varepsilon\alpha_2 \frac{1}{4n+1} + \varepsilon\alpha \frac{1}{4n+1} \right)
\end{aligned}$$

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$$
\begin{aligned}
&= \varepsilon\frac{1}{(2n+1)(4n+1)} + \frac{4n-1}{4n+1} + \varepsilon\alpha_1\frac{6n+1}{(2n+1)(4n+1)} + \varepsilon\alpha_1\frac{1}{4n+1}\\
&\leqslant \frac{24n+3}{24n+5} < 1,
\end{aligned}
$$

which is a contradiction. Thus, the point O must be the point O_2. However, the same argument shows that this is not possible either. This completes the proof. □

6.2. Sporadic cases

There are 38 sporadic families of quasi-smooth well-formed complete intersection log del Pezzo surfaces in weighted projective spaces and these are listed in Table 2. Here we describe the computations for the global log canonical thresholds of five sporadic cases.

Lemma 6.3. *Let X be a quasi-smooth complete intersection log del Pezzo surface defined by two quasi-homogeneous polynomials of degree 6 and 8 in the weighted projective space $\mathbb{P}(1,2,3,4,5)$. If the defining equation of degree 6 contains the monomial yt, then $\operatorname{lct}(X)=1$. If not, then $\operatorname{lct}(X)=\frac{7}{12}$.*

Proof. The surface X can be assumed to be defined by the quasi-homogeneous equations

$$
\begin{aligned}
xf(x,y,z,t) + y(y^2+at) + z^2 + xw &= 0,\\
xg(x,y,z,t,w) + t(by^2+t) + zw &= 0.
\end{aligned}
$$

Here, f and g are quasi-homogeneous equations of degrees 5 and 7, respectively; a and b are constants with $ab \neq 1$. The surface X is singular at the point P_w.

The curve C_x is defined by the quasi-homogeneous equations

$$
\begin{aligned}
y(y^2+at) + z^2 &= 0,\\
t(by^2+t) + zw &= 0
\end{aligned}
$$

in $\operatorname{Proj}(\mathbb{C}[y,z,t,w])$. The curve C_x is an irreducible and reduced curve on the surface X. It is smooth at $C_x \cap C_w$. Consider the open set $C_x \setminus C_w$ of the curve C_x that is a $\mathbb{Z}_5$-quotient of the affine curve

$$
\begin{aligned}
y(y^2+at) + z^2 &= 0,\\
t(by^2+t) + z &= 0
\end{aligned}
$$

in $\operatorname{Spec}(\mathbb{C}[y,z,t])$. This affine curve is isomorphic to the curve defined by the equation

$$
y(y^2+at) + t^2(by^2+t)^2 = 0
$$

in $\operatorname{Spec}(\mathbb{C}[y,t])$. From this equation, we can see that $c(X,C_x)=1$ if $a\neq 0$; $c(X,C_x)=\frac{7}{12}$ if $a=0$.

Put $c(X,C_x)=\lambda$ and suppose that $\operatorname{lct}(X)<\lambda$. There is then an effective $\mathbb{Q}$-divisor

$$
D \equiv_{\mathbb{Q}} -K_X
$$

such that the log pair $(X,\lambda D)$ is not log canonical at some point $P\in X$.

Table 2. *Sporadic cases.*

Weight	Multidegree	lct(X)	Singular points
$(1,1,1,1,1)$	$\{2,2\}$	$2/3$	
$(1,2,2,3,3)$	$\{4,6\}$	1	$P_t,\ P_w$
$(1,2,3,4,5)$	$\{6^*,\ 8\}$	1	P_w
$(1,2,3,4,5)$	$\{6^{**},\ 8\}$	$7/12$	P_w
$(1,3,3,5,5)$	$\{6,10\}$	1	$P_y,\ P_z,\ P_t,\ P_w$
$(1,4,5,7,11)$	$\{12,15\}$	1	P_t, R_w
$(1,4,7,10,13)$	$\{14,20\}$	1	P_w
$(1,5,8,12,19)$	$\{20,24\}$	1	P_w
$(1,5,9,13,17)$	$\{18,26\}$	1	$P_y,\ P_w$
$(1,7,11,17,27)$	$\{28,34\}$	1	$P_z,\ P_w$
$(1,7,12,17,23)$	$\{24,35\}$	1	$P_t,\ P_w$
$(1,8,13,19,31)$	$\{32,39\}$	1	$P_t,\ P_w$
$(1,9,15,23,23)$	$\{24,46\}$	1	$P_y,\ P_z,\ P_t,\ P_w$
$(2,2,3,3,3)$	$\{6,6\}$	$\geqslant 6/5$	$P_{xy} = 4 \times \frac{1}{3}(1,1)$
$(2,3,4,5,5)$	$\{8,10\}$	$\geqslant 9/8$	$P_y,\ P_t,\ P_w$
$(2,3,5,6,7)$	$\{10,12\}$	$\geqslant 3/2$	$P_w,\ P_{yt} = 2 \times \frac{1}{3}(1,1)$
$(3,3,5,5,7)$	$\{10,12\}$	$\geqslant 7/4$	$P_z,\ P_t,\ P_w,\ P_{xy} = 4 \times \frac{1}{3}(1,1)$
$(3,5,6,8,13)$	$\{16,18\}$	$\geqslant 5/3$	$P_y,\ P_w,\ P_{xz} = 3 \times \frac{1}{3}(1,1)$
$(3,5,7,9,11)$	$\{16,18\}$	$\geqslant 14/11$	$P_y,\ P_z,\ P_w,\ P_{xz} = 2 \times \frac{1}{3}(1,2)$
$(4,5,7,10,13)$	$\{18,20\}$	$\geqslant 2$	$P_z,\ P_w,\ P_{yt} = 2 \times \frac{1}{5}(3,4)$
$(5,7,10,14,23)$	$\{28,30\}$	$35/12$	$P_w,\ P_{xz} = 3 \times \frac{1}{5}(2,4),\ P_{yt} = 2 \times \frac{1}{7}(5,3)$
$(5,9,12,20,31)$	$\{36,40\}$	$55/24$	$P_w,\ P_{xy} = 2 \times \frac{1}{5}(3,4),\ P_{yz} = \frac{1}{3}(1,1)$
$(5,14,17,21,37)$	$\{42,51\}$	$10/3$	$P_x,\ P_w,\ P_{yt} = \frac{1}{7}(5,3)$
$(6,7,9,11,14)$	$\{18,28\}$	$\geqslant 7/2$	$P_t,\ P_{yw} = 2 \times \frac{1}{7}(5,4),\ P_{xz} = \frac{1}{3}(1,1)$
$(6,8,9,11,13)$	$\{22,24\}$	$\geqslant 3$	$P_z,\ P_w,\ P_{xz} = \frac{1}{3}(1,1)$
$(9,15,23,23,31)$	$\{46,54\}$	$\geqslant 23/6$	$P_y,\ P_w,\ P_{zt} = 2 \times \frac{1}{23}(18,7),\ P_{xy} = \frac{1}{3}(1,1)$
$(9,15,23,23,37)$	$\{46,60\}$	$45/8$	$P_x,\ P_z,\ P_t,\ P_w,\ P_{xy} = \frac{1}{3}(1,1)$
$(9,23,30,38,67)$	$\{76,90\}$	$81/14$	$P_y,\ P_w,\ P_{xz} = \frac{1}{3}(1,1)$
$(10,17,25,34,43)$	$\{60,68\}$	6	$P_z,\ P_w,\ P_{xz} = \frac{1}{5}(4,3),\ P_{yt} = 2 \times \frac{1}{17}(3,16)$
$(11,18,27,44,61)$	$\{72,88\}$	$77/16$	$P_z,\ P_w,\ P_{xt} = 2 \times \frac{1}{11}(7,5),\ P_{yz} = \frac{1}{9}(4,7)$
$(11,27,36,62,97)$	$\{108,124\}$	$121/24$	$P_x,\ P_w,\ P_1 = \frac{1}{9}(4,7)$
$(11,29,39,49,59)$	$\{88,98\}$	$\geqslant 117/16$	$P_y,\ P_z,\ P_w$
$(11,29,39,49,67)$	$\{78,116\}$	$77/10$	$P_x,\ P_t,\ P_w$
$(11,29,38,48,85)$	$\{96,114\}$	$99/14$	$P_x,\ P_y,\ P_w$
$(13,22,55,76,97)$	$\{110,152\}$	$117/20$	$P_x,\ P_w,\ P_{yz} = \frac{1}{11}(4,9)$
$(13,23,34,56,89)$	$\{102,112\}$	$104/15$	$P_x,\ P_y,\ P_w$
$(13,23,35,47,57)$	$\{70,104\}$	$65/8$	$P_y,\ P_t,\ P_w$
$(13,23,35,57,79)$	$\{92,114\}$	$91/12$	$P_x,\ P_z,\ P_w$
$(14,19,25,32,45)$	$\{64,70\}$	$28/3$	$P_y,\ P_z,\ P_w$

*The polynomial of degree 6 contains the monomial yt.

**The polynomial of degree 6 does not contain the monomial yt.

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By Lemma 3.6, we may assume that the curve C_x is not contained in the support of D. Suppose that the point P is the point P_w. Since the curve C_x is singular at the point P_w with multiplicity at least 2, we have

$$\frac{2}{\lambda} < 2\,\mathrm{mult}_{P_w}(D) \leqslant 5D \cdot C_x = 5\frac{1 \times 1 \times 6 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = 2.$$

This is a contradiction. Thus, the point P cannot be the point P_w.

For every point Q in $C_x \setminus \{P_w\}$, we have

$$\mathrm{mult}_Q(D) \leqslant D \cdot C_x = \frac{1 \times 1 \times 6 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = \frac{2}{5} < \frac{1}{\lambda}.$$

Therefore, $P \notin C_x$.

Let $\mathcal{L}$ be the pencil on X cut by the equation

$$\alpha x^2 + \beta y = 0,$$

where $[\alpha, \beta] \in \mathbb{P}^1$. There is a curve C in the pencil $\mathcal{L}$ that passes through the point P. Since the point P lies outside C_x, the curve C is cut by an equation $y = \alpha x^2$ for some α. Therefore, the curve C is defined by the quasi-homogeneous equations

$$\begin{aligned} y &= \alpha x^2, \\ xf(x,y,z,t) + y(y^2 + at) + z^2 + xw &= 0, \\ xg(x,y,z,t,w) + t(by^2 + t) + zw &= 0. \end{aligned}$$

The curve C is quasi-smooth at the point P_w at which the curve C_x and C meet. The affine piece of C defined by $x \neq 0$ is the curve

$$\begin{aligned} f(1,\alpha,z,t) + \alpha^3 + a\alpha t + z^2 + w &= 0, \\ g(1,\alpha,z,t,w) + b\alpha^2 t + t^2 + zw &= 0 \end{aligned}$$

in $\mathrm{Spec}(\mathbb{C}[z,t,w])$. This is isomorphic to the curve defined by

$$t^2 + b\alpha^2 t - z(f(1,\alpha,z,t) + \alpha^3 + a\alpha t + z^2) + g(1,\alpha,z,t,-(f(1,\alpha,z,t) + \alpha^3 + a\alpha t + z^2)) = 0$$

in $\mathrm{Spec}(\mathbb{C}[z,t])$. Since the equation keeps the monomial t^2 regardless of the constants α, a and b, we have $\mathrm{mult}_Q(C) \leqslant 2$ for any point Q on C. Thus, we have $c(X, \frac{1}{2}C) \geqslant \lambda$. Furthermore, the equation always has the monomials t^2 and z^3 and so the curve C must be irreducible. Therefore, we may assume that the curve C is not contained in the support of D. Then,

$$\frac{1}{\lambda} < \mathrm{mult}_P(D) \leqslant D \cdot C = \frac{1 \times 2 \times 6 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = \frac{4}{5}.$$

This is a contradiction. The obtained contradiction completes the proof. □

Lemma 6.4. *Let X be a quasi-smooth complete intersection log del Pezzo surface defined by two quasi-homogeneous polynomials of degrees* 18 *and* 26 *in the weighted projective space* $\mathbb{P} = \mathbb{P}(1,5,9,13,17)$. *Then,* $\mathrm{lct}(X) = 1$.

Proof. The surface X can be assumed to be defined by the quasi-homogeneous equations

$$xf(x,y,z,t)+z^2+yt+xw=0,$$
$$xg(x,y,z,t,w)+t^2+zw=0,$$

where f and g are quasi-homogeneous equations of degrees 17 and 25, respectively. The surface X is singular at the points P_y and P_w.

The curve C_x is defined by the quasi-homogeneous equations

$$z^2+yt=0,$$
$$t^2+zw=0$$

in $\mathrm{Proj}(\mathbb{C}[y,z,t,w])$. The divisor C_x consists of L_{yw} and an irreducible and reduced curve R_x. Note that the curves L_{yw} and R_x meet at the points P_y and P_w.

Consider the open subset $C_x \setminus C_w$ of the curve C_x that is a $\mathbb{Z}_{17}$-quotient of the affine curve

$$z^2+yt=0,$$
$$t^2+z=0$$

in $\mathrm{Spec}(\mathbb{C}[y,z,t])$. The affine curve is isomorphic to the curve defined by the equation

$$yt+t^4=0\subset \mathrm{Spec}(\mathbb{C}[y,t]).$$

It shows that the log pair (X, C_x) is log canonical along $C_x \setminus C_w$. Consider the open subset $C_x \setminus C_y$ of the curve C_x that is a $\mathbb{Z}_5$-quotient of the affine curve

$$z^2+t=0,$$
$$t^2+zw=0$$

in $\mathrm{Spec}(\mathbb{C}[z,t,w])$. The affine curve is isomorphic to the curve defined by the equation

$$zw+z^4=0\subset \mathrm{Spec}(\mathbb{C}[z,w]).$$

Therefore, the log pair (X, C_x) is log canonical along $C_x \setminus C_y$. Consequently, $c(X, C_x) = 1$.

Suppose that $\mathrm{lct}(X) < 1$. There is then an effective $\mathbb{Q}$-divisor

$$D \equiv_{\mathbb{Q}} -K_X$$

such that the log pair (X, D) is not log canonical at some point $P \in X$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(9))$ contains the monomials x^4y, x^9 and z, by Lemma 3.7, we have the inequality

$$\mathrm{mult}_Q(D) \leqslant \frac{9\times 18\times 26}{1\times 5\times 9\times 13\times 17}=\frac{36}{85}<1$$

for each point $Q \in X \setminus C_x$. Therefore, the point P must belong to C_x.

We have the following intersection numbers:

$$-K_X \cdot L_{yw} = \tfrac{1}{85}, \qquad -K_X \cdot R_x = \tfrac{3}{85}, \qquad -K_X \cdot C_x = \tfrac{4}{85},$$
$$L_{yw} \cdot R_x = \tfrac{22}{85}, \qquad L_{yw}^2 = -\tfrac{21}{85}, \qquad R_x^2 = -\tfrac{19}{85}.$$

By Lemma 3.6, we may assume that the support of the curve C_x is not contained in the support of $\mathrm{Supp}(D)$.

Suppose that $\mathrm{Supp}(D)$ does not contain the curve R_x and put $D = \mu L_{yw} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor with $L_{yw} \not\subset \mathrm{Supp}(\Delta)$. The inequality

$$\tfrac{3}{85} = D \cdot R_x \geqslant \mu L_{yw} \cdot R_x = \tfrac{22}{85}\mu$$

implies that $0 \leqslant \mu \leqslant \frac{3}{22}$. Therefore, the log pair $(X, L_{yw} + \Delta)$ is not log canonical at the point P and the point P lies on the curve L_{yw}. However, this is impossible because of the inequality

$$\mathrm{mult}_Q(\Delta|_{L_{yw}}) \leqslant 17(D - \mu L_{yw}) \cdot L_{yw} = 17(\tfrac{1}{85} + \mu\tfrac{21}{85}) < 1$$

for each point Q on L_{yw}.

Now we suppose that $\mathrm{Supp}(D)$ does not contain the curve L_{yw} and put $D = \nu R_x + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor with $R_x \not\subset \mathrm{Supp}(\Omega)$. Then the inequality

$$\tfrac{1}{85} = D \cdot L_{yw} \geqslant R_x \cdot L_{yw} = \tfrac{22}{85}\nu$$

implies that $0 \leqslant \nu \leqslant \frac{1}{22}$. For any point Q on R_x we have

$$\mathrm{mult}_Q(\Omega|_{R_x}) \leqslant 17(D - \nu R_x) \cdot R_x = 17(\tfrac{3}{85} + \nu\tfrac{19}{85}) < 1.$$

From this inequality we can derive a contradiction as before.

Consequently, $\mathrm{lct}(X) = 1$. □

Lemma 6.5. *Let X be a quasi-smooth complete intersection log del Pezzo surface defined by two quasi-homogeneous polynomials of degree 6 in the weighted projective space $\mathbb{P}(2, 2, 3, 3, 3)$. Then, $\mathrm{lct}(X) \geqslant \frac{6}{5}$.*

Proof. The surface X is defined by the quasi-homogeneous equations

$$f_3(x, y) + f_2(z, t, w) = 0,$$
$$g_3(x, y) + g_2(z, t, w) = 0,$$

where f_i and g_j are homogeneous polynomials of degrees i and j, respectively. For X to be quasi-smooth, the equation $f_3(x, y)g_3(x, y) = 0$ must define six distinct points in $\mathbb{P}^1$ and for any $[a, b] \in \mathbb{P}^1$, the rank of the quadratic form $af_2(z, t, w) + bg_2(z, t, w)$ must be at least 2.

The surface X is singular only at the points P_1, P_2, P_3 and P_4 of type $\frac{1}{3}(1, 1)$, which are contained in the set $\{x = y = f_2(z, t, w) = g_2(z, t, w) = 0\}$.

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Let $\mathcal{L}$ be the pencil on X cut out by the equation

$$\alpha x + \beta y = 0,$$

where $[\alpha, \beta] \in \mathbb{P}^1$. Let C_β be the member of the pencil $\mathcal{L}$ cut out by the equation $x - \beta y = 0$, that is, C_β is defined by the quasi-homogeneous equations

$$f_3(\beta, 1)y^3 + f_2(z, t, w) = 0,$$
$$g_3(\beta, 1)y^3 + g_2(z, t, w) = 0$$

in $\mathrm{Proj}(\mathbb{C}[y, z, t, w])$. Note that the quadratic form

$$g_3(\beta, 1)f_2(z, t, w) - f_3(\beta, 1)g_2(z, t, w) = 0$$

is of rank either 2 or 3. Furthermore, there is a constant β' such that the corresponding quadratic form is of rank 2. The curve $C_{\beta'}$ is defined by the equations

$$(\lambda_1 z + \mu_1 t + \nu_1 w)(\lambda_2 z + \mu_2 t + \nu_2 w) = 0,$$
$$g_3(\beta', 1)y^3 + g_2(z, t, w) = 0,$$

where the two points $[\lambda_1 : \mu_1 : \nu_1]$ and $[\lambda_2 : \mu_2 : \nu_2]$ are distinct in $\mathbb{P}^2$. The divisor $C_{\beta'}$ consists of two irreducible and reduced curves C_1 and C_2. Each curve C_i is defined by $x - \beta' y = \lambda_i z + \mu_i t + \nu_i w = g_3(\beta', 1)y^3 + g_2(z, t, w) = 0$. The curves C_1 and C_2 intersect transversally. Note that a member C in the pencil $\mathcal{L}$ is quasi-smooth if its corresponding quadratic form is of rank 3. Thus, for each member C of the pencil $\mathcal{L}$, we have $c(X, \frac{1}{2}C) = 2$.

Now, we claim that $\mathrm{lct}(X) \geqslant \frac{6}{5}$. Suppose not. Then there is an effective $\mathbb{Q}$-divisor

$$D \equiv_{\mathbb{Q}} -K_X$$

such that the log pair $(X, \frac{6}{5}D)$ is not log canonical at some point $P \in X$.

We have the following intersection numbers:

$$-K_X \cdot C_1 = -K_X \cdot C_2 = \tfrac{1}{3}, \qquad C_1 \cdot C_2 = 1, \qquad C_1^2 = C_2^2 = -\tfrac{1}{3}.$$

Suppose that $P \in X \setminus \mathrm{Sing}(X)$. Then there is a curve $C \in \mathcal{L}$ passing through the point P. If the curve C is irreducible then we have

$$\mathrm{mult}_P\, D \leqslant D \cdot C = \frac{1 \times 2 \times 6 \times 6}{2 \times 2 \times 3 \times 3 \times 3} = \frac{2}{3} < \frac{5}{6}.$$

Therefore, the curve C must be reducible. It consists of two irreducible curves C_1 and C_2. By Lemma 3.6, we may assume that the support of the curve C is not contained in $\mathrm{Supp}(D)$. Without loss of generality, we can assume that $C_1 \not\subset \mathrm{Supp}(D)$. Since we have

$$\mathrm{mult}_P\, C_1 \leqslant D \cdot C_1 = \tfrac{1}{3} < \tfrac{5}{6},$$

the point P must belong to C_2. Put $D = \mu C_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains neither C_1 nor C_2. The inequality

$$\tfrac{1}{3} = D \cdot C_1 \geqslant \mu C_1 \cdot C_2 = \mu$$

implies that $0 \leqslant \mu \leqslant \frac{1}{3}$. Then the log pair $(X, C_2 + \frac{6}{5}\Omega)$ is not log canonical at the point P. However,

$$\Omega \cdot C_2 = (D - \mu C_2) \cdot C_2 \leqslant \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{1}{3} = \tfrac{4}{9} < \tfrac{5}{6},$$

which is a contradiction. Thus, $P \in \operatorname{Sing}(X)$. We may assume that the point P is the point P_1 without loss of generality.

Let $\pi\colon \bar{X} \to X$ be the weighted blow-up at the point P_1 with weights $(1,1)$. Let E be the exceptional divisor of π. We then have

$$K_{\bar{X}} \equiv \pi^*(K_X) - \frac{1}{3}E, \qquad \pi^* D \equiv \bar{D} + \frac{\alpha}{3}E,$$

where $\bar{D}$ is the proper transform of D and α is a positive rational number. The log pull-back of the log pair $(X, \frac{6}{5}D)$ is the log pair

$$\left(\bar{X}, \frac{6}{5}\bar{D} + \left(\frac{1}{3} + \frac{6}{5} \times \frac{\alpha}{3}\right)E\right).$$

This is not log canonical at some point $O \in E$. We then obtain

$$\frac{5}{9} - \frac{\alpha}{3} < \operatorname{mult}_O(\bar{D}).$$

Let $\mathcal{N}$ be the sublinear system of $|\mathcal{O}_{\mathbb{P}}(3)|$ that consists of curves passing through the point P_1. We can find two constants a and b such that the quadratic form $af_2(z,t,w) + bg_2(z,t,w)$ is of rank 2. Then the surface X is defined by the quasi-homogeneous equations

$$(c_1x + d_1y)(c_2x + d_2y)(c_3x + d_3y) + (\lambda_1 z + \mu_1 t + \nu_1 w)(\lambda_2 z + \mu_2 t + \nu_2 w) = 0,$$
$$g_3(x,y) + g_2(z,t,w) = 0,$$

where $(c_1x+d_1y)(c_2x+d_2y)(c_3x+d_3y) = af_3(x,y)+bg_3(x,y)$ and $(\lambda_1 z+\mu_1 t+\nu_1 w)(\lambda_2 z+\mu_2 t+\nu_2 w) = af_2(z,t,w)+bg_2(z,t,w)$. Note that the equations $x = y = \lambda_1 z+\mu_1 t+\nu_1 w = g_2(z,t,w) = 0$ define two singular points, say P_1 and P_2, of the four singular points of X. Let N be the member of the linear system $\mathcal{N}$ cut out by the equation $\lambda_1 z+\mu_1 t+\nu_1 w = 0$. Then N consists of the three irreducible and reduced curves M_1, M_2 and M_3 intersecting each other at the points P_1 and P_2.

By Lemma 3.6, we can assume that the support of D does not contain one of M_1, M_2, M_3. Without loss of generality, we can assume that the curve M_1 is not contained in the support of D. The inequality

$$0 \leqslant \bar{D} \cdot \bar{M}_1 = D \cdot M_1 + \frac{\alpha}{9}E^2 = \frac{1}{3} - \frac{\alpha}{3}$$

implies that $\alpha \leqslant 1$. Since

$$\frac{1}{3} + \frac{6}{5} \times \frac{\alpha}{3} \leqslant 1,$$

the log pair $(\bar{X}, \frac{6}{5}\bar{D} + E)$ is not log canonical at the point O. For the point $O \in E$ there exists a member C in the pencil $\mathcal{N}$ whose proper transform $\bar{C}$ on $\bar{X}$ passes through the point O. Suppose that C is irreducible. Then we have

$$\pi^* C \equiv \bar{C} + E.$$

We have

$$1 = D \cdot C = \bar{D} \cdot \bar{C} + \alpha,$$

and hence $\bar{D} \cdot \bar{C} = 1 - \alpha$. Thus,

$$\frac{5}{9} - \frac{\alpha}{3} < \mathrm{mult}_O \bar{D} \leqslant \bar{D} \cdot \bar{C} = 1 - \alpha$$

implies that $\alpha < \frac{2}{3}$. However, by Lemma 3.5, we have

$$\tfrac{5}{6} < \mathrm{mult}_O(\bar{D}|_E) \leqslant \bar{D} \cdot E = \alpha,$$

which is a contradiction. Thus, the member C must be reducible. Then C consists of the three curves L_1, L_2 and L_3. We can assume that the support of the member C is not contained in the support of D. Without loss of generality, we can assume that the curve L_1 is not contained in the support of D. Put $D = \alpha_2 L_2 + \alpha_3 L_3 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that L_2 and L_3 are not contained in the support of Ω. We then have

$$\pi^* \Omega \equiv \bar{\Omega} + \frac{\beta}{3} E.$$

The inequality

$$\tfrac{1}{3} = D \cdot L_1 \geqslant \alpha_2 L_2 \cdot L_1 + \alpha_3 L_3 \cdot L_1$$

implies that $\alpha_2 + \alpha_3 \leqslant \frac{1}{3}$. If $O \in \bar{L}_1$, then

$$\frac{1}{3} - \frac{\alpha}{3} = \bar{D} \cdot \bar{L}_1 \geqslant \mathrm{mult}_O \bar{D} > \frac{5}{9} - \frac{\alpha}{3},$$

which is a contradiction. If $O \in \bar{L}_2$, then we consider the log pair

$$(\bar{X}, \tfrac{6}{5}(\alpha_2 \bar{L}_2 + \alpha_3 \bar{L}_3 + \bar{\Omega}) + \theta E),$$

where

$$\theta = \frac{1}{3} + \frac{6}{5}\left(\frac{\alpha_2}{3} + \frac{\alpha_3}{3} + \frac{\beta}{3}\right).$$

Since it is not log canonical at the point O that lies on $\bar{L}_2 \setminus \bar{L}_1$, we have

$$1 < \frac{6}{5}\bar{\Omega} \cdot \bar{L}_2 + \theta E \cdot \bar{L}_2 = \frac{6}{5}\left(\frac{1}{3} + \frac{1}{3}\alpha_2 - \alpha_3 - \frac{\beta}{3}\right) + \frac{1}{3} + \frac{6}{5}\left(\frac{\alpha_2}{3} + \frac{\alpha_3}{3} + \frac{\beta}{3}\right).$$

Thus, $\alpha_2 - \alpha_3 > \frac{1}{3}$, which is a contradiction. Similarly, if $O \in \bar{L}_3$, we can obtain a contradiction. □

Lemma 6.6. *Let X be a quasi-smooth complete intersection log del Pezzo surface defined by two quasi-homogeneous polynomials of degrees* 28 *and* 30 *in the weighted projective space* $\mathbb{P} = \mathbb{P}(5, 7, 10, 14, 23)$. *Then,* $\mathrm{lct}(X) = \frac{35}{12}$.

Proof. The surface X can be defined by the quasi-homogeneous equations

$$t(t - y^2) + xw = 0,$$
$$(z - b_1x^2)(z - b_2x^2)(z - b_3x^2) + yw = 0,$$

where the b_i, $i = 1, 2, 3$, are distinct constants. The surface X is singular at the point P_w. The surface X also has three singular points of type $\frac{1}{5}(2, 4)$ at $P_1 = [1:0:b_1:0:0]$, $P_2 = [1:0:b_2:0:0]$ and $P_3 = [1:0:b_3:0:0]$ and two singular points of type $\frac{1}{7}(5, 3)$ at $Q_1 = [0:1:0:0:0]$, $Q_2 = [0:1:0:1:0]$.

The curve C_x is defined by the quasi-homogeneous equations

$$t(t - y^2) = 0,$$
$$z^3 + yw = 0$$

in $\mathrm{Proj}(\mathbb{C}[y, z, t, w])$. It shows that the curve C_x consists of two irreducible and reduced curves $C_{x,1}$ and $C_{x,2}$. The curve $C_{x,1}$ is defined by $x = t = z^3 + yw = 0$ and the second curve $C_{x,2}$ is defined by $x = t - y^2 = z^3 + yw = 0$. Consider the open set $C_x \setminus C_w$ of the curve C_x that is a $\mathbb{Z}_{23}$-quotient of the affine curve

$$t(t - y^2) = 0,$$
$$z^3 + y = 0$$

in $\mathrm{Spec}(\mathbb{C}[y, z, t])$. The affine curve is isomorphic to the curve defined by

$$t(t - z^6) = 0 \subset \mathrm{Spec}(\mathbb{C}[z, t]).$$

It shows that the log canonical threshold of the log pair (X, C_x) at the point P_w is $\frac{7}{12}$ and the log pair is log canonical elsewhere. Therefore, $c(X, \frac{1}{5}C_x) = \frac{35}{12}$.

The curve C_y is defined by the quasi-homogeneous equations

$$t^2 + xw = 0,$$
$$(z - b_1x^2)(z - b_2x^2)(z - b_3x^2) = 0$$

in $\mathrm{Proj}(\mathbb{C}[x, z, t, w])$. Then $C_y = C_{y,1} + C_{y,2} + C_{y,3}$, where the $C_{y,j}$ are irreducible and reduced curves defined by the equations $y = t^2 + xw = z - b_jx^2 = 0$. Consider the open set $C_y \setminus C_w$ of the curve C_y that is a $\mathbb{Z}_{23}$-quotient of the affine curve

$$t^2 + x = 0,$$
$$(z - b_1x^2)(z - b_2x^2)(z - b_3x^2) = 0$$

in $\mathrm{Spec}(\mathbb{C}[x, z, t])$. The affine curve is isomorphic to the curve defined by

$$(z - b_1t^4)(z - b_2t^4)(z - b_3t^4) = 0 \subset \mathrm{Spec}(\mathbb{C}[z, t]).$$

This shows that the log canonical threshold of the log pair (X, C_y) at the point P_w is at least $\frac{5}{12}$ and the log pair is log canonical elsewhere. Therefore, the log pair $(X, \frac{5}{12}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < \frac{35}{12}$. Then there is an effective $\mathbb{Q}$-divisor

$$D \equiv_{\mathbb{Q}} -K_X$$

such that the log pair $(X, \frac{35}{12}D)$ is not log canonical at some point $P \in X$.

Since the space $H^0(\mathbb{P}, O_{\mathbb{P}}(35))$ contains the monomials x^7, y^5 and xz^3, for each point $Q \in X \setminus (C_x \cup \mathrm{Sing}(X))$ we obtain the inequality

$$\mathrm{mult}_Q(D) \leqslant \frac{35 \times 28 \times 30}{5 \times 7 \times 10 \times 14 \times 23} < \frac{12}{35}$$

from Lemma 3.7. It follows that the point P must belong to $C_x \cup \mathrm{Sing}(X)$.

We have the following intersection numbers:

$$-K_X \cdot C_{x,1} = -K_X \cdot C_{x,2} = \frac{3}{7 \times 23}, \qquad C_{x,1} \cdot C_{x,2} = \frac{6}{23}, \qquad C_{x,1}^2 = C_{x,2}^2 = -\frac{27}{7 \times 23},$$
$$-K_X \cdot C_{y,1} = -K_X \cdot C_{y,2} = -K_X \cdot C_{y,3} = \frac{2}{5 \times 23},$$
$$C_{y,1} \cdot C_{y,2} = C_{y,1} \cdot C_{y,3} = C_{y,2} \cdot C_{y,3} = \frac{4}{23}, \qquad C_{y,1}^2 = C_{y,2}^2 = C_{y,3}^3 = -\frac{26}{5 \times 23}.$$

By Lemma 3.6, we may assume that the support of the curve C is not contained in $\mathrm{Supp}(D)$. Suppose that $C_{x,2}$ is not contained in the support of D. Put $D = \mu C_{x,1} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve $C_{x,1}$. The inequality

$$\frac{3}{7 \times 23} = D \cdot C_{x,2} \geqslant \mu C_{x,1} \cdot C_{x,2} = \mu \frac{6}{23}$$

implies that $0 \leqslant \mu \leqslant \frac{1}{14}$. For any point O_2 other than P_w on $C_{x,2}$, we have

$$\mathrm{mult}_{O_2}(D) \leqslant 7(D \cdot C_{x,2}) = \tfrac{3}{23} < \tfrac{12}{35},$$

and for any point O_1 on $C_{x,1}$ other than P_w, we have

$$\mathrm{mult}_{O_1}(\Delta|_{C_{x,1}}) \leqslant 7(D - \mu C_{x,1}) \cdot C_{x,1} \leqslant 7\left(\frac{3}{7 \times 23} + \frac{1}{14} \times \frac{27}{7 \times 23}\right) < \frac{12}{35}.$$

These show that the point P must be one of the singular points of X other than Q_1 and Q_2. Similarly, if the curve $C_{x,1}$ is not contained in the support of D, then we can derive the same conclusion.

By Lemma 3.6, we may assume that the support of the curve C_y is not contained in $\mathrm{Supp}(D)$. Suppose that $C_{y,1}$ is not contained in the support of D. Put $D = \nu_2 C_{y,2} + \nu_3 C_{y,3} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains neither $C_{y,2}$ nor $C_{y,3}$. The inequality

$$\frac{2}{5 \times 23} = D \cdot C_{y,1} \geqslant \nu_2 C_{y,2} \cdot C_{y,1} + \nu_3 C_{y,3} \cdot C_{y,1} = \frac{4}{23}$$

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implies that $\nu_1+\nu_2\leqslant\frac{1}{10}$. We have

$$\begin{aligned}\operatorname{mult}_{P_1}(D)&\leqslant 5(D\cdot C_{y,1})=\frac{2}{23}<\frac{12}{35},\\ \operatorname{mult}_{P_2}(D|_{C_{y,2}})&\leqslant 5(D-\nu_2C_{y,2})\cdot C_{y,2}\leqslant 5\bigg(\frac{2}{5\times 23}+\frac{1}{10}\times\frac{26}{5\times 23}\bigg)<\frac{12}{35},\\ \operatorname{mult}_{P_3}(D|_{C_{y,3}})&\leqslant 5(D-\nu_3C_{y,3})\cdot C_{y,3}\leqslant 5\bigg(\frac{2}{5\times 23}+\frac{1}{10}\times\frac{26}{5\times 23}\bigg)<\frac{12}{35}.\end{aligned}$$

Thus, the point P cannot be any of the points P_1, P_2 or P_3. Similarly, if either $C_{y,2}$ or $C_{y,3}$ is not contained in the support of D, then we obtain the same result. Thus, the point P must be the point P_w.

Let $\pi\colon\bar{X}\to X$ be the weighted blow-up at the point P_w with weights $(4,1)$. As before, we may assume that D contains neither $C_{x,i}$ nor $C_{y,j}$ for some $i\in\{1,2\}$ and some $j\in\{1,2,3\}$. Let E be the exceptional divisor of π. Then,

$$\begin{aligned}K_{\bar{X}}&\equiv_{\mathbb{Q}}\pi^*(K_X)-\frac{18}{23}E, &\qquad \pi^*C_{x,i}&\equiv_{\mathbb{Q}}\bar{C}_{x,i}+\frac{1}{23}E,\\ \pi^*C_{y,j}&\equiv_{\mathbb{Q}}\bar{C}_{y,j}+\frac{4}{23}E, &\qquad \pi^*D&\equiv_{\mathbb{Q}}\bar{D}+\frac{\alpha}{23}E,\end{aligned}$$

where $\bar{C}_{x,i}$, $\bar{C}_{y,j}$ and $\bar{D}$ are the proper transforms of $C_{x,i}$, $C_{y,j}$ and D, respectively, and α is a positive rational number. The divisor E contains one singular point O_4 of type $\frac{1}{4}(1,1)$ on the surface $\bar{X}$. The point O_4 is contained in $\bar{C}_{x,i}$ but not in $\bar{C}_{y,1}$.

The log pull-back of the log pair $(X,\frac{35}{12}D)$ is the log pair

$$\bigg(\bar{X},\frac{35}{12}\bar{D}+\bigg(\frac{35}{12}\times\frac{\alpha}{23}+\frac{18}{23}\bigg)E\bigg).$$

This is not log canonical at some point $O\in E$. We have the inequality

$$0\leqslant\bar{C}_{y,j}\cdot\bar{D}=C_{y,j}\cdot D+\frac{4\alpha}{23^2}E^2=\frac{2}{5\times 23}-\frac{\alpha}{23}.$$

Hence, we have $\alpha\leqslant\frac{2}{5}$. Since

$$\frac{35}{12}\times\frac{\alpha}{23}+\frac{18}{23}<1,$$

the log pair

$$(\bar{X},\tfrac{35}{12}\bar{D}+E)$$

is not log canonical at the point O. Since

$$\frac{35}{12}\bar{D}\cdot E=\frac{35}{12}\times\frac{\alpha}{4}<1,$$

the point O must be the singular point of $\bar{X}$ on the exceptional curve E, which is the point O_4. The inequality

$$1<\operatorname{mult}_O\bigg(\frac{35}{12}\bar{D}+\bigg(\frac{35}{12}\times\frac{\alpha}{23}+\frac{18}{23}\bigg)E\bigg)$$

implies that

$$\frac{12}{161} - \frac{\alpha}{23} < \mathrm{mult}_O(\bar{D}).$$

However,

$$\mathrm{mult}_O(\bar{D}) \leqslant 4\bar{D} \cdot \bar{C}_{x,i} = 4\left(\frac{3}{161} - \frac{\alpha}{4 \times 23}\right),$$

which is a contradiction. The obtained contradiction completes the proof. □

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